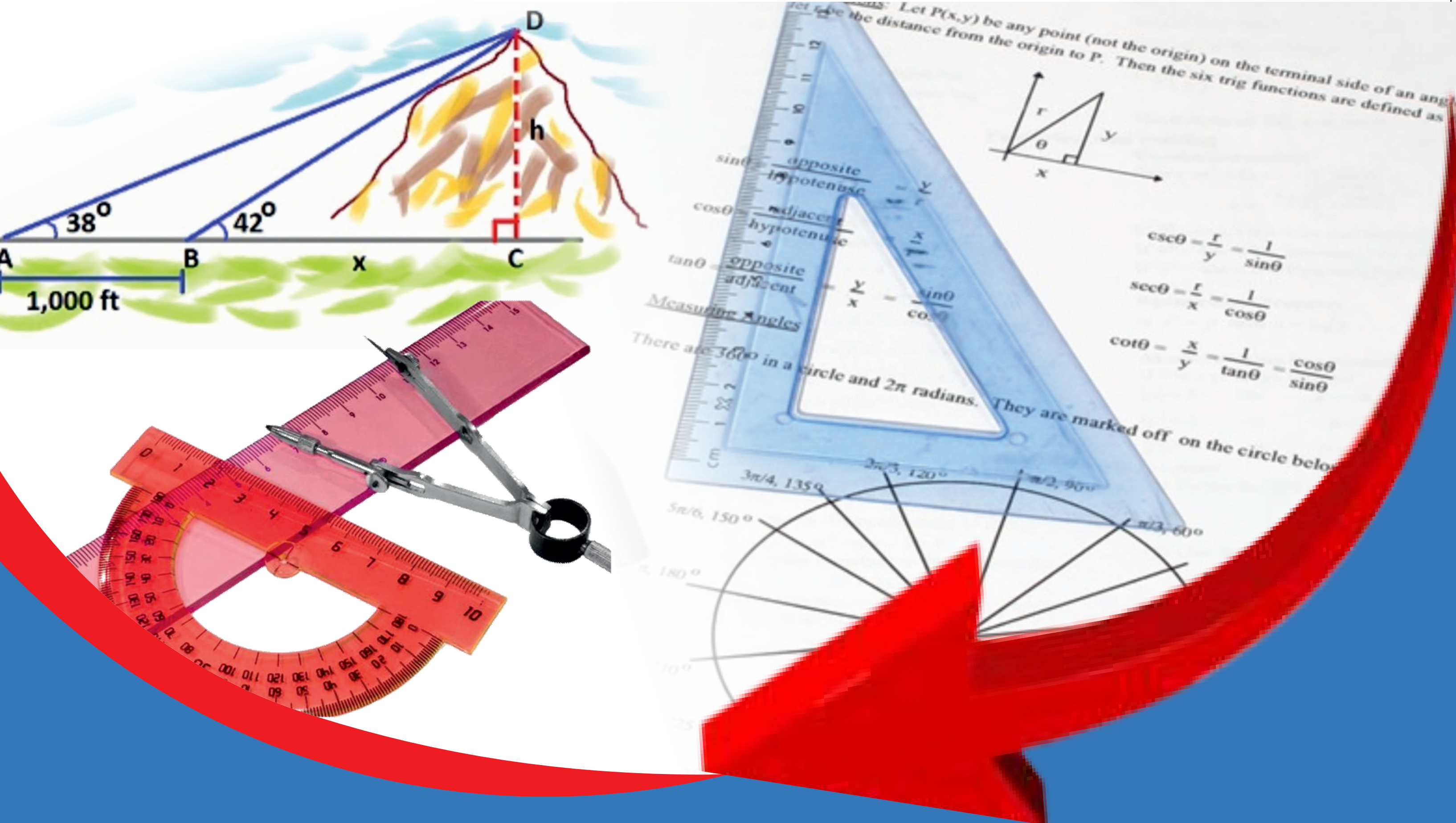


# Trigonometry Booster

*with Problems & Solutions*

*for*

# JEE

Main and Advanced

McGraw Hill Education

Rejaul Makshud

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# About the Author

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**Rejaul Makshud**
M. Sc. (Calcutta University, Kolkata)

**McGraw Hill Education (India) Private Limited**
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*Dedicated to*

*My Parents*

# Preface

*TRIGONOMETRY BOOSTER with Problems & Solutions for JEE Main and Advanced* is meant for aspirants preparing for the entrance examination of different technical institutions, especially NIT/IIT/BITSAT/IISc. In writing this book, I have drawn heavily from my long teaching experience at National Level Institutes. After many years of teaching I have realised the need of designing a book that will help the readers to build their base, improve their level of mathematical concepts and enjoy the subject.

This book is designed keeping in view the new pattern of questions asked in JEE Main and Advanced Exams. It has five chapters. Each chapter has the concept booster followed by a large number of exercises with the exact solutions to the problems as given below:

| | | |
|---|---|---|
| Level - I | : | Problems based on Fundamentals |
| Level - II | : | Mixed Problems (Objective Type Questions) |
| Level - III | : | Problems for JEE Advanced Exam |
| Level - IV | : | Tougher Problems for JEE Advanced Exams |
| (0.......9) | : | Integer type Questions |
| Passages | : | Comprehensive Link Passages |
| Matching | : | Matrix Match |
| Reasoning | : | Assertion and Reason |
| Previous years papers | : | Questions asked in Previous Years' IIT-JEE Exams |

Remember friends, no problem in mathematics is difficult. Once you understand the concept, they will become easy. So please don't jump to exercise problems before you go through the Concept Booster and the objectives. Once you are confident in the theory part, attempt the exercises. The exercise problems are arranged in a manner that they gradually require advanced thinking.

I hope  this book will help you to build your base, enjoy the subject and improve your confidence to tackle any type of problem easily and skilfully.

My special thanks goes to Mr. M.P. Singh (IISc. Bangalore), Mr. Manoj Kumar (IIT, Delhi), Mr. Nazre Hussain (B. Tech.), Dr. Syed Kashan Ali (MBBS) and Mr. Shahid Iqbal, who have helped, inspired and motivated me to accomplish this task. As a matter of fact, teaching being the best learning process, I must thank all my students who inspired me most for writing this book.

I would like to convey my affectionate thanks to my wife, who helped me immensely and my children who bore with patience my neglect during the period I remained devoted to this book.

I also convey my sincere thanks to Mr Biswajit Das of McGraw Hill Education for publishing this book in such a beautiful format.

I owe a special debt of gratitude to my father and elder brother, who taught me the first lesson of Mathematics and to all my learned teachers— Mr. Swapan Halder, Mr. Jadunandan Mishra, Mr. Mahadev Roy and Mr. Dilip Bhattacharya, who instilled the value of quality teaching in me.

I have tried my best to keep this book error-free. I shall be grateful to the readers for their constructive suggestions toward the improvement of the book.

**Rejaul Makshud**
M. Sc. (Calcutta University, Kolkata)

# Contents

CHAPTER

# 1 The Ratios and Identities

## CONCEPT BOOSTER

### 1.1 Introduction

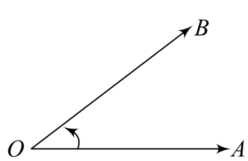
Trigonometry (from Greek trigonon “triangle” + metron “measure”) is a branch of mathematics that studies triangles and the relationships of the lengths of their sides and the angles between those sides.

Trigonometry defines the trigonometric functions, which describe those relationships and have applicability to cyclical phenomena, such as waves. This field, evolved during the third century BC as a branch of geometry, was used extensively for astronomical studies. It is also the foundation of the practical art of surveying.

Trigonometry basics are often taught in school either as a separate course or as part of a pre-calculus course. The trigonometric functions are pervasive in parts of pure mathematics and applied mathematics such as Fourier analysis and the wave equation, which are in turn essential to many branches of science and technology.

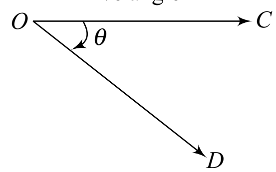
### 1.2 Measurement of Angles

1. **Angle**: The measurement of an angle is the amount of rotation from the initial side to the terminal side.
2. **Sense of an Angle**: The sense of an angle is +ve or –ve based on whether the initial side rotates in the anti-clock-wise or clockwise direction to get the terminal side.

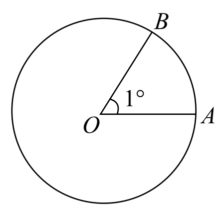



Positive angle

–ve angle




3. **System of measuring angles**
   There are three systems of measuring angles such as
   (i) Sexagesimal system
   (ii) Centisimal system
   (iii) Circular system
4. In sexagesimal system, we have
   1 right angle = $90^o$
   $1^o = 60'$
   $1' = 60''$
5. In centasimal system, we have
   1 right angle = $100^g$
   $1^g = 100'$
   $1' = 100''$
6. In circular system, the unit of measurement is radian
   **Radian**: One radian is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.
   Here, $\angle AOB$ = 1 radian = $1^e$.




***Notes***
(i) When an angle is expressed in radians, the word radian is omitted.

(ii) Since $180^{\circ} = \pi$ radian $= \left(\dfrac{22}{7 \times 180}\right)$ radian = 0.01746 radian

(iii) 1 radian $= \dfrac{180^{\circ}}{\pi} = \left(\dfrac{180}{22} \times 7\right) = 57^{\circ}16'22'$

(iv) The angle between two consecutive digits is $30^{\circ}$ $\left(\dfrac{\pi}{6}\text{radians}\right)$

(v) The hour hand rotates through an angle of 30° in one hour (i.e. $\left(\dfrac{1}{2}\right)$ in one minute)

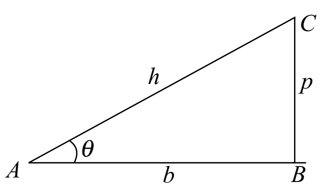
(vi) The minute hand rotates through an angle of 6° in one minute.

(vii) The relation amongst three systems of measurement of an angle is
$$\frac{D}{90^{\circ}} = \frac{G}{100} = \frac{2R}{\pi}$$

(viii) The number of radians in an angle subtended by an arc of a circle at the centre is $\dfrac{\text{Arc}}{\text{Radius}}$, i.e., $\theta = \dfrac{s}{r}$

## 1.3 TRIGONOMETRICAL RATIOS

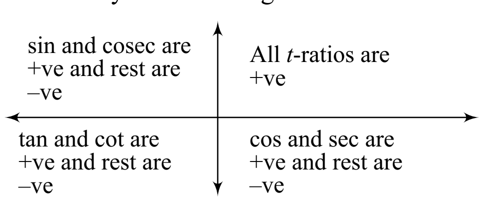
### 1.3.1 Definitions of Trigonometric Ratios




1. $\sin\theta = \dfrac{p}{h}$
2. $\text{cosec}\,\theta = \dfrac{h}{p}$
3. $\cos\theta = \dfrac{b}{h}$
4. $\sec\theta = \dfrac{h}{b}$
5. $\tan\theta = \dfrac{p}{b}$
6. $\cot\theta = \dfrac{b}{p}$

### 1.3.2 Signs of Trigonometrical Ratios

The signs of the trigonometrical ratios in different quadrants are remembered by the following chart.




It is also known as all, sin, tan, cos formula.

### 1.3.3 Relation between the Trigonometrical Ratios of an Angle

**Step-I:** (i) $\sin\theta \cdot \text{cosec}\,\theta = 1$

(ii) $\cos\theta \cdot \sec\theta = 1$

(iii) $\tan\theta \cdot \cot\theta = 1$

**Step-II:** (i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(ii) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

**Step-III:** (i) $\sin\theta \cdot \text{cosec}\,\theta = 1$

(ii) $\cos\theta \cdot \sec\theta = 1$

(iii) $\tan\theta \cdot \cot\theta = 1$

**Step-IV:** (i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $\sec^2\theta = 1 + \tan^2\theta$

(iii) $\text{cosec}^2\theta = 1 + \cot^2\theta$

**Step-V: Ranges of odd power t-ratios**

(i) $-1 \le \sin^{2n+1}\theta, \cos^{2n+1}\theta \le 1$

(ii) $-\infty < \tan^{2n+1}\theta, \cot^{2n+1}\theta < \infty$

(iii) $\text{cosec}^{2n+1}\theta, \sec^{2n+1}\theta \ge 1$

$\text{cosec}^{2n+1}\theta, \sec^{2n+1}\theta \le -1$

where $n \in W$

**Step-VI: Ranges of even power t-ratios**

(i) $0 \le \sin^{2n}\theta, \cos^{2n}\theta \le 1$

(ii) $0 \le \tan^{2n}\theta, \cot^{2n}\theta < \infty$

(iii) $1 \le \text{cosec}^{2n}\theta, \sec^{2n}\theta < \infty$

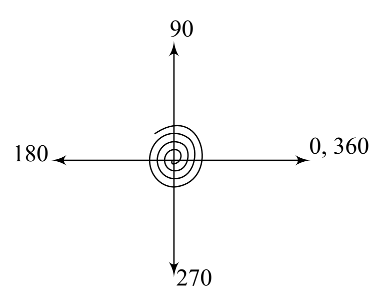
where $n \in N$

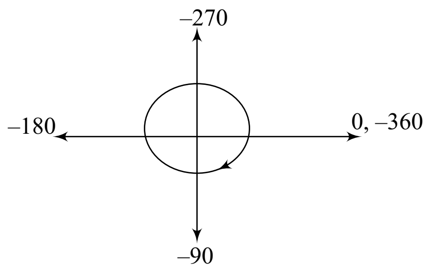
## 1.4 LIMITS OF THE VALUES OF TRIGONOMETRICAL FUNCTIONS

1. $-1 \le \sin\theta \le 1$
2. $-1 \le \cos\theta \le 1$
3. $\text{cosec}\,\theta \ge 1$ and $\text{cosec}\,\theta \le -1$
4. $\sec\theta \ge 1$ and $\sec\theta \le -1$
5. $-\infty < \tan\theta < \infty$
6. $-\infty < \cot\theta < \infty$

## 1.5 SIGN OF TRIGONOMETRIC RATIOS

(E) Rotation






## 1.6 T-RATIOS OF THE ANGLE $(-\theta)$, IN TERMS OF $\theta$, FOR ALL VALUES OF $\theta$

1. (i) $\sin(-\theta) = -\sin\theta$
   (ii) $\cos(-\theta) = \cos\theta$
   (iii) $\tan(-\theta) = -\tan\theta$
   (iv) $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$
   (v) $\sec(-\theta) = \sec\theta$
   (vi) $\cot(-\theta) = -\cot\theta$

## 1.7 T-RATIOS OF THE DIFFERENT ANGLES IN TERMS OF $\theta$, FOR ALL VALUES OF $\theta$

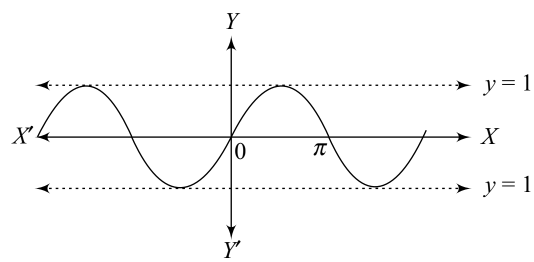
2. (i) $\sin(90-\theta) = \sin(90° \times 1 - \theta) = \cos\theta$
   (ii) $\sin(90+\theta) = \sin(90° \times 1 + \theta) = \cos\theta$
   (iii) $\sin(180-\theta) = \sin(90° \times 2 - \theta) = \sin\theta$
   (iv) $\sin(180+\theta) = \sin(90° \times 2 + \theta) = -\sin\theta$
   (v) $\sin(270-\theta) = \sin(90° \times 3 - \theta) = -\cos\theta$
   (vi) $\sin(270+\theta) = \sin(90° \times 3 + \theta) = -\cos\theta$
   (vii) $\sin(360-\theta) = \sin(90° \times 4 - \theta) = -\sin\theta$
   (viii) $\sin(360+\theta) = \sin(90° \times 4 + \theta) = \sin\theta$
3 (i) $\cos(90-\theta) = \cos(90° \times 1 - \theta) = \sin\theta$
   (ii) $\cos(90+\theta) = \cos(90° \times 1 + \theta) = -\sin\theta$
   (iii) $\cos(180-\theta) = \cos(90° \times 2 - \theta) = -\cos\theta$
   (iv) $\cos(180+\theta) = \cos(90° \times 2 + \theta) = -\cos\theta$
   (v) $\cos(270-\theta) = \cos(90° \times 3 - \theta) = -\sin\theta$
   (vi) $\cos(270+\theta) = \cos(90° \times 3 + \theta) = -\sin\theta$
   (vii) $\cos(360-\theta) = \cos(90° \times 4 - \theta) = \cos\theta$
   (viii) $\cos(360+\theta) = \cos(90° \times 4 + \theta) = \cos\theta$.
4. (i) $\tan(90-\theta) = \tan(90° \times 1 - \theta) = \cot\theta$
   (ii) $\tan(90+\theta) = \tan(90° \times 1 + \theta) = -\cot\theta$
   (iii) $\tan(180-\theta) = \tan(90° \times 2 - \theta) = -\tan\theta$
   (iv) $\tan(180+\theta) = \tan(90° \times 2 + \theta) = \tan\theta$
   (v) $\tan(270-\theta) = \tan(90° \times 3 - \theta) = \cot\theta$
   (vi) $\tan(270+\theta) = \tan(90° \times 3 + \theta) = -\cot\theta$
   (vii) $\tan(360-\theta) = \tan(90° \times 4 - \theta) = -\tan\theta$
   (viii) $\tan(360+\theta) = \tan(90° \times 4 + \theta) = \tan\theta$

***Note:* All the above results can be remembered by the following simple rule.**

1. If $\theta$ be measured with an even multiple of 90° by + or – sign, then the T-ratios remains unaltered (i.e., sine remains sine and cosine remains cosine, etc.) and treating $\theta$ as an acute angle, the quadrant in which the associated angle lies, is determined and then the sign of the T-ratio is determined by the All – Sin – Tan – Cos formula.
2. If $\theta$ be associated with an odd multiple of 90 by +ve or –ve sign, then the T-ratios is altered in form (i.e., sine becomes cosine and cosine becomes sine, tangent becomes cotangent and conversely, etc.) and the sign of the ratio is determined as in the previous paragraph.
3. If the multiple of 90 is more than 4, then divide it by 4 and find out remainder. If remainder is 0, then the degree lies on right of $x$-axis, if remainder is 1, then the degree lies on the +ve $y$-axis, if remainder is 2, then the degree lies on –ve of $x$-axis and if the remainder is 3, then the degree lies on the –ve of $y$-axis respectively.

## 1.8 GRAPH OF TRIGONOMETRIC FUNCTIONS

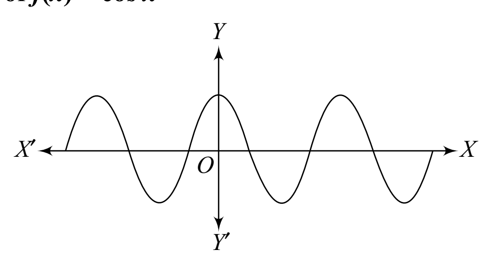
**1. Graph of $f(x) = \sin x$**




**Characteristics of Sine Function**

1. It is an odd function, since $\sin(-x) = -\sin x$
2. It is a periodic function with period $2\pi$
3. $\sin x = 1 \Rightarrow x = (4n+1)\dfrac{\pi}{2}, n \in I$
4. $\sin x = 0 \Rightarrow x = n\pi, n \in I$
5. $\sin x = -1 \Rightarrow x = (4n-1)\dfrac{\pi}{2}, n \in I$

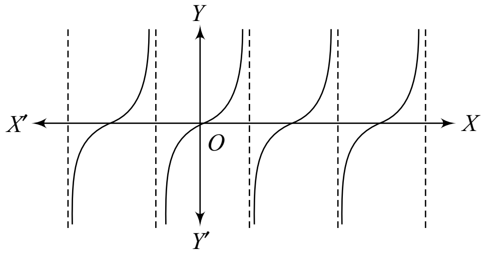
**Graph of $f(x) = \cos x$**




**Characteristics of cosine function**

1. It is an even function, since $\cos(-x) = \cos x$
2. It is a periodic function with period $2\pi$.
3. $\cos x = 1 \Rightarrow x = 2n\pi, n \in I$
4. $\cos x = 0 \Rightarrow x = (2n+1)\dfrac{\pi}{2}, n \in I$
5. $\cos x = -1 \Rightarrow x = (2n+1)\pi, n \in I$

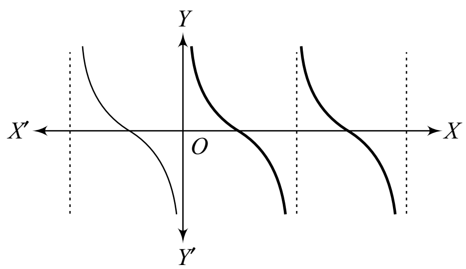
**3. Graph of *f*(*x*) = tan *x***




**Characteristics of tangent function**

1. It is an odd function, since tan (–$x$) = –tan $x$
2. It is a periodic function with period $\pi$
3. $\tan x = 1 \Rightarrow x = (4n+1)\dfrac{\pi}{4}, n \in I$
4. $\tan x = 0 \Rightarrow x = n\pi, n \in I$
5. $\tan x = -1 \Rightarrow x = (4n-1)\dfrac{\pi}{4}, n \in I$

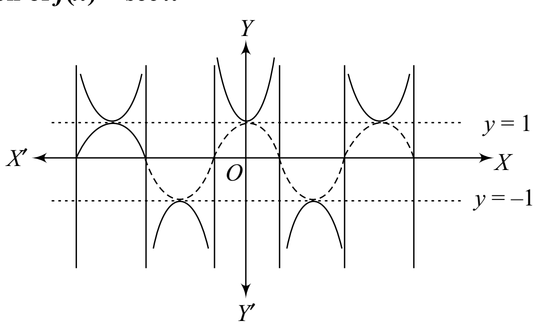
**4. Graph of *f*(*x*) = cot *x***




**Characteristics of cotangent function**

1. It is an odd function, since cot (–$x$) = –cot $x$
2. It is a periodic function with period $2\pi$
3. $\cot x = 1 \Rightarrow x = (4n+1)\dfrac{\pi}{4}, n \in I$
4. $\cot x = 0 \Rightarrow x = (2n+1)\dfrac{\pi}{2}, n \in I$
5. $\cot x = -1 \Rightarrow x = (4n-1)\dfrac{\pi}{4}, n \in I$

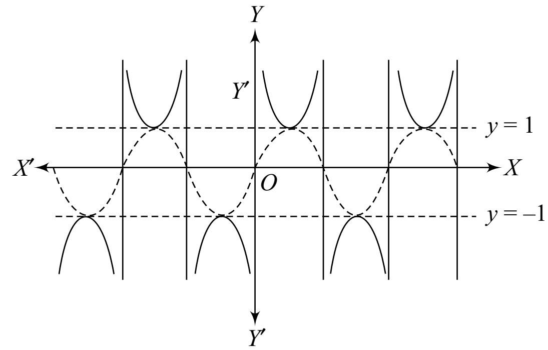
**Graph of *f*(*x*) = sec *x***




**Characteristics of secant function**

1. It is an even function, sec(–$x$) = sec $x$
2. It is a periodic function with period $2\pi$
3. sec $x$ can never be zero.
4. $\sec x = 1 \Rightarrow x = 2n\pi, n \in I$
5. $\sec x = -1 \Rightarrow x = (2n+1)\pi, n \in I$

**Graph of *f*(*x*) = cosec *x***




**Characteristics of co-secant function**

1. It is an odd function, since
   cosec (–$x$) = –cosec $x$
2. It is a periodic function with period $2\pi$
3. $\operatorname{cosec} x = 1 \Rightarrow x = (4n+1)\dfrac{\pi}{2}, n \in I$
4. cosec $x$ can never be zero.
5. $\sec x = -1 \Rightarrow x = (4n-1)\dfrac{\pi}{2}, n \in I$

## 1.9 T-ratios of Compound Angles

### 1.11 Definition

The algebraic sum or difference of two or more angles is called a compound angle such as

$A + B, A - B, A + B + C, A + B - C$, etc.

### 1.9.1 The Addition Formula

1. $\sin (A + B) = \sin A \cos B + \cos A \sin B$
2. $\cos (A + B) = \cos A \cos B - \sin A \sin B$
3. $\tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

### 1.9.2 Subtraction Formulae

1. $\sin (A - B) = \sin A \cos B - \cos A \sin B$
2. $\cos (A - B) = \cos A \cos B + \sin A \sin B$
3. $\tan (A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

## 1.10 Some Important Deductions

**Deduction 1**

$\sin (A + B) \sin (A - B)$
$= \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

**Proof**: We have $\sin (A + B) \sin (A - B)$

$= \{\sin A \cos B + \cos A \sin B\}$
$\times \{\sin A \cos B - \cos A \sin B\}$
$= \{\sin^2 A \cos^2 B - \cos^2 A \sin^2 B\}$
$= \{\sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B\}$
$= \{\sin^2 A - \sin^2 A \sin^2 B - \sin^2 B - \sin^2 A \sin^2 B$
$= \sin^2 A - \sin^2 B$

$= (1 - \cos^2 A) - (1 - \cos^2 B)$
$= \cos^2 B - \cos^2 A$

**Deduction 2**

$\cos (A + B) \cos (A - B)$
$= \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

**Proof:** We have, $\cos (A + B) \cos (A - B)$
$= \{\cos A \cos B + \sin A \sin B\}$
$\times \{\cos A \cos B - \sin A \sin B\}$
$= \{\cos^2 A \cos^2 B - \sin^2 A \sin^2 B\}$
$= \{\cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B\}$
$= \{\cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B\}$
$= \cos^2 A - \sin^2 B$

**Deduction-3**

$$\cot (A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

**Proof**: We have, $\cos (A + B)$

$$= \frac{\cos (A + B)}{\sin (A + B)}$$

$$= \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$$

$$= \frac{\frac{\cos A \cos B}{\sin A \sin B} - \frac{\sin A \sin B}{\sin A \sin B}}{\frac{\sin A \cos B}{\sin A \sin B} + \frac{\cos A \sin B}{\sin A \sin B}}$$

$$= \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

**Deduction 4**

$$\cot (A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

**Proof**: We have, $\cot (A - B)$

$$= \frac{\cos (A - B)}{\sin (A - B)}$$

$$= \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B}$$

$$= \frac{\frac{\cos A \cos B}{\sin A \sin B} + \frac{\sin A \sin B}{\sin A \sin B}}{\frac{\sin A \cos B}{\sin A \sin B} - \frac{\cos A \sin B}{\sin A \sin B}}$$

$$= \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

**Deduction 5**

$\sin (A + B + C)$
$= \cos A \cos B \cos C$
$(\tan A + \tan B + \tan C - \tan A \tan B \tan C)$

**Proof**: We have $\sin (A + B + C)$
$= \sin (A + B) \cos C + \cos (A + B) \sin C$
$= \{\sin A \cdot \cos B + \cos A \cdot \sin B\} \cos C$
$+ \{\cos A \cos B - \sin A \sin B\} \sin C$
$= \sin A \cdot \cos B \cdot \cos C + \sin B \cdot \cos A \cdot \cos C$
$+ \sin C \cdot \cos A \cos B - \sin A \cdot \sin B \cdot \sin C$
$= \cos A \cdot \cos B \cdot \cos C\, [\tan A + \tan B + \tan C]$
$-\tan A \cdot \tan B \cdot \tan C]$

**Deduction 6**

$\cos (A + B + C)$
$= \cos A \cos B \cos C$
$\times [1 - \tan A \tan B - \tan B \tan C - \tan C \tan A]$

**Proof**: We have, $\cos (A + B + C)$
$= \cos (A + B) \cos C - \sin (A + B) \sin C$
$= \{\cos A \cos B - \sin A \sin B\} \cos C$
$-\{\sin A \cos B + \cos A \sin B\} \sin C$
$= \cos A \cos B \cos C - \sin A \sin B \cos C$
$\{-\sin A \sin C \cos B - \cos A \sin B \sin C\}$
$= \cos A \cos B \cos C$
$\times [1 - \tan A \tan B - \tan B \tan C - \tan C \tan A]$

**Deduction 7**

$\tan (A + B + C)$

$$= \frac{\sin (A + B + C)}{\cos (A + B + C)}$$

$$= \frac{\cos A \cos B \cos C\, (\tan A + \tan B + \tan C - \tan A \tan B \tan C)}{\cos A \cos B \cos C\, (1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)}$$

**Proof**: We have, $\tan (A + B + C)$

$$= \frac{\sin (A + B + C)}{\cos (A + B + C)}$$

$$= \frac{\cos A \cos B \cos C \{\tan A + \tan B + \tan C - \tan A \tan B \tan C\}}{\cos A \cos B \cos C \{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A\}}$$

## 1.11 Transformation Formulae

### 1.11.1 Transformation of Products into Sums or Differences

1. $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$
2. $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
3. $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
4. $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

### 1.11.2 Transformations of Sums or Differences into Products

1. $\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}$
2. $\sin C - \sin B = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}$
3. $\cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}$
4. $\cos C - \tan D = -2 \sin \frac{C + D}{2} \sin \frac{C - D}{2}$

## 1.12 Multiple Angles

### 1.12.1 Definition

An angle is of the form $nA$, $n \in Z$, is called a multiple angle of $A$. Such as $2A$, $3A$, $4A$, etc. are each multiple angles of $A$.

### 1.12.2 Trigonometrical Ratios of 2A in Terms of *t*-ratio of A

1. $\sin 2A = 2 \sin A \cos A$
2. $\cos 2A = \cos^2 A - \sin^2 A$
   $= 2\cos^2 A - 1 = 1 - 2\sin^2 A$
3. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

### 1.12.3 T-ratios of Angle 2A

4. $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$,
5. $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
6. $1 - \cos 2A = 2\sin^2 A$,
7. $1 + \cos 2A = 2\cos^2 A$
8. $\tan A = \dfrac{\sin 2A}{1+\cos 2A}$,
9. $\tan A = \dfrac{1-\cos 2A}{\sin 2A}$

### 1.12.4 Trigonometrical Ratios of 3A in Terms of *t*-ratio of A

10. $\sin 3A = 3\sin A - 4\sin^3 A$
11. $\cos 3A = 4\cos^3 A - 3\cos A$
12. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

## 1.13 Some Important Deductions

**Deduction 1**

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

**Proof:** We have,

$$\sin^2 A = \frac{1}{2}(2\sin^2 A) = \frac{1}{2}(1-\cos 2A)$$

**Deduction 2**

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

**Proof**: We have, $\cos^2 A$

$$= \frac{1}{2}(2\cos^2 A) = \frac{1}{2}(1+\cos 2A)$$

**Deduction-3**

$$\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$$

**Proof:** We have, $\sin 3A = 3\sin A - 4\sin^3 A$

$\Rightarrow \quad 4\sin^3 A = 3\sin A - \sin 3A$

$\Rightarrow \quad \sin^3 A = \dfrac{1}{4}(3\sin A - \sin 3A)$

**Deduction-4**

$$\cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$$

**Proof:** We have, $\cos 3A = 4\cos^3 A - 3\cos A$

$\Rightarrow \quad 4\cos^3 A = \cos 3A + 3\cos A$

$\Rightarrow \quad \cos^3 A = \dfrac{1}{4}(\cos 3A + 3\cos A)$

**Deduction 5**

$$\sin A \sin(60 - A) \cdot \sin(60 + A) = \frac{1}{4}\sin 3A$$

**Proof**: We have,

$$\begin{aligned}
&\sin A \cdot \sin(60° - A) \cdot \sin(60° + A)\\
&= \sin A \cdot (\sin^2 60° - \sin^2 A)\\
&= \sin A \cdot \left(\frac{3}{4} - \sin^2 A\right)\\
&= \frac{\sin A}{4} \cdot \left(3 - 4\sin^2 A\right)\\
&= \frac{1}{4}(3\sin A - 4\sin^3 A)\\
&= \frac{1}{4} \times \sin 3A
\end{aligned}$$

**Deduction 6**

$$\cos A \cdot \cos(60 - A) \,.\, \cos(60 + A) = \frac{1}{4}\cos 3A$$

**Proof**: We have,

$$\begin{aligned}
&\cos A \cdot \cos(60° - A) \cdot \cos(60° + A)\\
&= \cos A \cdot (\cos^2 60° - \sin^2 A)\\
&= \cos A \cdot \left(\frac{1}{4} - 1 + \cos^2 A\right)\\
&= \cos A \cdot \left(-\frac{3}{4} + \cos^2 A\right)\\
&= \frac{\cos A}{4} \cdot (-3 + 4\cos^2 A)\\
&= \frac{1}{4} \cdot (-3\cos A + 4\cos^3 A)\\
&= \frac{1}{4} \cdot (4\cos^3 A - 3\cos A)\\
&= \frac{1}{4} \times \cos 3A
\end{aligned}$$

**Deduction 7**

$$\tan A \cdot \tan(60 - A) \,.\, \tan(60 + A) = \tan 3A$$

**Proof**: We have,

$\tan A \cdot \tan(60° - A) \cdot \tan(60° + A)$

$$= \frac{\sin A \cdot \sin (60° - A) \cdot \sin (60° + A)}{\cos A \cdot \cos (60° - A) \cdot \cos (60° + A)}$$

$$= \frac{\frac{1}{4}\sin 3A}{\frac{1}{4}\cos 3A}$$

$$= \frac{\sin 3A}{\cos 3A}$$

$$= \tan 3A$$

**Deduction 8**

$\sin 4A = 4 \sin \cos A - 8 \cos A \sin^3 A$

**Proof:** We have, $\sin 4A$

$= 2 \sin 2A \cdot \cos 2A$
$= 2(2 \sin A \cdot \cos A)(1 - 2 \sin^2 A)$
$= 4 \sin A \cdot \cos A(1 - 2 \sin^2 A)$
$= 4 \sin A \cdot \cos A - 8 \sin^3 A \cdot \cos A$

**Deduction 9**

$\cos 4A = 1 - 8 \sin^2 A + 8 \sin^4 A$

**Proof:** We have, $\cos 4A$

$= \cos 2 (2A)$
$= 1 - 2 \sin^2 (2A)$
$= 1 - 2 (2 \sin A \cdot \cos A)^2$
$= 1 - 8 \sin^2 A \cdot \cos^2 A$
$= 1 - 8 \sin^2 A (1 - \sin^2 A)$
$= 1 - 8 \sin^2 A + 8 \sin^4 A$

**Deduction 10**

$$\tan 4A = \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}$$

**Proof**: We have, $\tan 4A$

$$= \tan 2 \cdot (2A)$$

$$= \frac{2 \tan 2A}{1 + \tan^2 2A}$$

$$= \frac{\frac{4 \tan A}{1 - \tan^2 A}}{1 - \left(\frac{2 \tan A}{1 - \tan^2 A}\right)^2}$$

$$= \frac{4 \tan A(1 - \tan^2 A)}{(1 - \tan^2 A)^2 - 4 \tan^2 A}$$

$$= \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}$$

**Deduction 11**

$\sin 5A = 16 \sin^5 A - 20 \sin^3 A + 5 \sin A$

**Proof**: We have, $\sin 5A$

$= \sin (3A + 2A)$
$= \sin 3A \cos 2A + \cos 3A \cdot \sin 2A$
$= (3 \sin A - 4 \sin^3 A)(1 - 2 \sin^2 A)$
$\quad + 2(4 \cos^3 A - 3 \cos A) \sin A \cos A$
$= (3 \sin A - 4 \sin^3 A)(1 - 2 \sin^2 A)$
$\quad + 2(4 \cos^2 A - 3) \sin A \cos^2 A$
$= (3 \sin A - 4 \sin^3 A)(1 - 2 \sin^2 A)$
$\quad + 2(1 - 4 \sin^2 A)(\sin A - \sin^3 A)$
$= (3 \sin A - 4 \sin^3 A - 6 \sin^3 A + 8 \sin^5 A)$
$\quad + 2(\sin A - 4 \sin^3 A - \sin^3 A + 4 \sin^5 A)$
$= 5 \sin A - 20 \sin^3 A + 16 \sin^5 A$
$= 16 \sin^5 A - 20 \sin^3 A + 5 \sin A$

**Deduction 12**

$\cos 5A = 16 \cos^5 A - 20 \cos^3 A + 5 \cos A$

**Proof**: We have, $\cos 5A$

$= \cos (3A + 2A)$
$= \cos 3A \cos 2A - \sin 3A \sin 2A$
$= (4 \cos^3 A - 3 \cos A)(2 \cos^2 A - 1)$
$\quad - (3 \sin A - 4 \sin^3 A)(2 \sin A \cos A)$
$= 8 \cos^5 A - 6 \cos^3 A - 4 \cos^3 A$
$\quad + 3 \cos A - (3 - 4 \sin^2 A)\, 2 \cos A\, (1 - \cos^2 A)$
$= 8 \cos^5 A - 10 \cos^3 A + 3 \cos A -$
$\quad (4 \cos^2 A - 1)\, (2 \cos A - 2 \cos^3 A)$
$= 8 \cos^5 A - 10 \cos^3 A + 3 \cos A - 8 \cos^3 A$
$\quad + 2 \cos A + 8 \cos^5 A - 2 \cos^3 A$
$= 16 \cos^5 A - 20 \cos^3 A + 5 \cos A$

**Deduction 13**

$\sin 6A = (6 \sin A - 32 \sin^3 A + 32 \sin^5 A) \cos A$

**Proof**: We have, $\sin 6A$

$= \sin 2 (3A)$
$= 2 \sin 3A \cdot \cos 3A$
$= 2(3 \sin A - 4 \sin^3 A)(4 \cos^3 A - 3 \cos A)$
$= 2(3 \sin A - 4 \sin^3 A)(1 - 4 \sin^2 A) \cos A$
$= 2(3 \sin A - 4 \sin^3 A - 12 \sin^3 A + 16 \sin 5A) \cos A$
$= 2(3 \sin A - 16 \sin^3 A + 16 \sin^5 A) \cos A$
$= (6 \sin A - 32 \sin^3 A + 32 \sin^5 A) \cos A$

**Deduction 14**

$\cos 6 A = 32 \cos^6 A - 48 \cos^4 A + 18 \cos^2 A - 1$

**Proof**: We have, $\cos 6A$

$= \cos 2(3A)$
$= 2 \cos^2 (3A) - 1$
$= 2(4 \cos^3 A - 3 \cos A)^2 - 1$
$= 2(16 \cos^6 A - 24 \cos^4 A + 9 \cos^2 A) - 1$
$= 32 \cos^6 A - 48 \cos^4 A + 18 \cos^2 A - 1$

## 1.14 The Maximum and Minimum Values of

$$f(x) = a \cos x + b \sin x + c$$

We have, $f(x) = a \cos x + b \sin x + c$

Let $a = r \sin \theta$ and $b = r \cos \theta$

Then $r = \sqrt{a^2 + b^2}$ and $\tan (\theta) = \dfrac{a}{b}$

Now, $f(x) = a\cos x + b\sin x + c$
$= r(\sin\theta\cos x + \cos\theta\sin x)$
$= r\sin(\theta + x)$

As we know that, $-1 \le \sin(\theta + x) \le 1$

$\Rightarrow \quad -r + c \le r\sin(\theta + x) + c \le r + c$

$\Rightarrow \quad -r + c \le f(x) \le r + c$

$\Rightarrow \quad -\sqrt{a^2+b^2} + c \le f(x) \le \sqrt{a^2+b^2} + c$

Thus, the maximum value of

$f(x)$ is $\sqrt{a^2+b^2} + c$

and the minimum values of $f(x)$ is $-\sqrt{a^2+b^2} + c$.

## 1.15 Sub–Multiple Angles

### 1.15.1 Definition

An angle is of the form $\frac{A}{n}$, $n \in Z\ (\neq 0)$, is called a sub-multiple angle of $A$. Thus $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}, \frac{A}{5}$, etc. are each a sub-multiple angle of $A$.

### 1.15.2 T-ratios of angle $\left(\frac{A}{2}\right)$ and $\left(\frac{A}{3}\right)$

1. $\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) = \dfrac{2\tan\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)}$

2. $\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$

$= 2\cos^2\left(\frac{A}{2}\right) - 1$

$= 1 - 2\sin^2\left(\frac{A}{2}\right) = \dfrac{1-\tan^2\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)}$

3. $\tan A = \dfrac{2\tan\left(\frac{A}{2}\right)}{1-\tan^2\left(\frac{A}{2}\right)}$

4. $\sin A = 3\sin\left(\frac{A}{3}\right) - 4\sin^3\left(\frac{A}{3}\right)$

5. $\cos A = 4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right)$

6. $\tan A = \dfrac{3\tan\left(\frac{A}{3}\right) - \tan^3\left(\frac{A}{3}\right)}{1-3\tan^2\left(\frac{A}{3}\right)}$

### 1.15.3 Values of sin 18°, cos 18° and tan 18°

1. $\sin(18°) = \left(\frac{\sqrt{5}-1}{4}\right)$

**Proof :** Let $A = 18°$

$\Rightarrow \quad 5A = 90°$

$\Rightarrow \quad 2A = 90° - 3A$

$\Rightarrow \quad \sin 2A = \sin(90° - 3A) = \cos 3A$

$\Rightarrow \quad 2\sin A\cos A = 4\cos^3 A - 3\cos A$

$\Rightarrow \quad 2\sin A = 4\cos^2 A - 3$

$\Rightarrow \quad 2\sin A = 4 - 4\sin^2 A - 3 = 1 - 4\sin^2 A$

$\Rightarrow \quad 4\sin^2 A + 2\sin A - 1 = 0$

$\Rightarrow \quad \sin A = \frac{-2 \pm \sqrt{20}}{8} = \frac{-2 \pm 2\sqrt{5}}{8}$

$\Rightarrow \quad \sin A = \frac{-1 \pm \sqrt{5}}{4}$

$\Rightarrow \quad \sin A = \frac{\sqrt{5}-1}{4}, \frac{-\sqrt{5}-1}{4}$

$\Rightarrow \sin(18°) = \frac{\sqrt{5}-1}{4}$, since 18° lies on the first quadrant.

2. $\cos 18° = \frac{1}{4}\sqrt{10+2\sqrt{5}}$

**Proof**: We have, $\cos(18°)$

$= \sqrt{1-\sin^2(18°)}$

$= \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2}$

$= \sqrt{1-\left(\frac{5+1-2\sqrt{5}}{16}\right)}$

$= \sqrt{\left(\frac{16-5-1+2\sqrt{5}}{16}\right)}$

$= \frac{1}{4}\sqrt{10+2\sqrt{5}}$

3. $\tan 18° = \dfrac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$

**Proof**: We have, $\tan(18°)$

$= \dfrac{\sin(18°)}{\cos(18°)}$

$= \dfrac{\left(\frac{\sqrt{5}-1}{4}\right)}{\frac{\sqrt{10+2\sqrt{5}}}{4}}$

$= \left(\dfrac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}\right)$

***Notes:***

(i) $\sin 72° = \cos 18° = \frac{1}{4}\sqrt{10+2\sqrt{5}}$

(ii) $\cos 72° = \sin 18° = \frac{\sqrt{5}-1}{4}$

## 1.15.4 Values of sin 36°, cos 36° and tan 36°

1. $\cos 36° = \frac{\sqrt{5}+1}{4}$

**Proof:** We have, cos (36°)

$$= \cos 2\,(18°)$$
$$= 1 - 2\sin^2(18°)$$
$$= 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2$$
$$= 1 - 2\left(\frac{5+1-2\sqrt{5}}{16}\right)$$
$$= \left(\frac{8-5-1+2\sqrt{5}}{8}\right)$$
$$= \left(\frac{2+2\sqrt{5}}{8}\right)$$
$$= \left(\frac{\sqrt{5}+1}{4}\right)$$

2. $\sin 36° = \frac{1}{4}\sqrt{10-2\sqrt{5}}$

**Proof:** We have, $\sin(36°) = \sqrt{1-\cos^2(36°)}$

$$= \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2}$$
$$= \sqrt{1-\left(\frac{5+1+2\sqrt{5}}{16}\right)}$$
$$= \sqrt{\left(\frac{16-5-1-2\sqrt{5}}{16}\right)}$$
$$= \sqrt{\left(\frac{10-2\sqrt{5}}{16}\right)}$$
$$= \frac{1}{4}\sqrt{10-2\sqrt{5}}$$

3. $\tan 36° = \frac{1}{4} \times (\sqrt{5}-1) \times \sqrt{10-2\sqrt{5}}$.

**Proof:** We have tan (36°)

$$= \frac{\sin(36°)}{\cos(36°)}$$
$$= \frac{\frac{\sqrt{10-2\sqrt{5}}}{4}}{\frac{\sqrt{5}+1}{4}}$$
$$= \frac{\sqrt{10-2\sqrt{5}}}{(\sqrt{5}+1)}$$
$$= \frac{(\sqrt{5}-1)\times(\sqrt{10-2\sqrt{5}})}{4}$$

***Notes:***

(i) $\sin 54° = \cos 36° = \frac{\sqrt{5}+1}{4}$

(ii) $\cos 54° = \sin 36° = \frac{1}{4}\sqrt{10-2\sqrt{5}}$

## 1.15.5 Some Important Deductions

**Deduction 1**

$$\tan\left(7\frac{1}{2}°\right) = \sqrt{6} - \sqrt{4} - \sqrt{3} + \sqrt{2}$$

**Proof**: As we know that, $\tan\theta = \frac{1-\cos(2\theta)}{\sin(2\theta)}$

Put $\theta = 7\frac{1}{2}°$, then

$$\tan\left(7\frac{1}{2}°\right) = \frac{1-\cos(15°)}{\sin(15°)}$$
$$= \frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$
$$= \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$$
$$= \frac{(2\sqrt{2}-\sqrt{3}-1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$
$$= \frac{(2\sqrt{2}-\sqrt{3}-1)(\sqrt{3}+1)}{2}$$
$$= \frac{2\sqrt{6}-3-\sqrt{3}+2\sqrt{2}-\sqrt{3}-1}{2}$$
$$= \frac{2(\sqrt{6}-\sqrt{4}-\sqrt{3}+\sqrt{2})}{2}$$
$$= (\sqrt{6}-\sqrt{4}-\sqrt{3}+\sqrt{2})$$

**Deduction 2**

$$\cot\left(7\frac{1}{2}^\circ\right) = \sqrt{6} + \sqrt{4} + \sqrt{3} + \sqrt{2}$$

**Proof**: As we know that, $\cot(\theta) = \dfrac{1 + \cos(2\theta)}{\sin(2\theta)}$

Put $\theta = 7\frac{1}{2}^\circ$,

Now, $\cot\left(7\frac{1}{2}^\circ\right)$

$$= \frac{1 + \cos(15^\circ)}{\sin(15^\circ)}$$

$$= \frac{1 + \dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \frac{2\sqrt{2} + \sqrt{3} + 1}{\sqrt{3} - 1}$$

$$= \frac{(2\sqrt{2} + \sqrt{3} + 1)(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)}$$

$$= \frac{(2\sqrt{6} + 3 + \sqrt{3} + 2\sqrt{2} + \sqrt{3} + 1)}{2}$$

$$= \frac{2(\sqrt{6} + \sqrt{4} + \sqrt{3} + \sqrt{2})}{2}$$

$$= (\sqrt{6} + \sqrt{4} + \sqrt{3} + \sqrt{2})$$

**Deduction 3**

$$\sin\left(22\frac{1}{2}^\circ\right) = \frac{1}{2}\sqrt{2 - \sqrt{2}}$$

**Proof**: As we know that,

$2\sin^2(\theta) = 1 - \cos 2\theta$

Put, $\theta = 22\frac{1}{2}^\circ$,

$$2\sin^2\left(22\frac{1}{2}^\circ\right) = 1 - \cos(45^\circ)$$

$$= 1 - \left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{2} - 1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(22\frac{1}{2}^\circ\right) = \pm\sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}$$

$$\Rightarrow \quad \sin\left(22\frac{1}{2}^\circ\right) = \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}$$

since, $22\frac{1}{2}^\circ$ lies in the first quadrant.

$$\Rightarrow \quad \sin\left(22\frac{1}{2}^\circ\right) = \frac{1}{2}\sqrt{2 - \sqrt{2}}$$

**Deduction 4**

$$\cos\left(22\frac{1}{2}^\circ\right) = \frac{1}{2}\sqrt{2 + \sqrt{2}}$$

**Proof**: As we know that, $2\cos^2(\theta) = 1 - \cos 2\theta$

Put, $\theta = 22\frac{1}{2}^\circ$,

$$\Rightarrow \quad 2\cos^2\left(22\frac{1}{2}^\circ\right) = 1 + \cos(45^\circ)$$

$$= 1 + \left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{2} + 1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(22\frac{1}{2}^\circ\right) = \pm\sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}$$

$$\Rightarrow \quad \cos\left(22\frac{1}{2}^\circ\right) = \sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}$$

since, $22\frac{1}{2}^\circ$ lies in the first quadrant.

Thus, $\cos\left(22\frac{1}{2}^\circ\right) = \frac{1}{2}\sqrt{2 + \sqrt{2}}$

**Deduction 5**

$$\tan\left(22\frac{1}{2}^\circ\right) = \sqrt{2} - 1$$

**Proof:** As we know that, $\tan\theta = \dfrac{1 - \cos(2\theta)}{\sin(2\theta)}$

Put $\theta = 22\frac{1}{2}^\circ$, $\tan\left(22\frac{1}{2}^\circ\right) = \dfrac{1 - \cos(45^\circ)}{\sin(45^\circ)}$

$$= \frac{1 - \dfrac{1}{\sqrt{2}}}{\dfrac{1}{\sqrt{2}}} = \sqrt{2} - 1$$

**Deduction 6**

$$\cot\left(22\frac{1}{2}^\circ\right) = \sqrt{2} + 1$$

**Proof**: As we know that,

$$\cot(\theta) = \frac{1 + \cos(2\theta)}{\sin(2\theta)}$$

Put $\theta = 22\frac{1}{2}^\circ$,

$$\Rightarrow \quad \cot\left(22\frac{1}{2}^\circ\right) = \frac{1+\cos(45^\circ)}{\sin(45^\circ)}$$

$$= \frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}+1$$

**Deduction 7**

$$\sin\left(112\frac{1}{2}^\circ\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

**Proof**: We have, $\sin\left(112\frac{1}{2}^\circ\right)$

$$= \sin\left(90^\circ\times 1 + 22\frac{1}{2}^\circ\right)$$

$$= \cos\left(22\frac{1}{2}^\circ\right)$$

$$= \frac{1}{2}\sqrt{2+\sqrt{2}}$$

**Deduction 8**

$$\cos\left(112\frac{1}{2}^\circ\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

**Proof**: We have, $\cos\left(112\frac{1}{2}^\circ\right)$

$$= \cos\left(90^\circ\times 1 + 22\frac{1}{2}^\circ\right)$$

$$= -\sin\left(22\frac{1}{2}^\circ\right)$$

$$= -\frac{1}{2}\sqrt{2+\sqrt{2}}$$

**Deduction 9**

$$\tan\left(112\frac{1}{2}^\circ\right) = -(\sqrt{2}+1)$$

**Proof**: We have, $\tan\left(112\frac{1}{2}^\circ\right)$

$$= \tan\left(90^\circ\times 1 + 22\frac{1}{2}^\circ\right)$$

$$= -\cot\left(22\frac{1}{2}^\circ\right)$$

$$= -(\sqrt{2}+1)$$

## 1.16 Conditional Trigonometrical Identities

Here, we shall deal with trigonometrical identities involving two or more angles. In establishing such identities we will be frequently using properties of supplementary and complementary angles and hence students are advised to go through all the above formulae, starting from the Ist topic.

We have certain trigonometrical identities like, $\sin^2\theta + \cos^2\theta = 1$ and $\sec^2\theta = 1 + \tan^2\theta$, etc. Such identities are identities in the sense that they hold for all values of the angles which satisfy the given condition amongst them and they are called **Conditional Identities**.

If $A$, $B$, $C$ denote the angles of a triangle $ABC$, then the relation $A + B + C = \pi$ enables us to establish many important identities involving trigonometric ratios of these angles.

(i) If $A + B + C = \pi$, then $A + B = \pi - C$, $B + C = \pi - A$ and $C + A = \pi - B$

(ii) If $A + B + C = \pi$, then
$\sin(A + B) = \sin(\pi - C) = \sin C$
similarly, $\sin(B - C) = \sin(\pi - A) = \sin A$
and $\sin(C + A) = \sin(\pi - B) = \sin B$

(iii) If $A + B + C = \pi$, then
$\cos(A + B) = \cos(\pi - C) = -\cos C$
Similarly, $\cos(B + C) = \cos(\pi - A) = -\cos A$
and $\cos(C + A) = \cos(\pi - B) = -\cos B$

(iv) If $A + B + C = \pi$, then
$\tan(A + B) = \tan(\pi - C) = -\tan C$
Similarly, $\tan(B + C) = \tan(\pi - A) = -\tan A$
and $\tan(C + A) = \tan(\pi - B) = -\tan B$

(v) If $A + B + C = \pi$, then $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$, $\frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$ and $\frac{C+A}{2} = \frac{\pi}{2} - \frac{B}{2}$

Therefore,

$$\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\left(\frac{C}{2}\right)$$

$$\cos\left(\frac{A+B}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin\left(\frac{C}{2}\right)$$

$$\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\left(\frac{C}{2}\right)$$

***Note:*** Dear students, please recollect the following formulae from basic trigonometry

**Step I:**
(i) $\sin 2A = 2\sin A\cos A$
(ii) $\cos 2A = 2\cos^2 A - 1$
(iii) $\cos 2A = 1 - 2\sin^2 A$
(iv) $\cos 2A = \cos^2 A - \sin^2 A$

**Step II:**
(i) $1 + \cos 2A = 2\cos^2 A$
(ii) $1 - \cos 2A = 2\sin^2 A$
(iii) $1 + \cos A = 2\cos^2(A/2)$
(iv) $1 - \cos A = 2\sin^2(A/2)$

**Step III:**
(i) $\cos(A + B) + \cos(A - B) = 2\cos A\cos B$
(ii) $\cos(A - B) - \cos(A + B) = 2\sin A\sin B$

**Step IV:**

(i) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

(iii) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

## 1.17 Trigonometrical Series

### 1.17.1 Introduction

In this section, we are mainly concerned with different procedures to find out the summation of trigonometrical series.

To find out the sum of different trigonometrical series, first we observe the nature of the angles of the trigonometrical terms. We must observe whether the angles form any sequence or not. If they form any sequences, then we must check, what kind of sequence it is. We also observe the sequence formed (if any) by the coefficients of terms of the series. So, our main attempt will be

(i) to express each term as a difference of the two terms directly or by manipulation and then addition, or
(ii) to arrange the series in such a way that it follows some standard trigonometrical expansion.

### 1.17.2 Different Types of the Summation of a Trigonometrical Series

**1.** A trigonometrical series involved with the terms of sines or cosines.

**Rule:** Whenever angles are in AP and the trigonometrical terms involved with sines or cosines having power 1.

1. We must multiply each term by $2\sin\left(\frac{\text{common diffrence of angles}}{2}\right)$
2. and then express each term as a difference of two terms,
3. And finally add them.

### 1.17.3 A Trigonomeytrical Series Based on Method of Difference

**Rules:**

1. Express each term of the series as a difference of two expressions.
2. Finally adding them and we shall get the required result.

## Exercises

### Level I

**(Problems Based on Fundamentals)**

**MEASUREMENT OF ANGLES**

1. If the radius of the earth 4900 km, what is the length of its circumference?
2. The angles of a triangle are in the ratio 3 : 4 : 5. Find the smallest angle in degrees and the greatest angle in radians.
3. The angles of a triangle are in AP and the number of degrees in the least is to the number of radians in the greatest as 60 to $\pi$, find the angles in degrees.
4. The number of sides in two regular polygons are 5 : 4 and the difference between their angles is 9. Find the number of sides of the polygon.
5. The angles of a quadrilateral are in AP and the greatest is double the least. Express the least angles in radians.
6. Find the angle between the hour hand and the minute hand in circular measure at half past 4.
7. Find the length of an arc of a circle of radius 10 cm subtending an angle of 30° at the centre.
8. The minute hand of a watch is 35 cm long. How far does its tip move in 18 minutes?
9. At what distance does a man, whose height is 2 m subtend an angle of 10°?
10. Find the distance at which a globe $5\frac{1}{2}$ cm in diameter, will subtend an angle of 6°.
11. The radius of the earth being taken to 6400 km and the distance of the moon from the earth being 60 times the radius of the earth. Find the radius of the moon which subtends an angle of 16° at the earth.
12. The difference between the the acute angles of a right angled triangle is $\frac{2\pi}{3}$ radians. Express the angles in degrees.
13. The angles of a quadrilateral are in AP and the greatest angle is 120°. Find the angles in radians.
14. At what distance does a man $5\frac{1}{2}$ ft in height, subtend an angle of 15″?
15. Find the angle between the hour hand and minute-hand in circular measure at 4 o'clock.

## TRIGONOMETRICAL RATIOS AND IDENTITIES

16. If sec $\theta$ + tan $\theta$ = 3, where $\theta$ lies in the first quadrant, then find the value of cos $\theta$.
17. If $\operatorname{cosec}\theta - \cot\theta = \frac{1}{5}$, then find the value of sin $\theta$
18. If $a = c\cos\theta + d\sin\theta$ and
$b = c\sin\theta - d\cos\theta$ such that
$a^m + b^n = c^p + d^q$, where $m, n, p, q \in N$
then find the value of $m + n + p + q + 42$.
19. If $3\sin\theta + 4\cos\theta = 5$, then find the value of
$3\cos\theta - 4\sin\theta$.
20. If $x = r\cos\theta\sin\varphi$, $y = r\cos\theta\cos\varphi$ and $z = r\sin\theta$ such that $x^m + y^n + z^p = r^2$, where $m, n, p \in N$, then find the value of $(m + n + p - 4)^{m+n+p+4}$.
21. If $x = \dfrac{2\sin\alpha}{1+\cos\alpha+3\sin\alpha}$, then find the value of
$\dfrac{\sin\alpha - 3\cos\alpha + 3}{2 - 2\cos\alpha}$.
22. If $P = \sec^6\theta - \tan^6\theta - 3\sec^2\theta\tan^2\theta$,
$Q = \operatorname{cosec}^6\theta - \cot^6\theta - 3\operatorname{cosec}^2\theta\cot^2\theta$ and
$R = \sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$, then find the value of $(P + Q + R)^{(P+Q+R)}$.
23. If $3\sin x + 4\cos x = 5$, for all $x$ in $\left(0, \dfrac{\pi}{2}\right)$,
then find the value of $2\sin x + \cos x + 4\tan x$
24. If $\sin A + \sin B + \sin C + 3 = 0$, then find the value of $\cos A + \cos B + \cos C + 10$.
25. If $(1+\sin\theta)(1+\cos\theta) = \dfrac{5}{4}$,
then find the value of $(1 - \sin\theta)(1 - \cos\theta)$.
26. Find the minimum value of the expression $f(x) = \dfrac{9x^2\sin^2 x + 4}{x\sin x}$, for all $x$ in $(0, \pi)$.
27. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, then
prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$
28. If $\tan^2\theta = 1 - e^2$ then prove that
$\sec\theta + \tan^3\theta\cdot\operatorname{cosec}\theta = (2 - e^2)^{3/2}$
29. If $\sin\theta + \sin^2\theta + \sin^3\theta = 1$,
then prove that,
$\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$
30. If $x = \dfrac{2\sin\theta}{1+\cos\theta+\sin\theta}$, then prove
that $\dfrac{1-\cos\theta+\sin\theta}{1+\sin\theta} = x$
31. Prove that $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$.
32. If $\sin x + \sin^2 x = 1$, then find the value of
$\cos^8 x + 2\cos^6 x + \cos^4 x$
33. If $0 \le \theta \le 180°$ and $81^{\sin^2\theta} + 81^{\cos^2\theta} = 30$, then find the value of $\theta$
34. Let $f_k(\theta) = \sin^k(\theta) + \cos^k(\theta)$.
Then find the value of $\dfrac{1}{6}f_6(\theta) - \dfrac{1}{4}f_4(\theta)$
35. If $x\sin^3\alpha + y\cos^3\alpha = \sin\alpha\cos\alpha$ and $x\sin\alpha = y\cos\alpha$ then prove that $x^2 + y^2 = 1$.
36. If $\tan\theta + \sin\theta = m$, $\tan\theta - \sin\theta = n$, then prove that $m^2 - n^2 = 4\sqrt{mn}$.
37. If $\dfrac{\cos^4 x}{\cos^2 y} + \dfrac{\sin^4 x}{\sin^2 y} = 1$, then prove that
$\dfrac{\cos^4 y}{\cos^2 x} + \dfrac{\sin^4 y}{\sin^2 x} = 1$
38. If $f_n(\theta) = \sin^n\theta + \cos^n\theta$, prove that
$2f_6(\theta) - 3f_4(\theta) + 1 = 0$
39. If $\dfrac{\sin A}{\sin B} = p$, $\dfrac{\cos A}{\cos B} = q$, prove that
$\tan A\cdot\tan B = \dfrac{p}{q}\left(\dfrac{q^2-1}{1-p^2}\right)$.
40. If $\dfrac{\sin^4\alpha}{a} + \dfrac{\cos^4\alpha}{b} = \dfrac{1}{a+b}$, then
prove that $\dfrac{\sin^8\alpha}{a^3} + \dfrac{\cos^8\alpha}{b^3} = \dfrac{1}{(a+b)^3}$

## MEASUREMENT OF DIFFERENT T-RATIOS

41. Find the value of
(i) sin 120°
(ii) sin 150°
(iii) sin 210°
(iv) sin 225°
(v) sin 300°
(vi) sin 330°
(vii) sin 405°
(viii) sin 650°
(ix) sin 1500°
(x) sin 2013°
42. Find the value of
$\cos(1°)\cdot\cos(2°)\cdot\cos(3°)\ldots\cos(189°)$
43. Find the value of
$\tan(1°)\cdot\tan(2°)\cdot\tan(3°)\ldots\tan(89°)$
44. Find the value of
$\tan 35°\cdot\tan 40°\cdot\tan 45°\cdot\tan 50°\cdot\tan 55°$
45. Find the value of
$\sin(10°) + \sin(20°) + \sin(30°) + \sin(40°) + \ldots + \sin(360°)$
46. Find the value of
$\cos(10°) + \cos(20°) + \cos(30°) + \cos(40°) + \ldots + \cos(360°)$
47. Find the value of
$\sin^2 5° + \sin^2 10° + \sin^2 15° + \ldots + \sin^2 90°$

48. Find the value of
$$\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\sin^2\left(\frac{4\pi}{9}\right)+\sin^2\left(\frac{7\pi}{18}\right)$$
49. Find the value of
$$\tan(20°)\tan(25°)\tan(45°)\tan(65°)\tan(70°)$$
50. Find the value of $\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)$ if $\sin(\theta_1)+\sin(\theta_2)+\sin(\theta_3)=3$
51. Find the value of
$$\sin^2 6°+\sin^2 12°+\ldots+\sin^2 90°$$
52. Find the value of
$$\sin^2 10°+\sin^2 20°+\ldots+\sin^2 90°$$
53. Find the value of
$$\sin^2 9°+\sin^2 18°+\ldots+\sin^2 90°$$
54. Find the value of
$$\tan 1°\cdot\tan 2°\cdot\tan 3°\ldots\tan 89°$$
55. Find the value of
$$\cos 1°\cdot\cos 2°\cdot\cos 3°\ldots\cos 189°$$
56. Solve for $\theta$; $2\sin^2\theta+3\cos\theta=0$ where $0<\theta<360°$.
57. Solve for $\theta$; $\cos\theta+\sqrt{3}\sin\theta=2$, where $0<\theta<360°$.
58. If $4n\alpha=\pi$, then prove that
$$\tan\alpha\tan 2\alpha\tan 3\alpha\ldots\tan(2n-1)\alpha=1.$$
59. Find the value of
$$\cos(18°)+\cos(234°)+\cos(162°)+\cos(306°).$$
60. Find the value of
$$\cos(20°)+\cos(40°)+\cos(60°)+\ldots+\cos(180°)$$
61. Find the value of
$$\sin(20°)+\sin(40°)+\sin(60°)+\ldots+\sin(360°)$$
62. Draw the graphs of
   (i) $f(x)=\sin x+1$
   (ii) $f(x)=\sin x-1$
   (iii) $f(x)=-\sin x$
   (iv) $f(x)=1-\sin x$
   (v) $f(x)=-1-\sin x$
   (vi) $f(x)=\sin 2x, \sin 3x$
   (vii) $f(x)=\sin^2 x$
   (viii) $f(x)=\cos^2 x$
   (ix) $f(x)=\max(\sin x, \cos x\}$
   (x) $f(x)=\min\{\sin x, \cos x\}$
   (xi) $f(x)=\min\left\{\sin x, \frac{1}{2}, \cos x\right\}$
   (xii) $f(x)=\max\{\tan x, \cot x\}$
   (xiii) $f(x)=\min\{\tan x, \cot x\}$
63. Find the number of solutions of
   (i) $\sin x=\frac{1}{2}$, $\forall x\in[0, 2\pi]$
   (ii) $\cos x=\frac{\sqrt{3}}{2}$, $\forall x\in[0, 3\pi]$
   (iii) $4\sin^2 x-1=0$, $\forall x\in[0, 3\pi]$
   (iv) $\sin^2 x-3\sin x+2=0$, $\forall x\in[0, 3\pi]$
   (v) $\cos^2 x-\cos x-2=0$, $\forall x\in[0, 3\pi]$

## COMPOUND ANGLES

64. Find the values of
   (i) $\sin(15°)$,
   (ii) $\cos(15°)$,
   (iii) $\tan(15°)$
65. Find the value of $\tan(75°)+\cot(75°)$
66. Prove that $\cos(9°)+\sin(9°)=\sqrt{2}\sin(54°)$
67. Prove that $\tan(70°)=2\tan(50°)+\tan(20°)$
68. Prove that $\dfrac{\cos 20°-\sin 20°}{\cos 20°+\sin 20°}=\tan 25°$
69. Prove that $\dfrac{\cos 7°+\sin 7°}{\cos 7°-\sin 7°}=\tan 52°$.
70. Prove that $\tan 20°+\tan 25°+\tan 20°\tan 25°=1$.
71. If $A+B=45°$, then find the value of $(1+\tan A)(1+\tan B)$
72. Find the value of
$$(1+\tan 245°)(1+\tan 250°)(1+\tan 260°)$$
$$(1-\tan 200°)(1-\tan 205°)(1-\tan 215°)$$
73. Prove that $\tan 13A-\tan 9A-\tan 4A$
$$=\tan 4A\cdot\tan 9A\cdot\tan 13A$$
74. Prove that $\tan 9A-\tan 7A-\tan 2A$
$$=\tan 2A\cdot\tan 7A\cdot\tan 9A$$
75. If $\tan\alpha=\dfrac{m}{m+1}$, $\tan\beta=\dfrac{1}{2m+1}$,
then prove that $\alpha+\beta=\dfrac{\pi}{4}$.
76. Prove that $\sin^2 B$
$$=\sin^2 A+\sin^2(A-B)-2\sin A\cos B\sin(A-B)$$
77. Prove that $\cos(2x+2y)$
$$=\cos 2x\cos 2y+\cos^2(x+y)-\cos^2(x-y)$$
78. If $\sin\theta=\dfrac{x-y}{x+y}$,
then prove that $\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)=\pm\sqrt{\dfrac{x}{y}}$
79. If $\tan\alpha=\dfrac{Q\sin\beta}{P+Q\cos\beta}$,
prove that $\tan(\beta-\alpha)=\dfrac{P\sin\alpha}{Q+P\cos\alpha}$
80. If $\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=-\dfrac{3}{2}$
prove that $\cos\alpha+\cos\beta+\cos\gamma=0$
and $\sin\alpha+\sin\beta+\sin\gamma=0$
81. If $\tan(\alpha+\theta)=n\tan(\alpha-\theta)$, show that $\dfrac{\sin 2\theta}{\sin 2\alpha}=\dfrac{n-1}{n+1}$
82. If $\sin\alpha+\sin\beta=a$ and $\cos\alpha+\cos\beta=b$, then show that
   (i) $\cos(\alpha+\beta)=\dfrac{b^2-a^2}{b^2+a^2}$
   (ii) $\sin(\alpha+\beta)=\dfrac{2ab}{b^2+a^2}$

83. If $\alpha$ and $\beta$ are the roots of $a \cos \theta + b \sin \theta = c$, then prove that

(i) $\cos(\alpha+\beta) = \dfrac{a^2-b^2}{a^2+b^2}$

(ii) $\cos(\alpha-\beta) = \dfrac{2c-(a^2+b^2)}{a^2+b^2}$

84. If $\alpha$ and $\beta$ are the roots of $a \tan \theta + b \sec \theta = c$, then show that

$\tan(\alpha+\beta) = \dfrac{2ac}{a^2-c^2}$

85. If $\tan(\pi \cos \theta) = \cot(\pi \sin \theta)$, prove that $\cos\left(\theta - \dfrac{\pi}{4}\right) = \pm\dfrac{1}{2\sqrt{2}}$.

86. If $\tan\theta = \dfrac{x \sin\varphi}{1 - x\cos\varphi}$ and $\tan\varphi = \dfrac{y\sin\theta}{1-y\cos\theta}$,

then prove that, $x \sin \varphi = y \sin \theta$.

87. If $\tan \alpha + \tan \beta = a$, $\cot \alpha + \cot \beta = b$ and $\tan(\alpha + \beta) = c$ then find a relation in $a$, $b$ and $c$.

88. If $\tan\beta = \dfrac{n \sin\alpha \cos\alpha}{1 - n\sin^2\alpha}$,

then prove that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$

89. If $x + y + z = 0$, then prove that

$$\cot(x+y-z)\cdot\cot(y+z-x) + \cot(y+z-x)\cdot\cot(z+x-y) + \cot(z+x-y)\cdot\cot(x+y-z) = 1$$

90. If $2\tan\alpha = 3\tan\beta$, then show that $\tan(\alpha-\beta) = \dfrac{\sin 2\beta}{5-\cos 2\beta}$

## TRANSFORMATION FORMULAE

91. Prove that:

(i) $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$

(ii) $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$

(iii) $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A+B}{2}\right)$

92. Prove that:

(i) $\sin 38° + \sin 22° = \sin 82°$

(ii) $\sin 105° + \cos 105° = \cos 45°$

(iii) $\cos 55° + \cos 65° + \cos 175° = 0$

(iv) $\cos 20° + \cos 100° + \cos 140° = 0$

(v) $\sin 50° - \sin 70 + \sin 10° = 0$

93. Prove that

$$\sin(47°) + \cos(77°) = \cos(17°)$$

94. Prove that

$$\cos(80°) + \cos(40°) - \cos(20°) = 0$$

95. Prove that

$$\sin(10°) + \sin(20°) + \sin(40°) + \sin(50°) - \sin(70°) - \sin(80°) = 0$$

96. Prove that $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\dfrac{\alpha-\beta}{2}\right)$

97. Prove that $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\dfrac{\alpha-\beta}{2}\right)$

98. Prove that $(\sin\alpha - \sin\beta)^2 + (\cos\alpha - \cos\beta)^2 = 4\sin^2\left(\dfrac{\alpha-\beta}{2}\right)$

99. Prove that $\cos 20 \cdot \cos 40 \cdot \cos 80 = \dfrac{1}{8}$

100. Prove that $\sin 20 \cdot \sin 40 \cdot \sin 80 = \dfrac{\sqrt{3}}{8}$

101. Prove that $\sin 10 \cdot \sin 50 \cdot \sin 60 \cdot \sin 70 = \dfrac{\sqrt{3}}{16}$

112. Prove that $\cos 10 \cdot \cos 30. \cos 50 \cdot \cos 70° = \dfrac{3}{16}$

113. Prove that:

(i) $\dfrac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A$

(ii) $\dfrac{\cos 4x + \cos 3x + \cos 12x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

(iii) $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$

(iv) $\dfrac{\sin A + \sin 2A + \sin 4A + \sin 5A}{\cos A + \cos 2A + \cos 4A + \cos 5A} = \tan 3A$

114. If $\sin A - \sin B = \dfrac{1}{2}$ and $\cos A - \cos B = \dfrac{1}{3}$, then find $\tan\left(\dfrac{A+B}{2}\right)$

115. If $\cos A + \cos B = \dfrac{1}{2}$, $\sin A + \sin B = \dfrac{1}{4}$,

then prove that $\tan\left(\dfrac{A+B}{2}\right) = \dfrac{1}{2}$

116. If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$, then

prove that $\tan A \tan B = \cot\left(\dfrac{A+B}{2}\right)$

117. If $\sin 2A = \lambda \sin 2B$, then prove that

$$\frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$$

118. Find the value of $\sqrt{3}\cot(20°) - 4\cos(20°)$

119. If $\sin A + \sin B = a$ and $\cos A + \cos B = b$, then find $\cos(A + B)$

120. If $2\cos A = x + \frac{1}{x}$, $2\cos B = y + \frac{1}{y}$, then find $\cos(A - B)$ Ans. $\frac{1}{2}\left(xy + \frac{1}{xy}\right)$

121. Prove that
$\sin(47°) + \sin(61°) - \sin(11°) - \sin(25°) = \cos(7°)$.

122. If $\tan\alpha = \frac{m}{m+1}$, and $\tan\beta = \frac{1}{2m+1}$, then find $\tan(\alpha + \beta)$

123. Find the number of integral values of $k$ for which $7\cos x + 5\sin x = 2k + 1$ has a solution.

124. Prove that
$$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma) = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right)$$

**MULTIPLE ANGLES**

125. Prove that $\left(\frac{1-\cos 2\theta}{\sin 2\theta}\right) = \tan\theta$

126. Prove that $\left(\frac{1+\cos 2\theta}{\sin 2\theta}\right) = \cot\theta$

127. Prove that $\cot\theta - \tan\theta = 2\cot(2\theta)$

128. Prove that
$$\tan\theta + 2\tan(2\theta) + 4\tan(4\theta) + 8\cot 8\theta = \cot\theta$$

129. If $\tan\theta = \frac{b}{a}$,
prove that $a\cos(2\theta) + b\sin(2\theta) = a$

130. Prove that $\sqrt{3}\,\text{cosec}(20°) - \sec(20°) = 4$

131. Prove that
$$\tan(9°) - \tan(27°) - \tan(63°) + \tan 81°) = 4$$

132. Prove that $\left(\frac{\sec 8A - 1}{\sec 4A - 1}\right) = \frac{\tan 8A}{\tan 2A}$

133. Prove that
(i) $\cos^2(\theta) + \cos^2\left(\frac{2\pi}{3} - \theta\right) + \cos^2\left(\frac{2\pi}{3} + \theta\right) = \frac{3}{2}$
(ii) $\cos^2\theta + \cos^2\left(\frac{\pi}{3} - \theta\right) + \cos^2\left(\frac{\pi}{3} + \theta\right) = \frac{3}{2}$

134 Prove that
$$\sin^2\theta + \sin^2(120° + \theta) + \sin^2(240° + \theta) = \frac{3}{2}$$

135. Prove that
$$4\sin(\theta)\sin(60° + \theta)\sin(60° - \theta) = \sin 3\theta$$

136. Prove that $\sin(20°)\cdot\sin(40°)\cdot\sin(80°) = \frac{\sqrt{3}}{8}$

137. Prove that
$$4\cos(\theta)\cdot\cos(60° - \theta)\cdot\cos 60° + \theta = \cos(3\theta)$$

138. Prove that
$$\cos(10°)\cdot\cos(50°)\cdot\cos(70°) = \frac{\sqrt{3}}{8}$$

139. Prove that
$$\tan(\theta) + \tan(60° + \theta) - \tan(60° - \theta) = 3\tan(3\theta)$$

140. Prove that
$$\cos(\theta)\cos(2\theta)\cdot\cos(2^2\theta)\cdot\cos(2^3\theta)\ldots\cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n\sin\theta}$$

141. Prove that $\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{8\pi}{7}\right) = \frac{1}{8}$

142. Prove that $\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$.

143. If $M = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ such that maximum $(M^2) = m_1$ and minimum $(M^2) = m_2$, then find the value of $m_1 - m_2$.

144. Prove that $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

145. Prove that
$$\left(\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x}\right) = \frac{1}{2}(\tan 27x - \tan x)$$

146. Prove that
$$\tan\left(\frac{\theta}{2}\right)(1 + \sec\theta)(1 + \sec 2\theta)(1 + \sec 2^2\theta)\ldots(1 + \sec(2^n\theta)) = \tan(2^n\theta)$$

Maximum or minimum values of $f(x) = \theta\cos + b\sin c)$

147. Find the maximum and minimum values of
(i) $f(x) = 3\sin x + 4\cos x + 10$
(ii) $f(x) = 3\sin(100)x + 4\cos(100)x + 10$
(iii) $f(x) = 3\sin x + 4$
(iv) $f(x) = 2\cos x + 5$
(v) $f(x) = \sin x + \cos x$
(vi) $f(x) = \sin x - \cos x$
(vii) $f(x) = \sin(\sin x)$
(viii) $f(x) = \cos(\cos x)$
(ix) $f(x) = \sin(\sin x) + \cos(\sin x)$
(x) $f(x) = \cos(\sin x) + \sin(\cos x)$

148. Find the range of $f(x) = \sin x + \cos x + 3$

149. Find the greatest and the least values of $2\sin^2\theta + 3\cos^2\theta$

150. Prove that
$$-4 < 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 < 10$$

151. Find the maximum and minimum values of

$$\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$$

152. Find the least value of

$$\operatorname{cosec}^2 x + 25\sec^2 x$$

153. Find the ratio of the greatest value of

$2 - \cos x + \sin^2 x$ to its least value.

154. If $y = 4\sin^2\theta - \cos 2\theta$, then $y$ lies in the interval............

155. If $m$ is the minimum value of $f(x) = 3\sin x + 5$ and $n$ is the maximum value of $g(x) = 3 - 2\sin x$ then find the value of $(m + n + 2)$.

156. Find the maximum and the minimum values of

$$f(x) = \sin^2 x + \cos^4 x.$$

157. Find the maximum and the minimum values of

$$f(x) = \cos^2 x + \sin^4 x.$$

158. Find the maximum and minimum values of

$$f(x) = \sin^4 x + \cos^4 x.$$

159. Find the maximum and minimum values of

$$f(x) = \sin^6 x + \cos^6 x.$$

160. If $A = \cos^2\theta + \sin^4\theta$ and

$B = \cos^4\theta + \sin^2\theta$ such that

$m_1$ = Maximum of $A$ and $m_2$ = Minimum of $B$

then find the value of $m_1^2 + m_2^2 + m_1 m_2$

161. Find the maximum and minimum values of

$$f(x) = (\sin x + \cot + \operatorname{cosec} 2x)^3$$

where $x \in \left(0, \dfrac{\pi}{2}\right)$

162. Find the maximum and minimum values of

$$f(x) = \frac{5}{\sin^2\theta - 6\sin\theta\cos\theta + 3\cos^2\theta}$$

163. Find the minimum value of

$$f(x) = \frac{a^2}{\cos^2 x} + \frac{b^2}{\sin^2 x},\ x \in \left(0, \frac{\pi}{2}\right)$$

164. Find the minimum value of

$$f(x) = \frac{x^2\sin^2 x + 4}{x\sin x},$$

where $x \in \left(0, \dfrac{\pi}{2}\right)$

165. Find the maximum and minimum values of

$f(x) = \log_x y + \log_y x$, where $x > 1$, $y > 1$

166. Find the minimum values of

$$f(x) = 2\log_{10} x - \log_x (0.01),\ x > 1$$

167. Find the minimum value of

$$f(x, y, z) = \frac{(x^2+1)(y^2+1)(z^2+1)}{xyz},\ x, y, z > 0$$

168. Find the minimum value of

$$f(x, y, z) = \frac{(x^3+2)(y^3+2)(z^3+2)}{xyz},\ x, y, z > 0$$

169. Find the minimum value of

$$f(a, b, c, d) = \frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd}$$

where $a, b, c, d > 0$

## SUB-MULTIPLE ANGLES

170. Prove that

(i) $\tan\dfrac{\theta}{2} + \cot\dfrac{\theta}{2} = 2\operatorname{cosec}\theta$

(ii) $\operatorname{cosec}\theta - \cot\theta = \tan\dfrac{\theta}{2}$

(iii) $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \tan\dfrac{\theta}{2}$

(iv) $\sec\theta + \tan\theta = \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$

(v) $(\cos A + \cos B)^2 + (\sin A + \sin B)^2$

$$= 4\cos^2\left(\frac{A - B}{2}\right)$$

171. Prove that $\sin^2(24°) - \sin^2(6°) = \dfrac{(\sqrt{5} - 1)}{8}$

172. Prove that $\sin^2(48°) - \cos^2(12°) = \dfrac{(\sqrt{5} + 1)}{8}$

173. Prove that $\sin(12°)\cdot\sin(48°)\cdot\sin(54°) = \dfrac{1}{8}$

174. Prove that $\sin(6°)\cdot\sin(42°)\cdot\sin(66°)\cdot\sin(78°) = \dfrac{1}{16}$

175. Prove that

$$4(\sin(24°) + \cos(6°)) = (1 + \sqrt{5})$$

176. Prove that

$$\tan(6°)\cdot\tan(42°)\cdot\tan(66°)\cdot\tan(78°) = 1$$

177. Prove that

$$\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right) = \frac{1}{8}$$

178. Prove that

$$\sin^4\left(\frac{\pi}{8}\right) + \sin^4\left(\frac{3\pi}{8}\right) + \sin^4\left(\frac{5\pi}{8}\right) + \sin^4\left(\frac{7\pi}{8}\right) = \frac{3}{2}$$

179. Prove that

$$\cos^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{3\pi}{8}\right) + \cos^4\left(\frac{5\pi}{8}\right) + \cos^4\left(\frac{7\pi}{8}\right) = \frac{3}{2}$$

180. Prove that $\tan 20°\tan 80° = \sqrt{3}\tan 50°$

181. Prove that $\tan(10°)\tan(70°) = \dfrac{1}{\sqrt{3}} \times \tan(40°)$

182. Prove that

$$\sin 55° - \sin 19° + \sin 53° - \sin 17° = \cos 1°$$

183. Prove that $\cos\dfrac{2\pi}{7}\cdot\cos\dfrac{4\pi}{7}\cdot\cos\dfrac{6\pi}{7} = \dfrac{1}{8}$

184. Prove that: $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$

185. Find the value of $\tan\left(7\frac{1}{2}^\circ\right)+\cot\left(7\frac{1}{2}^\circ\right)$.

186. If $\cos\left(\frac{x}{2}\right)-\sqrt{3}\sin\left(\frac{x}{2}\right)$ takes its minimum value then find its $x$.

187 If $\alpha$ and $\beta$ be two different roots $a\cos\theta+b\sin\theta=c$, then prove that $\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}$.

## CONDITIONAL IDENTITIES

188. If $A+B+C=\pi$, then prove that,
$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$$

189. If $A+B+C=\pi$, then prove that,
$\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$

190. If $A+B+C=\pi$, then prove that
$$\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$$

191. If $A+B+C=\pi$, then prove that,
(i) $\sin^2 A+\sin^2 B+\sin^2 C=2+\cos A\cos B\cos C$
(ii) $\cos^2 A+\cos^2 A+\cos^2 A=2+\sin A\sin B\sin C$

192 If $A+B+C=\pi$, then prove that
$$\frac{\sin 2A+\sin 2B+\sin 2C}{\cos A+\cos B+\cos C-1}=8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

193. In a $\Delta ABC$ prove that
(i) $\tan A+\tan B+\tan C=\tan A\cdot\tan B\cdot\tan C$
(ii) $\cot A\cdot\cot B+\cot B\cdot\cot C+\cot C\cdot\cot A=1$.

194. If $A$, $B$, $C$ and $D$ be the angles of a quadrilateral, then prove that
$$\frac{\tan A+\tan B+\tan C+\tan D}{\cot A+\cot B+\cot C+\cot D}=\tan A\cdot\tan B\cdot\tan C\cdot\tan C$$

194. In a $\Delta ABC$ prove that
$$(\cot A+\cot B)(\cot B+\cot C)(\cot C+\cot A)=\text{cosec}\,A\,\text{cosec}\,B\,\text{cosec}\,C.$$

195. If $xy+yz+zx=1$, then prove that
$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

196. If $xy+yz+zx=1$, then prove that,
$$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}=\frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}$$

197 Prove that
$$\tan(\alpha-\beta)+\tan(\beta-\gamma)+\tan(\gamma-\alpha)=\tan(\alpha-\beta)\tan(\beta-\gamma)\tan(\gamma-\alpha)$$

198 In a $\Delta ABC$, if $\cot A+\cot B+\cot C=\sqrt{3}$, then prove that the triangle is an equilateral.

199 If $x+y+z=xyz$, then prove that
$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}$$

200. Prove that $1+\cos 56^\circ+\cos 58^\circ-\cos 66^\circ=4\cos 28^\circ\cos 29^\circ\sin 33^\circ$.

## TRIGONOMETRICAL SERIES

201. Prove that
$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\sin(\alpha+3\beta)+\ldots+\sin(\alpha+(n-1)\beta)=\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\times\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]$$

202. Prove that
$$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots+\sin n\alpha=\frac{\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\times\sin(n+1)\left(\frac{\alpha}{2}\right)$$

203. Prove that
$$\sin\theta+\sin 3\theta+\ldots+\sin(2n-1)\theta=\frac{\sin^2 n\theta}{\sin\theta}$$

204 Prove that
$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+\beta)+\cos(\alpha+\beta)+\ldots+\cos(\alpha+(n-1)\beta)=\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\times\cos\left(\alpha+(n-1)\frac{\beta}{2}\right)$$

205. Prove that $\cos\alpha+\cos 2\alpha+\cos 3\alpha+\ldots$
$$\ldots.+\cos n\alpha=\frac{\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\times\cos\left(\frac{(n+1)\alpha}{2}\right)$$

206. Find the sum of $n$-terms of the series
$$\frac{\sin x}{\sin 2x\cdot\sin 3x}+\frac{\sin x}{\sin 3x\cdot\sin 4x}+\frac{\sin x}{\sin 4x\cdot\sin 5x}+\ldots.\text{ to}-n-\text{terms}$$

207. Prove that $\frac{1}{\cos\theta+\cos 2\theta}+\frac{1}{\cos\theta+\cos 5\theta}$
$$+\frac{1}{\cos\theta+\cos 7\theta}+\ldots.\text{ to}-n-\text{terms}=\text{cosec}\,\theta\,[\tan(n+1)\,\theta-\tan\theta]$$

208. Prove that

$\tan x \cdot \tan 2x + \tan 2x \cdot \tan 3x$
$+ \ldots + \tan nx \cdot \tan (n + 1)x$
$= \cot x [\tan (n + 1)x - \tan x ] - n$

209. Prove that

$$\tan^{-1}\left(\frac{x}{1+2x^2}\right)+\tan^{-1}\left(\frac{x}{1+6x^2}\right)$$
$$+\tan^{-1}\left(\frac{x}{1+12x^2}\right)\cdots+\tan^{-1}\left(\frac{x}{1+n(n+1)x^2}\right)$$
$= \tan^{-1} (n + 1) x - \tan^{-1} x$

210. Prove that

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\tan^{-1}\left(\frac{1}{19}\right)$$
$$+\ldots..+\tan^{-1}\left(\frac{1}{n^2+3n+3}\right)$$
$= \tan^{-1} (n + 2) - \tan^{-1} 2.$

211. Prove that $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$
$$+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right)+\ldots\ldots\text{to }\infty=\frac{\pi}{2}$$

212. Prove that $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{2}{9}\right)$
$$+\tan^{-1}\left(\frac{4}{23}\right)+\ldots\ldots\text{ to }\infty=\frac{\pi}{4}$$

## Level II

### (Mixed Problems)

1. If $\sec x = p + \frac{1}{p}$, then $\sec x + \tan x$ is
(a) $p$ (b) $2p$ (c) $\frac{1}{4p}$ (d) $\frac{4}{p}$
2. If $\operatorname{cosec} x - \sin x = a^3$, $\sec x - \cos x = b^3$, then $a^2b^2(a^2 + b^2)$ is
(a) 0 (b) 1 (c) –1 (d) $ab$
3. If $\sec x + \cos x = 2$, then the value of $\sec^3 x(1+ \sec^3 x) + \cos^3 x(1 + \cos^3 x)$ is
(a) 2 (b) 4 (c) 6 (d) 8
4. The value of
$\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)$ is
(a) 1/2 (b) –1/2 (c) 0 (d) None
5. Which of the following is smallest?
(a) sin 1 (b) sin 2 (c) sin 3 (d) sin 4
6. Which of the following is greatest?
(a) sin 1 (b) cos 1 (c) tan 1 (d) cot 1
7. If A= cos (cos $x$) + sin (cos $x$), then the least and greatest value of A are
(a) 0, 2 (b) –1, 1
(c) $-\sqrt{2}, \sqrt{2}$ (d) $0, \sqrt{2}$
8. If $A + B = \frac{\pi}{3}$, $A, B > 0$ then the maximum value of tan $A \cdot \tan B$ is
(a) 1/3 (b) 1 (c) 1/2 (d) 2/3
9. The maximum value of $a \sin 2x + b \cos 2x$ for all real $x$ is
(a) $a + b$ (b) $\sqrt{a^2+b^2}$
(c) maximum $\{|a|, |b|\}$ (d) maximum $\{a, b\}$
10. Which of the following is/are true?
(a) sin 1 > sin 1°(b) tan 1 > tan 1°
(c) sin 4 > sin 4°(d) tan 4 > tan 4°
11. If $\cos 5x = a \cos^5 x + b \cos^3 x + c \cos x + d$, then
(a) $a = 16$ (b) $b – 20$
(c) $c = 5$ (d) $d = 2$
12. If $\sin^3 x \sin 3x = c_0 + c_1 \cos x + c_2\cos 2x + c_3 \cos 3x + \ldots\ldots\ldots + c_n \cos nx$, then
(a) Highest value of n is 6
(b) $c_0 = 1/8$
(c) $c_2 = –c_4$
(d) $c_1 = c_3 = c_5$.
13. If $f(x) = \cos[\pi] x + \sin [\pi] x$, where [,] is the greatest integer function, then
$f\left(\frac{\pi}{2}\right)$ is
(a) 0 (b) cos 3 (c) cos 4 (d) None
14. Let $f(x)$
$$=\begin{vmatrix}1+\sin^2 x & \cos^2 x & 4\sin 2x\\ \sin^2 x & 1+\cos^2 x & 4\sin 2x\\ \sin^2 x & \cos^2 x & 1+4\sin 2x\end{vmatrix},$$
then the maximum value of $f(x)$ is
(a) 0 (b) 2 (c) 6 (d) None
15. For any real $x$, the maximum value of
$\cos^2 (\cos x) + \sin^2 (\sin x)$ is
(a) 1 (b) $1 + \sin^2 1$
(c) $1 + \cos^2 1$ (d) None
16. If $a = \sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$ and $x$ is the solution of the equation $y = 2[x] + 2$ and $y = 3 [x – 2 ]$, where [,] = GIF, then $a$ is
(a) $[x]$ (b) $1/[x]$ (c) $2[x]$ (d) $[x]^2$
17. The minimum value of $\sin^8 x + \cos^8 x$ is
(a) 0 (b) 1 (c) 1/8 (d) 2

18. If $\frac{\sin^4\theta}{a}+\frac{\cos^4\theta}{b}=\frac{1}{a+b}$, then $\frac{\sin^8\theta}{a^3}+\frac{\cos^8\theta}{b^3}$ is
(a) $\frac{1}{a^3+b^3}$ (b) $\frac{1}{(a+b)^3}$
(c) $\frac{1}{(a-b)^3}$ (d) None

19. The value of $\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right)$ is
(a) 1 (b) $\frac{1}{\sqrt{7}}$ (c) $\sqrt{7}$ (d) None

20. If $\alpha$ and $\beta$ are the solutions of $\sin^2 x + a\sin x + b = 0$ as well as that of $\cos^2 x + c\cos x + d = 0$, then $\sin(\alpha+\beta)$ is
(a) $\frac{2bd}{b^2+d^2}$ (b) $\frac{a^2+c^2}{2ac}$
(c) $\frac{b^2+d^2}{2bd}$ (d) $\frac{2ac}{a^2+c^2}$

21. If $\sec\theta+\tan\theta=1$, then one of the roots of the equation
$$a(b-c)x^2+b(c-a)x+c(a-b)=0$$
is
(a) $\tan\theta$ (b) $\sec\theta$ (c) $\cos\theta$ (d) $\sin\theta$

22. If $\alpha$ is the common positive root of the equation $x^2 - ax + 12 = 0$, $x^2 - bx + 15 = 0$ and $x^2 - (a-b)x + 36 = 0$ and $\cos x + \cos 2x + \cos 3x = 0$, then $\sin x + \sin 2x + \sin 3x$ is
(a) 3 (b) –3 (c) 0 (d) 2

23. For any real $\theta$, the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ is
(a) 1 (b) $1+\sin^2 1$
(c) $1+\cos^2 1$ (d) $1-\cos^2 1$

24. If $\alpha+\beta=\pi/2$ and $\beta+\gamma=\alpha$, then $\tan\alpha$ is
(a) $2(\tan\beta+\tan\gamma)$ (b) $(\tan\beta+\tan\gamma)$
(c) $(\tan\beta+2\tan\gamma)$ (d) $(2\tan\beta+\tan\gamma)$

25. The maximum value of
$$\cos\alpha_1\cdot\cos\alpha_2\cdot\cos\alpha_3\ldots\cos\alpha_n$$
under the restriction
$$0\le\alpha_1,\alpha_2,\alpha_3,\ldots\ldots\ldots\ldots,\alpha_n\le\frac{\pi}{2} \text{ and}$$
$$\cot\alpha_1\cdot\cot\alpha_2,\cot\alpha_3\ldots\cot\alpha_n=1, \text{ is}$$
(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$ (c) $\frac{1}{2n}$ (d) 1

26. If $A > 0$ and $B > 0$ and $A+B=\frac{\pi}{3}$, then the maximum value of tan A. tan B is
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{2}}$

27. If $\tan\beta = 2\sin\alpha\times\sin\gamma\times\operatorname{cosec}(\alpha+\gamma)$, then $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in
(a) AP (b) GP (c) HP (d) AGP

28. The minimum value of the expression $\sin\alpha+\sin\beta+\sin\gamma$, where $\alpha,\beta,\gamma$ are real positive angles satisfying $\alpha+\beta+\gamma=\pi$, is
(a) positive (b) negative (c) 0 (d) –3

29. The value of $4\cos 20^\circ-\sqrt{3}\cot 20^\circ$ is
(a) 1 (b) –1 (c) –1/2 (d) 1/4

30. The maximum value of
$$4\sin^2 x+3\cos^2 x+\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right) \text{ is}$$
(a) $4+\sqrt{2}$ (b) $3+\sqrt{2}$
(c) $4-\sqrt{2}$ (d) 4

31. The value of the expression
$(\sqrt{3}\sin 75^\circ-\cos 75^\circ)$ is
(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) 2

32. The value of $(4+\sec 20^\circ)\sin 20^\circ$ is
(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) $2\sqrt{3}$

33. If $(1+\tan 1^\circ)(1+\tan 2^\circ)\ldots(1+\tan 45^\circ)=2^\circ$ then the value of $n$ is
(a) 20 (b) 21 (c) 22 (d) 23

34. The number of all possible triplets $(a_1, a_2, a_3)$ such that $a_1 + a_2\cos(2x) + a_3\sin^2 x = 0$ is
(a) 0 (b) 1 (c) 2 (d) infinite

35. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, then the value of $\sin(2\theta)$ is
(a) –1/2 (b) –1/3 (c) –2/3 (d) –3/4

36. A real root of the equation $8x^3 - 6x - 1 = 0$ is
(a) $\cos\left(\frac{\pi}{5}\right)$ (b) $\cos\left(\frac{\pi}{9}\right)$
(c) $\cos\left(\frac{\pi}{18}\right)$ (d) $\cos\left(\frac{\pi}{36}\right)$

37. The value of $(\sqrt{3}\cot(20^\circ)-4\cos(20^\circ))$ is
(a) 1 (b) –1 (c) $-\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{2}$

38. If $\tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{a}{b}$, then $\sin(\theta)$ is
(a) $\left(\frac{a-b}{a+b}\right)$ (b) $-\left(\frac{a-b}{a+b}\right)$
(c) $\left(\frac{a+b}{a-b}\right)$ (d) $-\left(\frac{a+b}{a-b}\right)$

39. The least value of $\operatorname{cosec}^2 x + 25\sec^2 x$ is
(a) 26 (b) 36 (c) 16 (d) 12

40. Let $y=\frac{\sin^3 x}{\cos x}-\frac{\cos^3 x}{\sin x}, 0<x<\frac{\pi}{2}$
then the minimum value of $y$ is
(a) 0 (a) 1 (c) 3/2 (d) 2

41. The expression $\tan(55^\circ)\tan(65^\circ)\tan(75^\circ)$ simplifies to $\cot(x^\circ)$, $0 < x < 90$, then $x$ is
(a) 5 (b) 8 (c) 9 (d) 10

42. If $x_1$ and $x_2$ are the roots of $x^2 + (1 - \sin\theta)x - \frac{1}{2}\cos^2\theta = 0$, then the maximum value of $x_1^2 + x_2^2$ is
(a) 2 (b) 3 (c) 9/4 (d) 4
43. The value of the expression
$\cos^2\left(\frac{\pi}{8}\right) + \cos^2\left(\frac{3\pi}{8}\right) + \cos^2\left(\frac{5\pi}{8}\right) + \cos^2\left(\frac{7\pi}{8}\right)$ is
(a) rational (b) integral
(c) prime (d) composite
44. If $a = \tan x$, then the value of $\cot\left(\frac{\pi}{4} - a\right)$ is
(a) $\left(\frac{a-1}{a+1}\right)$ (b) $\left(\frac{a^2-1}{a^2+1}\right)$
(c) $\left(\frac{a^2+1}{a^2-1}\right)$ (d) $\left(\frac{a+1}{a-1}\right)$
45. If $\sin\theta + \cos\theta = \frac{1}{5}, 0 \le \theta \le \pi$, then $\tan\theta$ is
(a) 3/4 (b) 4/3 (c) –3/4 (d) –4/3

## Level III

### (Problems for JEE Advanced)

1. Prove that
$$\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$$
2. Prove that
$$\tan 9° - \tan 27° - \tan 63° + \tan 81° = 4$$
3. Prove that
$$\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = \frac{1}{2}(\tan 27x - \tan x)$$
4. If $\frac{\sin x}{\sin y} = \frac{1}{2}, \frac{\cos x}{\cos y} = \frac{3}{2}$ where $x, y \in R$, then find the value of $\tan(x + y)$.
5. Prove that
$$\sin^4\left(\frac{\pi}{16}\right) + \sin^4\left(\frac{3\pi}{16}\right) + \sin^4\left(\frac{5\pi}{16}\right) + \sin^4\left(\frac{7\pi}{16}\right) = \frac{3}{2}$$
6 If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$, then prove that $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$.
7. If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, then prove that 1 cot $\alpha$ tan $\beta = 0$
8. If $\alpha + \beta = 90°$ and $\beta + \gamma = \alpha$, then prove that
$$\tan\alpha = \tan\beta + 2\tan\gamma$$
9. If $\tan\left(\frac{\pi}{24}\right) = (\sqrt{a} - \sqrt{b})(\sqrt{c} - \sqrt{d})$ where $a, b, c, d$ are positive integers, then find the value of $(a + b + c + d + 2)$
10. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta + \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta - \frac{2\pi}{3}\right)}$, then find the value of $x + y + z$.
11. If $\sin(25°)\sin(35°)\sin(85°) = \frac{\sqrt{a} + \sqrt{b}}{\sqrt{c}}$ where $a, b, c \in I^+$, find the value of $(a + b + c - 2)$
12. Find the value of $\sqrt{3}\cot(20°) - 4\cos(20°)$.
13. Prove that
sin (2°) + sin (4°) + sin (6°)
sin (8°) + … + sin (180°) = cot (1°)
14. Find the value of
$$\sin\left(\frac{\pi}{2013}\right) + \sin\left(\frac{3\pi}{2013}\right) + \sin\left(\frac{5\pi}{2013}\right) + \sin\left(\frac{7\pi}{2013}\right) + \text{... upto (2013) trems}$$
15. If $\tan y = \left(\frac{1+\sqrt{1+y}}{1+\sqrt{1-y}}\right)$,
then prove that $\sin(4y) = y$
16. If $\tan\left(\frac{\pi}{4} + \frac{y}{2}\right) = \tan^3\left(\frac{\pi}{4} + \frac{x}{2}\right)$,
prove that $\frac{\sin y}{\sin x} = \frac{3 + \sin^2 x}{1 + 3\sin^2 x}$
17. If $\tan\beta = \frac{\tan\alpha + \tan\gamma}{1 + \tan\alpha\tan\gamma}$,
prove that $\sin(2\beta) = \frac{\sin(2\alpha) + \sin(2\gamma)}{1 + \sin(2\alpha)\cdot\sin(2\gamma)}$
18. If $4\sin(27°) = (a + \sqrt{b})^{1/2} - (c - \sqrt{d})^{1/2}$ where $a, b, c, d \in N$, find the value of $(a + b + c + d + 2)$
19. If $(1 + \sin\theta)(1 + \cos\theta) = \frac{5}{4}$, find the value of $(1 - \sin\theta)(1 - \cos\theta)$
20. If $3\sin x + 4\cos x = 5$ where $x \in \left(0, \frac{\pi}{2}\right)$, then find the value of $2\sin x + \cos x + 4\tan x$
21. If $\cos A = \tan B$, $\cos B = \tan C$, $\cos C = \tan A$, prove that $\sin A = \sin B = \sin C = 2\sin(18°)$
22. Find the value of
$$\tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{3\pi}{16}\right) + \tan^2\left(\frac{5\pi}{16}\right) + \tan^2\left(\frac{7\pi}{16}\right)$$
23 If $\sin(1°)\cdot\sin(3°)\cdot\sin(5°)\ldots\sin(89°) = \frac{1}{2^n}$ then find the value of $n$
24. If $(1 + \tan(1°))(1 + \tan(2°))(1 + \tan(3°)) \ldots (1 + \tan(45°))$, then find $n$

25. Prove that

$$\left(1+\cos\left(\frac{\pi}{10}\right)\right)\left(1+\cos\left(\frac{3\pi}{10}\right)\right)\left(1+\cos\left(\frac{7\pi}{10}\right)\right)\left(1+\cos\left(\frac{9\pi}{10}\right)\right)=\frac{1}{16}$$

26 Prove that

$$\cos(60°)\cos(36°)\cos(42°)\cos(78°)=\frac{1}{16}$$

27. Let $f_k(\theta) = \sin^k(\theta) + \cos^k(\theta)$.

Then find the value of $\frac{1}{6}f_6(\theta)-\frac{1}{4}f_4(\theta)$

28. Find the maximum and minimum values of

$f(\theta) = \sin^2(\sin\theta) + \cos^2(\cos\theta)$

29. Find the minimum value of

$f(\theta) = (3\sin(\theta) - 4\cos(\theta) - 10)$
$(3\sin(\theta) + 4\cos(\theta) - 10)$

30. Find the range of $A = \sin^{2010}\theta + \cos^{2014}\theta$

31. If $\frac{\sin A}{\sin B}=p, \frac{\cos A}{\cos B}=q$, prove that

$$\tan A\cdot\tan B=\frac{p}{q}\left(\frac{q^2-1}{1-p^2}\right).$$

32. If $\frac{\tan(\alpha-\beta)}{\tan\alpha}+\frac{\sin^2\gamma}{\sin^2\alpha}=1$, then prove that $\tan^2\gamma = \tan\alpha\cdot\tan\beta$

33. If $\tan\left(\frac{\theta}{2}\right)=\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\varphi}{2}\right)$, then

prove that $\cos\varphi=\frac{\cos\theta-e}{1-e\cos\theta}$

34. If $\cos\theta=\frac{a\cos\varphi+b}{a+b\cos\varphi}$, then prove that

$$\tan\left(\frac{\theta}{2}\right)=\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{\varphi}{2}\right).$$

35. If $\sin x + \sin y = a$, $\cos x + \cos y = b$

then prove that $\tan\left(\frac{x-y}{2}\right)=\pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

36. If $\tan\left(\frac{\pi}{16}\right)=(a+b\sqrt{2})^{1/2}-(\sqrt{c}+d)$, where $a, b, c, d$ are +ve integers, then find the value of $(a + b + c + d + 1)$.

37. If $\alpha$ and $\beta$ are two values of $\theta$ satisfying the equation $\frac{\cos\theta}{a}+\frac{\sin\theta}{b}=\frac{1}{c}$. Prove that $\cot\left(\frac{\alpha+\beta}{2}\right)=\frac{b}{a}$

38. Prove that $\sin\left(\frac{\pi}{14}\right)$ is a root of

$$8x^3 - 4x^2 - 4x + 1 = 0$$

39 If $x + y + z = xyz$, prove that

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}$$ **[Roorkee, 1983]**

40. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that $\cos(3\alpha) + \cos(3\beta) + \cos(3\gamma) = 3\cos(\alpha+\beta+\gamma)$ and $\sin(3\alpha) + \sin(3\beta) + \sin(3\gamma) = 3\sin(\alpha+\beta+\gamma)$ **[Roorkee, 1985]**

41. Show that (without using tables) $\tan 9° - \tan 27° - \tan 63° + \tan 81° = 4$ **[Roorkee, 1989]**

42. Find '$a$' and '$b$' such that the inequality $a \le \cos x + 5\sin\left(x-\frac{\pi}{6}\right)\le b$ holds good for all $x$. **[Roorkee, 1989]**

43. If $A = \cos^2\theta + \sin^4\theta$, then for all values of $\theta$, find range of $A$. **[Roorkee, 1992]**

44. Given the product $p$ of sines of the angles of a triangle and the product $q$ of their cosines, find the cubic equation, whose co-efficients are functions of $p$ and $q$ and whose roots are the tangents of the angles of the triangle. **[Roorkee, 1992]**

45 If $x = \cos(10°)\cos(20°)\cos(40°)$, then find the value of $x$. **[Roorkee, 1995]**

46. Find the real values of $x$ for which $27^{\cos 2x}\cdot 81^{\sin 2x}$ is minimum and also find its minimum value. **[Roorkee, 2000]**

47. If $e^{i\theta - \log\cos(x-iy)} = 1$, then find the values of $x$ and $y$ in terms of $\theta$. **[Roorkee, 2001]**

## Level IV

### (Tougher Problems for JEE Advanced)

1. Prove that the sum of
$\tan x\tan 2x + \tan 2x\tan 3x + \dots + \tan x\tan(n+1)x = \cot x\tan(n+1)x - (n-1)$

2. Prove that
$\text{cosec}\, x + \text{cosec}\, 2x + \text{cosec}\, 4x + 5$ to $n$ terms

$$=\cot\left(\frac{x}{2}\right)-\cot(2^{n-1}x).$$

3. Prove that

$\cot(16°)\cot(44°) + \cot(44°)\cot(76°) - \cot(76°)\cot(16°) = 3$

5. If $\theta=\frac{\pi}{2^n-1}$, prove that $2^n\cos\theta$

$$\cos(2\theta)\cdot\cos(4\theta)\cdot\cos(8\theta)\dots\cos(2^{n-1}\theta) = -1$$

6. Prove that

$$\sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)+\sin\left(\frac{8\pi}{7}\right)=\frac{\sqrt{7}}{2}$$

7. Prove that
$$\tan^2\left(\frac{\pi}{16}\right)+\tan^2\left(\frac{2\pi}{16}\right)+....+\tan^2\left(\frac{7\pi}{16}\right)=35$$
8. Prove that $\left(\tan^2\left(\frac{\pi}{7}\right)+\tan^2\left(\frac{2\pi}{7}\right)+\tan^2\left(\frac{3\pi}{7}\right)\right)$
$$\times\left(\cot^2\left(\frac{\pi}{7}\right)+\cot^2\left(\frac{2\pi}{7}\right)+\cot^2\left(\frac{3\pi}{7}\right)\right)=105$$
9. Prove that $\frac{3+\cot(76°)\cot(16°)}{\cot(76°)+\cot(16°)}=\cot(44°)$
10. If $\cos x+\cos y+\cos z=0$, then prove that $\cos(3x)+\cos(3y)+\cos(3z)=12\cos x\cos y\cos z$
11. Prove that $\tan^6\left(\frac{\pi}{9}\right)-33\tan^4\left(\frac{\pi}{9}\right)+27\tan^2\left(\frac{\pi}{9}\right)=3$
12. If $\cos A+\cos B+\cos C=0=\sin A+\sin B+\sin C$ then prove that $\sin^2 A+\sin^2 B+\sin^2 C=\cos^2 A+\cos^2 B+\cos^2 C=\frac{3}{2}$.
13. Let $A$, $B$, $C$ be three angles such that $A=\frac{\pi}{4}$ and $\tan B$. $\tan C=p$. Find all possible values of $p$ such that $A$, $B$, $C$ are three angles of a triangle.
14. If $\frac{\tan 3A}{\tan A}=k$, show that $\frac{\sin 3A}{\sin A}=\frac{2k}{k-1}$ and $k$ cannot lie between 1/3 and 3.
15. If $A+B+C=\pi$, then prove that $\cot A+\cot B+\cot C-\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C=\cot A\cdot\cot B\cdot\cot C$
16. If $\tan\alpha=\frac{1}{\sqrt{x(x^2+x+1)}}$
$$\tan\beta=\frac{\sqrt{x}}{\sqrt{(x^2+x+1)}}$$
and $\tan\gamma=\frac{\sqrt{(x^2+x+1)}}{x\sqrt{x}}$
then prove that $\alpha+\beta=\gamma$
17. If $\alpha$ and $\beta$ are acute angles and $\cos 2\alpha=\frac{3\cos 2\beta-1}{3-\cos 2\beta}$, prove that $\tan\alpha:\tan\beta=\sqrt{2}:1$
18. If $\tan^3\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)=\tan\left(\frac{\beta}{2}+\frac{\pi}{4}\right)$, prove that
$$\sin\beta=\frac{(3+\sin^2\alpha)\sin\alpha}{1+3\sin^2\alpha}$$
19. If $\sin\beta=\frac{1}{5}\sin(2\alpha+\beta)$, prove that
$$\tan(\alpha+\beta)=\frac{3}{2}\tan\alpha$$
20. If $\sin x+\sin y=3(\cos x-\cos y)$, prove that $\sin(3x)+\sin(3y)=0$
21. If $\sec(\varphi-\alpha)$, $\sec\varphi$, $\sec(\varphi+\alpha)$ are in AP then prove that $\cos(\varphi)=\sqrt{2}\cos\left(\frac{\alpha}{2}\right)$
22. If $\tan\left(\frac{x+y}{2}\right)$, $\tan z$, $\tan\left(\frac{x-y}{2}\right)$ are in GP then prove that $\cos(x)=\cos(y)\cos(2z)$
23. Prove that $\frac{\sec^2\theta-\tan\theta}{\sec^2\theta+\tan\theta}$ lies between 1/3 and 3 for all real $\theta$
24. If $\theta=\frac{\pi}{2^n+1}$, find the value of
$2^n\cos(\theta)\cos(2\theta)\cos(2^2\theta)\ldots\cos(2^{n-1}\theta)$
25. Find the value of
$\tan(6°)\tan(42°)\tan(66°)\tan(78°)$
26. If $\frac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)}=\frac{\tan\gamma}{\tan\beta}$, prove that $\sin(\beta-\gamma)=0$ or $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$
27. If $A+B+C=\pi$, prove that
$$\cot A+\frac{\sin A}{\sin B\sin C}=\cot B+\frac{\sin B}{\sin A\sin C}$$
$$=\cot C+\frac{\sin C}{\sin A\sin B}$$
28. If $\frac{\sin(\theta+A)}{\sin(\theta+B)}=\sqrt{\frac{\sin(2A)}{\sin(2B)}}$, then prove that $\tan^2\theta=\tan A\tan B$
29. If $\cos(X-y)=-1$, then prove that $\cos x+\cos y=0$ and $\sin x+\sin y=0$.
30. If $\sqrt{2}\cos A=\cos B+\cos^3 B$
and $\sqrt{2}\sin A=\sin B-\sin^3 B$,
prove that $\sin(A-B)=\pm\frac{1}{3}$
31. Prove that $\sin(9°)=\frac{1}{4}(\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}})$
32. Find the range of $f(x)=\sin\left(\sqrt{\frac{\pi^2}{36}-x^2}\right)$
33. Find the value of $\sum_{k=1}^{6}\left(\sin\left(\frac{2k\pi}{7}\right)-i\cos\left(\frac{2k\pi}{7}\right)\right)$ where $i=\sqrt{-1}$
34. If $\cos\theta+\cos\varphi=\alpha$ and $\sin\theta+\sin\varphi=b$, find the value of $\tan\left(\frac{\theta}{2}\right)+\tan\left(\frac{\varphi}{2}\right)$
35. If $\frac{\tan\theta}{\tan\theta-\tan 3\theta}=\frac{1}{3}$, find the value of
$$\frac{\cot\theta}{\cot(\theta)-\cot(3\theta)}$$

## Integer Type Questions

1. If $\dfrac{x}{\cos\theta} = \dfrac{y}{\cos\left(\theta - \dfrac{2\pi}{3}\right)} = \dfrac{z}{\cos\left(\theta + \dfrac{2\pi}{3}\right)}$, find the value of $(x + y + z + 4)$
2. Find the numerical value of
$$\sum_{r=0}^{9} \sin^2\left(\frac{\pi r}{18}\right)$$
3. If $\dfrac{\sin x}{\sin y} = \dfrac{1}{2}$ and $\dfrac{\cos x}{\cos y} = \dfrac{3}{2}$, where $x, y \in \left(0, \dfrac{\pi}{2}\right)$, find the value of $\dfrac{\tan^2(x + y)}{5}$
4. If $\cos(x - y)$, $\cos x$, $\cos(x + y)$ are in HP such that $\left|\sec x \cdot \cos\left(\dfrac{y}{2}\right)\right| = m$, find the value of $(m^2 + 2)$.
5. If $\tan x + \tan\left(\dfrac{\pi}{3} + x\right) + \tan\left(\dfrac{2\pi}{3} + x\right)$
$= k \tan 3x$, find $k$
6. Let $f(\theta) = \sin^2\theta + \sin^2\left(\dfrac{2\pi}{3} + \theta\right) + \sin^2\left(\dfrac{4\pi}{3} + \theta\right)$, find the value of $2f\left(\dfrac{\pi}{15}\right)$
7. If $m = \sqrt{3}\,\text{cosec}(20°) - \sec(20°)$ and $n = \sin(12°)\sin(48°)\sin(5°)$ where $m, n \in N$, find the value of $(m + 8n + 2)$
8. Let $\tan(15°)$ and $\tan(30°)$ are the roots of $x^2 + px + q = 0$, find the value of $(2 + q - p)$
9. Let $x = \dfrac{\sum_{n=1}^{44} \cos(n°)}{\sum_{n=1}^{44} \sin(n°)}$, find the value $[x + 3]$, where [,] = GIF
10. If the value of the expression sin (25°) sin (35°) sin (85) can be expressed as $\dfrac{\sqrt{a} + \sqrt{b}}{c}$ where $a, b, c \in N$ and are in their lowest form, find the value of $\left(\dfrac{c}{a + b} + 2\right)$
11. Let $m = \sum_{k=1}^{17} \cos\left(\dfrac{k\pi}{9}\right)$, find the value of $(m^2 + m + 2)$
12. If the expression tan (55°) tan (65°) tan (75°) simplifies to cot $(x°)$ and $m$ is the numerical value of the expression tan (27°) + tan (18°) + tan (27°) tan (18°), find the value of $(m + x + 1)$

## Comprehensive Link Passages

**Passage-I**

Increasing product with angles are in GP

$$\cos\alpha \times \cos 2\alpha \times \cos 2^2\alpha \ldots \cos 2^{n-1}\alpha$$
$$= \begin{cases} \dfrac{\sin 2^n\alpha}{2^n \sin\alpha} : \text{if } \alpha \neq np \\ \dfrac{1}{2^n} \quad : \text{if } \alpha = \dfrac{\pi}{2^n + 1}, \\ -\dfrac{1}{2^n} : \text{if } \alpha = \dfrac{\pi}{2^n - 1} \end{cases}$$

where $n$ is an integer.

On the basis of above information, answer the following questions:

1. The value of $\cos\left(\dfrac{2\pi}{7}\right)\cos\left(\dfrac{4\pi}{7}\right)\cos\left(\dfrac{6\pi}{7}\right)$ is
(a) –1/2 (b) 1/2 (c) ¼ (d) 1/8
2. If $\alpha = \dfrac{\pi}{13}$, the value of $\prod_{r=1}^{6}(\cos(r\alpha))$ is
(a) 1/64 (b) –1/64 (c) 1/32 (d) –1/8
3. The value of $\sin\left(\dfrac{\pi}{14}\right)\sin\left(\dfrac{3\pi}{14}\right)\sin\left(\dfrac{5\pi}{14}\right)\sin\left(\dfrac{7\pi}{14}\right)\sin\left(\dfrac{9\pi}{14}\right)\sin\left(\dfrac{11\pi}{14}\right)\sin\left(\dfrac{13\pi}{14}\right)$ is
(a) 1 (b) 1/8 (c) 1/32 (d) 1/64
4. The value of $\sin\left(\dfrac{\pi}{18}\right)\sin\left(\dfrac{5\pi}{18}\right)\sin\left(\dfrac{7\pi}{18}\right)$ is
(a) 1/16 (b) 1/8 (c) – 1/8 (d) –1
5. The value of
$64\sqrt{3}\sin\left(\dfrac{\pi}{48}\right)\cos\left(\dfrac{\pi}{48}\right)\cos\left(\dfrac{\pi}{24}\right)\cos\left(\dfrac{\pi}{12}\right)\cos\left(\dfrac{\pi}{6}\right)$ is
(a) 8 (b) 6 (c) 4 (d) –1

**Passage-II**

If $\cos\left(\dfrac{\pi}{7}\right), \cos\left(\dfrac{3\pi}{7}\right), \cos\left(\dfrac{5\pi}{7}\right)$ are the roots of the equation $8x^3 - 4x^2 - 4x + 1 = 0$,

on the basis of the above information, answer the following questions.

1. The value of $\sec\left(\dfrac{\pi}{7}\right) + \sec\left(\dfrac{3\pi}{7}\right) + \sec\left(\dfrac{5\pi}{7}\right)$ is
(a) 2 (b) 4 (c) 8 (d) None

2. The value of $\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)$ is

(a) ¼ (b) 1/8 (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{8}$

3. The value of $\cos\left(\frac{\pi}{14}\right)\cos\left(\frac{3\pi}{14}\right)\cos\left(\frac{5\pi}{14}\right)$ is

(a) ¼ (b) 1/8 (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{7}}{8}$

4. The equation whose roots are

$$\tan^2\left(\frac{\pi}{7}\right), \tan^2\left(\frac{3\pi}{7}\right), \tan^2\left(\frac{5\pi}{7}\right) \text{ is}$$

(a) $x^3 - 35x^2 + 7x - 21 = 0$
(b) $x^3 - 35x^2 + 21x - 7 = 0$
(c) $x^3 - 35x^2 + 35x - 7 = 0$
(d) $x^3 - 21x^2 + 7x - 35 = 0$

5. The value of

$$\sum_{r=1}^{3}\left\{\tan^2\left(\frac{2r-1}{7}\right)\pi\right\} \times \sum_{r=1}^{3}\left\{\cot^2\left(\frac{2r-1}{7}\right)\pi\right\} \text{ is}$$

(a) 15 (b) 105 (c) 21 (d) 147

**Passage-III**

Let $x^2 + y^2 = 1$ for every $x, y$ in $R$.
Then,

1. The value of $P = (3x - 4x^3)^2 + (3y - 4y^3)^2$is
(a) 2 (b) 1 (c) 0 (d) –1
2. The minimum value of $Q = x^6 + y^6$ is
(a) 1 (b) 1/2 (c) 1/4 (d) –1
3. The maximum value of $R = x^2 + y^4$ IS
(a) 0 (b) 1 (c) 1/2 (d) 3/4

**Passage – IV**

Consider the polynomial

$$P(x) = (x - \cos 36°)(x - \cos 84°)(x - \cos 156°)$$

Then,

1. The co-efficient of $x^2$ is
(a) 0 (b) 1
(c) –1/2 (d) $\left(\frac{\sqrt{5}-1}{2}\right)$
2. The co-efficient of $x$ is
(a) 3/2 (b) –3/2 (c) –3/4 (d) 2
3. The constant term in $P(x)$ is
(a) $\left(\frac{\sqrt{5}-1}{4}\right)$ (b) $\left(\frac{\sqrt{5}-1}{16}\right)$
(c) $\left(\frac{\sqrt{5}+1}{16}\right)$ (d) $\frac{1}{16}$

**Passage-V**

If $a \sin x + b \cos x = 1$ such that $a^2 + b^2 = 1$ for all $a, b$ Î $\in (0, 1)$ then,

1. The value of $\sin x$ is
(a) $a$ (b) $b$ (c) $a/b$ (d) $b/a$
2. The value of $\cos x$ is
(a) $a$ (b) $b$ (c) $a/b$ (d) $b/a$
3. The value of tan x is
(a) $a$ (b) $b$ (c) $a/b$ (d) $b/a$

**Passage-VI**

Let $\sec x + \tan x = \frac{22}{7}$, where $0 < x < \frac{\pi}{2}$

then

1. The value of $\tan\left(\frac{x}{2}\right)$ is
(a) 15/29 (b) 13/25 (c) 14/29 (d) –15/29
2. The value of $\left(1 - \sqrt{\frac{1-\cos x}{1+\cos x}}\right)$ is
(a) 15/29 (b) 14/29 (c) 0 (d) 12/25
3. The value of (cosec $x$ + cot $x$) is
(a) 29/14 (b) 15/28 (c) 29/15 (d) 15/29

## Matrix Match (For JEE-Advanced Examination Only)

**1.** Match the following columns:

| | Column I | | Column II |
|---|---|---|---|
| (A) | If $\theta + \varphi = \frac{\pi}{2}$, where $\theta$ and $\phi$ are positive, then $(\sin\theta + \sin\varphi)\sin\left(\frac{\pi}{4}\right)$ is always less than | (P) | 1 |
| (B) | If $\sin\theta - \sin\varphi = a$ and $\cos\theta + \cos\varphi = b$, then $a^2$ $b^2$ can not exceed | (Q) | 2 |
| (C) | If $3\sin\theta + 5\cos\theta = 5$, $(\theta \neq 0)$, then the value of $5\sin\theta - 3\cos\theta$ is | (R) | 3 |
| | | (S) | 4 |
| | | (T) | 5 |

2. Match the following columns:

| | Column I | | Column II |
|---|---|---|---|
| (A) | The value of cos (20°), cos (40°), cos (80°) is | (P) | $\sqrt{3}/8$ |
| (B) | The value of cos (20°), cos (40°), cos (60°), cos (80°) is | (Q) | $\sqrt{3}/16$ |
| (C) | The value of sin (20°), sin (40°), sin (80°) is | (R) | $\sqrt{3}/32$ |
| (D) | The value of sin (20°), sin (40°), sin (60°), sin (80°) is | (S) | 1/16 |
| | | (T) | 1/8 |

3. Match the following columns :

| Column I | | Column II | |
|---|---|---|---|
| (A) | If maximum and minimum values of $\dfrac{7+6\tan\theta-\tan^2\theta}{1+\tan^2\theta}$ | (P) | $\lambda+\mu=2$ |
| | For all real values of $\theta\left(\neq\dfrac{\pi}{2}\right)$ are $\lambda$ and $\mu$ respectively, then | (Q) | $\lambda-\mu=6$ |
| (B) | If maximum and minimum values of $5\cos\theta+3\cos\left(\theta+\dfrac{\pi}{3}\right)+3$ | (R) | $\lambda+\mu=6$ |
| | For all real values of $\theta$ are $\lambda$ and $\mu$ respectively, then | (S) | |
| (C) | If maximum and minimum values of $1+\sin\left(\dfrac{\pi}{4}+\theta\right)+2\cos\left(\dfrac{\pi}{4}-\theta\right)$ | (T) | $\lambda-\mu$ = 14 |
| | For all real values of $\theta$ are l and $\mu$ respectively, then | | |

4. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| | In a triangle $ABC$ | | |
| (A) | $\sin 2A+\sin 2B+\sin 2C$ is | (P) | $4\sin A\cdot\sin B\cdot\sin C$ |
| (B) | $\cos 2A+\cos 2B+\cos 2C$ is | (Q) | $-1-4\cos A\cdot\cos B\cdot\cos C$ |
| (C) | $\sin^2 A+\sin^2 B+\sin^2 C$ is | (R) | $2+2\cos A\cdot\cos B\cdot\cos C$ |
| (D) | $\cos^2 A+\cos^2 B+\cos^2 C$ is | (S) | $1-2\cos A\cdot\cos B\cdot\cos C$ |

5. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | The value of $\cos^4\left(\dfrac{\pi}{8}\right)+\cos^4\left(\dfrac{3\pi}{8}\right)+\cos^4\left(\dfrac{5\pi}{8}\right)+\cos^4\left(\dfrac{7\pi}{8}\right)$ is | (P) | 1/8 |
| | | (Q) | –3/2 |
| (B) | The value of sin (12°), sin (48°), sin (54°) is | (R) | 3/2 |
| (C) | The value of | | |
| | sin (6°), sin (42°), sub(66°), sin (78°) is | (S) | 1/16 |
| (D) | The value of tan (6°), tan (42°), tan (66°), tan (78°) is | (T) | 1 |

6. Match the following columns :

| Column I | | Column II | |
|---|---|---|---|
| | If $A+B=\dfrac{\pi}{4}$, then the value of (1 + tan $A$) (1 + tan $B$) is 2 | (P) | 2 |
| (A) | The value of (1 + tan 21°) (1 + tan 22°) (1 + tan 23°) (1 + tan 24°) is | (Q) | 4 |
| (B) | The value of ( 1 + tan 2058°), (1 – tan 2013°) is | (R) | 8 |
| (C) | The value of $\left(1+\tan\left(\dfrac{\pi}{8}-x\right)\right)\cdot\left(1+\tan\left(x+\dfrac{\pi}{8}\right)\right)$. is | (S) | – 8 |
| (D) | The value of (1 + tan 235°), (1– tan 190°) is | (T) | –4 |

7. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | The value of 2 tan (50°) + tan (20°) is | (P) | 3 |
| (B) | The value of tan (40°) + 2 tan (10°) is | (Q) | 5 |
| (C) | The value of tan (20°) tan (40°), tan (60°), tan (80°) is | (R) | tan (70°) |
| (D) | If 3 sin $x$ + 4 cos $x$ = 5, then the value of 2 sin $x$ + cos $x$ + 4 tan $x$ is | (S) | tan (50°) |

8. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | The minimum value of $2\sin^2\theta+3\cos^2\theta$ is | (P) | 1 |
| (B) | The maximum value of $\sin^2\theta+\cos^4\theta$ is | (Q) | 3/4 |
| (C) | The least value of $\sin^4\theta+\cos^2\theta$ is | (R) | 2 |
| (D) | The greatest value of $\sin^{2014}\theta+\cos^{2010}\theta$ is | (S) | 4 |

9. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| If $\alpha$ and $\beta$ are the solutions of $a\cos\theta+b\sin\theta=c$, then | | | |
| (A) | the value of $\sin\alpha+\sin\beta$ is | (P) | $\dfrac{c^2-b^2}{a^2+b^2}$ |
| (B) | the value of $\sin\alpha\cdot\sin\beta$ is | (Q) | $\dfrac{2ac}{a^2+b^2}$ |

| | | | |
|---|---|---|---|
| (C) | the value of $\cos\alpha + \cos\beta$ is | (R) | $\frac{c^2 - a^2}{a^2 + b^2}$ |
| (D) | the value of $\cos\alpha \cdot \cos\beta$ is | (S) | $\frac{2bc}{a^2 + b^2}$ |

10. Match the following columns:

| | Column I | Column II | |
|---|---|---|---|
| (A) | The value of $\cos(12°) + \cos(84°) + \cos(156°) + \cos(132°)$ is | (P) | 0 |
| (B) | The value of $2\tan\left(\frac{\pi}{10}\right) + 3\sec\left(\frac{\pi}{10}\right) - 4\cos\left(\frac{\pi}{10}\right)$ is | (Q) | 1 |
| (C) | The value of $\sqrt{3}\cot(20°) - 4\cos(20°)$ is | (R) | 2 |
| (D) | The value of $\tan(20°) + 2\tan(50°) - \tan(70°)$ is | (S) | –1/2 |
| | | (T) | –1 |

## Assertion and Reason

**Codes**

(A) Both A and R are individually true and R is the correct explanation of A.
(B) Both A and R are individually true and R is not the correct explanation of A.
(C) A is true but R is false.
(D) A is false but R is true.

1. Assertion (A): $\sin\theta = x + \frac{1}{x}$ is impossible if $x \in R - \{0\}$.
Reason (R): AM $\geq$ GM
(a) A (b) B (c) C (d) D

2. Assertion (A): A is an obtuse angle in $\Delta ABC$, then $\tan B . \tan C > 1$
Reason (R): In $\Delta ABC$, $\tan A = \frac{\tan B + \tan C}{\tan B \tan C - 1}$
(a) A (b) B (c) C (d) D

3. Assertion (A): $\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$
Reason (R): $\cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ is complex $7^{\text{th}}$ root of unity.
(a) A (b) B (c) C (d) D

4. Assertion (A): $\tan\alpha + 2\tan(2\alpha) + 4\tan(4\alpha) + 8\tan(8\alpha) - 16\cot(16\alpha) = \cot\alpha$
Reason (R): $\cot\alpha - \tan\alpha = 2\cot 2\alpha$
(a) A (b) B (c) C (d) D

5. Assertion (A):
$$\cos^2\alpha + \cos^2\left(\alpha + \frac{\pi}{3}\right) + \cos^2\left(\alpha + \frac{4\pi}{3}\right) = 3\cos\alpha\cos\left(\alpha + \frac{2\pi}{3}\right)\cos\left(\alpha + \frac{4\pi}{3}\right)$$
Reason (R): If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$
(a) A (b) B (c) C (d) D

6. Assertion (A): $\tan(5\theta) - \tan(3\theta) - \tan(\theta) = \tan(5\theta), \tan(3\theta), \tan(\theta)$
Reason (R): If $x = y + z$, then $\tan x - \tan y - \tan z = \tan x \cdot \tan y \cdot \tan z$
(a) A (b) B (c) C (d) D

7. Assertion (A): The maximum value of $\sin\theta + \cos\theta$ is 2
Reason (R): The maximum value of $\sin\theta$ is 1 and that of $\cos\theta$ is also 1.
(a) A (b) B (c) C (d) D

8. Assertion (A): The maximum value of $\prod_{i=1}^{n}\cos(\alpha_i)$ under the restriction $0 \leq \alpha_1, \alpha_2, \alpha_3, ..., \alpha_n \leq \frac{\pi}{2}$ is $\frac{1}{2^{n/2}}$
Reason (R): $\prod_{i=1}^{n}\cot(\alpha_i) = 1$
(a) A (b) B (c) C (d) D

9. Assertion (A): If $A + B + C = \pi$, then the maximum value of $\tan A \cdot \tan B \cdot \tan C$ is $3\sqrt{3}$
Reason (R): AM $\geq$ GM
(a) A (b) B (c) C (d) D

10. Assertion (A): $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is positive for all real values of $x$ and $y$ only when $x = y$
Reason (R): $\sec^2\theta \geq 1$
(a) A (b) B (c) C (d) D

## Questions Asked In Previous Years' JEE-Advanced Examinations

1. Prove that
$\sin x \sin \cdot \sin(x - y + \sin y \sin z \sin(y - z) + \sin z \sin x \sin(z - x) + \sin(x - y)\sin(y - z)\sin(z - x) = 0$
**[IIT-JEE, 1978]**

2. If $\cos(\alpha + \beta) = \frac{4}{5}, \sin(\alpha - \beta) = \frac{5}{13}$ and $\alpha, \beta$ lie between 0 and $\frac{\pi}{4}$, find $\tan 2\alpha$ **[IIT-JEE, 1979]**

3. Given $A = \sin^2\theta + \cos^4\theta$ for all values of $\theta$, then
**[IIT-JEE, 1980]**
(a) $1 \leq A \leq 2$ (b) $3/4 \leq A \leq 1$
(c) $13/6 \leq A \leq 1$ (d) $3/4 \leq A \leq 13/6$

4. If $\tan A = \frac{1 - \cos B}{\sin B}$, then $\tan 2A = \tan B$. Is it true/false ? **[IIT-JEE, 1980]**

5. Suppose $\sin^3 x \sin 3x = \sum_{m=0}^{n} C_m \cos(mx)$ is an identity in $x$, when $C_0, C_1, \ldots C_n$ are constants and $C_n \neq 0$ is the value of $n = \ldots$

**[IIT-JEE, 1981]**

6. Without using the tables, prove that

$\sin 12° \sin 54° \sin 48° = \frac{1}{8}$ **[IIT-JEE, 1982]**

7. If $\alpha + \beta + \gamma = \pi$, then prove that $\sin^2 \alpha + \sin^2 \beta - \sin^2 \gamma = 2 \sin \alpha \times \sin \beta \cdot \sin \gamma$ **[IIT-JEE, 1983]**

8. Prove that

$$16 \cos\left(\frac{2\pi}{15}\right) \cos\left(\frac{4\pi}{15}\right) \cos\left(\frac{8\pi}{15}\right) \cos\left(\frac{16\pi}{15}\right) = 1$$

**[IIT-JEE, 1983]**

9. The value of

$$\left(1 + \cos\left(\frac{\pi}{8}\right)\right)\left(1 + \cos\left(\frac{3\pi}{8}\right)\right)$$
$$\left(1 + \cos\left(\frac{5\pi}{8}\right)\right)\left(1 + \cos\left(\frac{7\pi}{8}\right)\right)$$

is equal to

(a) 1/2 (b) $\cos\frac{\pi}{8}$ (c) 1/8 (d) $\frac{1+\sqrt{2}}{2\sqrt{2}}$

**[IIT-JEE, 1984]**

10. No questions asked in 1985.

11. The expression $3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$ is equal to

(a) 0 (b)* 1
(c) 3 (d) $\sin 4\alpha + \cos \alpha$

**[IIT-JEE, 1986]**

12. No questions asked in 1987.

13. The value of the expression $\sqrt{3}$ cosec (20°) – sec (20°) is equal to

(a) 2 (b) $\frac{2 \sin 20°}{\sin 40°}$
(c) 4 (d) $\frac{4 \sin 20°}{\sin 40°}$

**[IIT-JEE, 1988]**

14. Prove that

$$\tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha = \cot \alpha$$

**[IIT-JEE, 1988]**

15. No questions asked between 1989 – 1990.

16. Find the value of

$$\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\sin\left(\frac{7\pi}{14}\right)$$
$$\sin\left(\frac{9\pi}{14}\right)\sin\left(\frac{11\pi}{14}\right)\sin\left(\frac{13\pi}{14}\right)$$

**[IIT-JEE, 1991]**

17. If $f(x) = \cos[\pi^2] x + \cos[-\pi^2]$, where [,] = G.I.F., then

(a) $f\left(\frac{\pi}{2}\right) = 1$ (b) $f(\pi) = 1$
(c) $f(-\pi) = 0$ (d) $f\left(\frac{\pi}{4}\right) = 1$

**[IIT-JEE-1991]**

18. Match the following columns:

| | Column I | Column II | |
|---|---|---|---|
| (i) | Positive | (A) | $\left(\frac{13\pi}{48}, \frac{14\pi}{48}\right)$ |
| (ii) | Negative | (B) | $\left(\frac{14\pi}{48}, \frac{18\pi}{48}\right)$ |
| | | (C) | $\left(\frac{18\pi}{48}, \frac{23\pi}{48}\right)$ |
| | | (D) | $\left(0, \frac{\pi}{2}\right)$ |

**[IIT-JEE, 1992]**

19. If $k = \sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$, the numerical value of $k$ is ______ **[IIT-JEE, 1993]**

20. If $A > 0$, $B > 0$ and $A + B = \frac{\pi}{3}$, then the maximum value of $\tan A \tan B$ is ______ **[IIT-JEE, 1993]**

21. Let $0 < x < \frac{\pi}{4}$, then ($\sec 2x - \tan 2x$) equals

(a) $\tan\left(x - \frac{\pi}{4}\right)$ (b) $\tan\left(\frac{\pi}{4} - x\right)$
(c) $\tan\left(\frac{\pi}{4} + x\right)$ (d) $\tan^2\left(\frac{\pi}{4} + x\right)$

**[IIT-JEE, 1994]**

22. The value of the expression $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$ is

(a) 11 (b) 12 (c) 13 (d) 14

**[IIT-JEE, 1995]**

23. The minimum value of the expression $\sin \alpha + \sin \beta + \sin \gamma$, where $\alpha, \beta, \gamma$ are real numbers satisfying $\alpha + \beta + \gamma = \pi$ is

(a) positive (b) 0
(c) negative (d) –3

**[IIT-JEE, 1995]**

24. $\sec^2 \theta = \left(\frac{4xy}{(x+y)^2}\right)$ is true if and only if

(a) $x + y = 0$ (b) $x = y, x \neq 0$
(c) $x = y$ (d) $x \neq 0, y \neq 0$

**[IIT-JEE, 1996]**

25. The graph of the function $\cos x \cos(x + 2) - \cos^2(x + 1)$ is

(a) a straight line passing through $(0, -\sin^2 \theta)$ with slope 2.

(b) a straight line passing through (0, 0)
(c) a parabola with vertex $(1, -\sin^2 1)$
(d) a straight line passing through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$ and parallel to $x$-axis.

**[IIT-JEE,1997]**

26. Which of the following numbers is/are rational?
(a) sin 15° (b) cos 15°
(c) sin 15° cos 15° (d) sin 15° cos 75°

**[IIT-JEE, 1998]**

27. For a positive integer $n$, let

$$f_n(\theta) = \tan\left(\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)(1+\sec 2^2\theta)$$

$\ldots (1 + \sec (2^n \theta))$, then

(a) $f_2\left(\frac{\pi}{16}\right)=1$ (b) $f_3\left(\frac{\pi}{32}\right)=1$
(c) $f_4\left(\frac{\pi}{64}\right)=1$ (d) $f_5\left(\frac{\pi}{128}\right)=1$

**[IIT-JEE, 1998]**

28. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$, then $f(\theta)$
(a) $\geq 0$ when $\theta \geq 0$ (b) $\leq 0$ for all real $\theta$
(c) $\geq 0$ for all real $\theta$ (d) $\leq 0$ only when $\theta \leq 0$

**[IIT-JEE, 2000]**

29. The maximum value of cos $\alpha_1 \times$ cos $\alpha_n$ under the restriction $0 \leq \alpha_1, \alpha_2, \ldots\ldots, \alpha_n \leq \frac{\pi}{2}$ and cot $\alpha_1, \cdot$ cot $\alpha_2 \ldots$, cot $\alpha_n = 1$ is
(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$ (c) $\frac{1}{2n}$ (d) 1

**[IIT-JEE-2001]**

30. No questions asked in 2002.

31. If $\alpha + \beta = \frac{\pi}{2}$ and $\alpha = \beta + \gamma$, then tan $\alpha$ is
(a) 2 (tan $\beta$ + tan $\gamma$) (b) tan $\beta$ + tan $\gamma$
(c) (tan $\beta$ + 2 tan $\gamma$) (d) 2 tan $\beta$ + tan $\gamma$

**[IIT-JEE, 2003]**

32. If $\alpha \in \left(0, \frac{\pi}{2}\right)$, then the expression

$y = \sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}}$ is always greater than or equal to
(a) 2 tan $\alpha$ (b) 2 (c) 1 (d) $\sec^3 \alpha$

**[IIT-JEE, 2003]**

33 Given that both $\theta$ and $\varphi$ are acute angles and $\sin\theta = \frac{1}{2}, \cos\varphi = \frac{1}{3}$, then the value of $(\theta + \varphi)$ belongs to the interval
(a) $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$
(c) $\left(\frac{2\pi}{3}, \frac{5\pi}{6}\right)$ (d) $\left(\frac{5\pi}{6}, \pi\right)$

**[IIT-JEE, 2004]**

34. Find the range of values of $t$ for which $2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1}, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

**[IIT-JEE, 2005]**

35. $\cos(\alpha - \beta) = 1$ and $\cos(\alpha+\beta) = \frac{1}{e}$, where $\alpha, \beta \in [-\pi, \pi]$. Values of $\alpha, \beta$ which satisfy the equations is/are
(a) 0 (b) 1 (c) 2 (d) 4.

**[IIT-JEE, 2005]**

36. Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and
$t_1 = (\tan\theta)^{\tan\theta}, t_2 = (\tan\theta)^{\cot\theta}$
$t_3 = (\cot\theta)^{\tan\theta}, t_4 = (\cot\theta)^{\cot\theta}$, then
(a) $t_1 > t_2 > t_3 > t_4$ (b) $t_4 > t_3 > t_1 > t_2$
(c) $t_3 > t_1 > t_2 > t_4$ (d) $t_2 > t_3 > t_1 > t_4$

**[IIT-JEE, 2006]**

Note: No questions asked in 2007, 2008.

37. If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then
(a) $\tan^2 x = \frac{2}{3}$ (b) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$
(c) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$ (d) $\tan^2 x = \frac{1}{3}$

**[IIT-JEE, 2009]**

38. The maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is ______

**[IIT-JEE, 2010]**

39. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then,
(a) $P \subset Q$ and $Q - P \neq \varphi$ (b) $Q \not\subset P$
(c) $P \not\subset Q$ (d) $P = Q$

**[IIT-JEE, 2011]**

40. The positive integral value of $n > 3$ satisfying the equation $\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is…

**[IIT-JEE, 2011]**

41. Let $f : (-1, 1) \to R$ be such that $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the values of $f\left(\frac{1}{3}\right)$ is /are
(a) $1-\sqrt{\frac{3}{2}}$ (b) $1+\sqrt{\frac{3}{2}}$ (c) $1-\sqrt{\frac{2}{3}}$ (d) $1+\sqrt{\frac{2}{3}}$

**[IIT-JEE, 2012]**

No questions asked in between 2013 to 2016.

# ANSWERS

## LEVEL II

1. (b) 2. (b) 3. (b) 4. (b) 5. (d)
6. (c) 7. (c) 8. (a, b) 9. (d)
10. (a,b, d) 11. (a,c) 12. (a,c, d)
13. (c) 14. (c) 15. (b) 16. (b) 17. (a)
18. (b) 19. (c) 20. (d) 21. (b, c) 22. (c)
23. (b) 24. (c) 25. (a) 26. (b) 27. (c)
28. (a) 29. (b) 30. (a) 31. (c) 32. (c)
33. (d) 34. (d) 35. (d) 36. (b) 37. (a)
38. (a) 39. (b) 40. (b) 41. (a) 42. (d)
43. (a, b, c) 44. (d) 45. (d)

## LEVEL IV

13. $p \in (-\infty, 3-2\sqrt{2}] \cup [3+2\sqrt{2}, \infty)$
16. $qx^3 - px^2 + (1+q)\ x - p = 0$
32. $R_f = \left[0, \frac{1}{2}\right]$
33. $i$
34. $\left(\frac{4b}{a^2+2a+b^2}\right)$
35. 2/3

### INTEGER TYPE QUESTIONS

1. 4 2. 5 3. 3 4. 4 5. 3
6. 3 7. 7 8. 3 9. 5 10. 4
11. 2 12. 7

### COMPREHENSIVE LINK PASSAGES

Passage-I: 1. (d) 2. (a) 3. (d) 4. (b) 5. (b)
Passage-II: 1. (b) 2. (b) 3. (d) 4. (c) 5. (b)
Passage-III: 1. (b) 2. (c) 3. (b)
Passage-IV: 1. (a) 2. (c) 2. (b)
Passage-V: 1. (a) 2. (b) 3. (c)
Passage-VI: 1. (a) 2. (b) 3. (c)

### MATRIX MATCH

1. (A) $\to$ (P, Q, R, S, T): (B) $\to$ (S, T); (C) $\to$ (R)
2. (A) $\to$ T; (B) $\to$ S; (C) $\to$ P; (D) $\to$ Q
3. (A) $\to$ (R, S); (B) $\to$ (R, T); (C) $\to$ (P, Q)
4. (A) $\to$ P; (B) $\to$ Q; (C) $\to$ R, (D) $\to$ S
5. (A) $\to$ (R); (B) $\to$ (P);(C) $\to$ (S), (D) $\to$ (T)
6. (A) $\to$ (Q); (B) $\to$ (P); (C) $\to$ (P), (D) $\to$ (P)
7. (A) $\to$ (R); (B) $\to$ (S); (C) $\to$ (P), (D) $\to$ (Q)
8. (A) $\to$ (R); (B) $\to$ (P); (C) $\to$ (Q); (D) $\to$ (P)
9. (A) $\to$ (S); (B) $\to$ (R); (C) $\to$ (Q), (D) $\to$ (P)
10. (A) $\to$ (S); (B) $\to$ (P); (C) $\to$ (Q), (D) $\to$ (P)

### ASSERTION AND REASON

1. (a) 2. (d) 3. (d) 4. (a) 5. (a)
6. (a) 7. (d) 8. (a) 9. (a) 10. (a)

# HINTS AND SOLUTIONS

## LEVEL I

1. Given $r = 4900$ km
Circumference $= 2\pi r$
$$= 2 \times \frac{22}{7} \times 4900$$
$$= 44 \times 700$$
$$= 30800 \text{ km}$$

2. Let the three angles be $3x$, $4x$ and $5x$, respectively
Thus, $3x + 4x + 5x = 180°$
$\Rightarrow \quad 12x = 180°$
$\Rightarrow \quad x = 15°$
Therefore, the smallest angle
$= 3x = 3 \times 15° = 45°$
and the greatest angle
$= 5x = 5 \times 15° = 75°$
$$= \left(75 \times \frac{\pi}{180}\right) \text{radians}$$
$$= \left(\frac{5\pi}{12}\right) \text{radians}$$

3. Let the three angles be $a + d$, $a$, $a - d$
Thus, $a + d + a + a - d = 180°$
$\Rightarrow \quad 3a = 180°$
$\Rightarrow \quad a = \frac{180°}{3} = 60°$
It is given that,
$$(a-d)° : (a+d) \times \frac{\pi}{180} = \frac{60}{\pi}$$
$$\Rightarrow \quad \frac{(a-d)}{(a+d)} \times \frac{180}{\pi} = \frac{60}{\pi}$$

$$\Rightarrow \quad \frac{(a-d)}{(a+d)} = \frac{1}{3}$$

$\Rightarrow \quad a + d = 3a - 3d$

$\Rightarrow \quad 4d = 2a$

$$\Rightarrow \quad d = \frac{a}{2} = 30°$$

Hence, the three angles are 90°, 60°, 30°.

4. Let the number of sides of the given polygons be $5x$ and $4x$ respectively.
It is given that,

$$\left(\frac{2\times 5x-4}{5x} - \frac{2\times 4x-4}{4x}\right)\times 90 = 9$$

$$\Rightarrow \quad \left(\frac{10x-4}{5x} - \frac{2x-1}{x}\right) = \frac{1}{10}$$

$$\Rightarrow \quad \left(\frac{10x-4-10x+5}{5x}\right) = \frac{1}{10}$$

$$\Rightarrow \quad \left(\frac{1}{x}\right) = \frac{1}{2}$$

$\Rightarrow \quad x = 2$

Hence, the number of sides of the polygons would be 10 and 8 respectively.

5. Let the angles of the quadrilateral be
$a - 3d, a - d, a + d, a + 3d$
It is given that, $a + 3d = 2(a - 3d)$
$\Rightarrow \quad a + 3d = 2a - 6d$
$\Rightarrow \quad a = 9d$
Also, $a + 3d + a - d + a + d + a + 3d = 360$
$\Rightarrow \quad 4a = 360$
$\Rightarrow \quad a = 90$
and $\quad d = 10$
Hence, the smallest angle = 90° – 30°
= 60°

$$= \left(\frac{\pi}{3}\right)^c$$

6. Clearly, at half past 4, the hour hand will be at $4\frac{1}{2}$ and minute hand will be at 6.
In 1 hour angle made by the hour hand 30°.
In $4\frac{1}{2}$ hours angle made by the hour hand

$$= \frac{9}{2} \times 30° = 135°$$

In 1 minute angle made by the minute hand = 6°
In 30° minutes, angle made by the minute hand = 6 × 30° = 180°
Thus, the angle between the hour hand and the minute hand = 180° – 135°
= 45°

7. Angle subtended at the centre

$$= 30° = \left(30 \times \frac{\pi}{180}\right) = \frac{\pi}{6}$$

Hence, $l = 10 \times \frac{\pi}{6} = \frac{5\pi}{3}$.

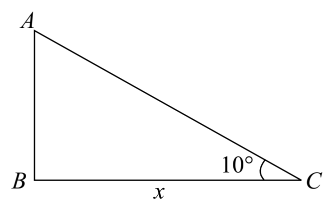
8. The angle traced by a minute hand in 60 minutes
= 360° = $2\pi$ radians
Thus the angle traced by minute hand in 18 minutes

$$= 2\pi \times \frac{18}{60} = \frac{3\pi}{5} \text{radians}$$

Hence, the distance moved by the tip in 18 minutes

$$= l = 35 \times \frac{3\pi}{5} = 21 \times \frac{22}{7} = 66 \text{ cm}$$

9. Let $AB$ be the height of the man and the required distance be $x$, where $BC = x$




Therefore, $\frac{2}{x} \times \frac{180}{\pi} = \frac{10}{60}$

$$\Rightarrow \quad x = \frac{2}{10} \times \frac{180}{\pi} \times 60$$

$$\Rightarrow \quad x = \frac{12 \times 180}{\pi}$$

$$\Rightarrow \quad x = \frac{12 \times 180}{\frac{22}{7}} = \frac{12 \times 180 \times 7}{22}$$

$$\Rightarrow \quad x = \frac{42 \times 180}{11} = 687.3$$

10. Let the required distance be $x$ cm.
According to the question,

$$6° = \frac{11}{2 \times x} \times \frac{180}{\pi}$$

$$\Rightarrow \quad \frac{6}{60} = \frac{11}{2 \times x} \times \frac{180}{\pi}$$

$$\Rightarrow \quad x = \frac{11}{2} \times \frac{180}{\pi} \times \frac{60}{6}$$

$$\Rightarrow \quad x = \frac{11}{2} \times \frac{180 \times 7}{22} \times 10$$

$\Rightarrow \quad x = 45 \times 7 \times 10 = 3150$
Hence, the required distance be 3150 cms.

11. Let the radius of the moon be $x$ km
It is given that, $\frac{16}{60} = \frac{2x}{60 \times 6400} \times \frac{180}{\pi}$

$\Rightarrow \quad x = \dfrac{16 \times 6400 \times \pi}{180 \times 2}$

$\Rightarrow \quad x = \dfrac{4 \times 640 \times \pi}{9}$

$\Rightarrow \quad x = \dfrac{4 \times 640 \times 22}{9 \times 7}$

$\Rightarrow \quad x = 894$

Hence, the radius of the moon be 894 km.

12. The difference between the the acute angles of a right angled triangle is $\dfrac{2\pi}{3}$ radians. Express the angles in degrees. Ans. $81^o, 9^o$

13. The angles of a quadrilateral are in AP and the greatest angle is $120^o$. Find the angles in radians.
Ans. $\dfrac{\pi}{3}, \dfrac{4\pi}{9}, \dfrac{5\pi}{9}, \dfrac{2\pi}{3}$.

14. At what distance does a man $5\dfrac{1}{2}$ ft in height, subtend an angle of 15°? Ans. 14.32 miles

15. Find the angle between the hour hand and minute-hand in circular measure at 4 O'clock.
Ans. $\dfrac{4\pi}{3}$

16. Given $\sec\theta + \tan\theta = 3$ …(i)

$\Rightarrow \quad (\sec\theta - \tan\theta) = \dfrac{1}{(\sec\theta + \tan\theta)} = \dfrac{1}{3}$ …(ii)

Adding (i) and (ii) we get,

$2\sec\theta = 3 + \dfrac{1}{3} = \dfrac{10}{3}$

$\Rightarrow \quad \sec\theta = \dfrac{5}{3}$

$\Rightarrow \quad \cos\theta = \dfrac{5}{3}$

17. Given, $\operatorname{cosec}\theta - \cot\theta = \dfrac{1}{5}$ …(i)

$\Rightarrow \quad \operatorname{cosec}\theta - \cot\theta = \dfrac{1}{\operatorname{cosec}\theta + \cot\theta} = 5$ …(ii)

Adding (i) and (ii) we get,

$2\operatorname{cosec}\theta = 5 + \dfrac{1}{5} = \dfrac{26}{5}$

$\Rightarrow \quad \operatorname{cosec}\theta = \dfrac{13}{5}$

$\Rightarrow \quad \sin\theta = \dfrac{5}{13}$

18. Given, $a = c\cos\theta + d\sin\theta$ …(i)
and $b = c\sin\theta - d\cos\theta$ …(ii)
Squaring and adding (i) and (ii) we get,

$a^2 + b^2 = (c\cos\theta + d\sin\theta)^2 + (c\sin\theta - d\cos\theta)^2$

$\Rightarrow \quad a^2 + b^2 = (c^2\cos^2\theta + d^2\sin^2\theta) + (c^2\sin^2\theta + d^2\cos^2\theta)$

$\Rightarrow \quad a^2 + b^2 = c^2 + d^2$

$\Rightarrow \quad m = 2, n = 2, p = 2, q = 2$

Hence, the value of $m + n + p + q + 42 = 50$

19. Let $x = 3\cos\theta - 4\sin\theta$ …(i)
and $5 = 3\sin\theta - 4\cos\theta$ …(ii)
Squaring and adding (i) and (ii) we get,

$x^2 + 5^2 = (3\cos\theta + 4\sin\theta)^2 + (3\sin\theta - 4\cos\theta)^2$

$\Rightarrow \quad x^2 + 5^2$
$= (9\cos^2\theta + 16\sin^2\theta + 24\sin\theta\cos\theta)$
$+ (9\sin^2\theta + 16\cos^2\theta - 24\sin\theta\cos\theta)$
$= (9\cos^2\theta + 16\sin^2\theta) + (9\sin^2\theta + 16\cos^2\theta)$
$= 9(\cos^2\theta + \sin^2\theta) + 16(\cos^2\theta + \sin^2\theta)$

$\Rightarrow \quad x^2 + 25 = 25$

$\Rightarrow \quad x^2 = 0$

$\Rightarrow \quad x = 0$

$\Rightarrow \quad 3\cos\theta - 4\sin\theta = 0$

20. We have $x^2 + y^2 + z^2$
$= (r\cos\theta\cos\varphi)^2 + (r\cos\theta\sin\varphi)^2 + (r\sin\theta)^2$

$\Rightarrow \quad x^2 + y^2 + z^2$
$= (r^2\cos^2\theta\cos^2\varphi) + (r^2\cos^2\theta\sin^2\varphi) + (r^2\sin^2\theta)$

$\Rightarrow \quad x^2 + y^2 + z^2$
$= r^2\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + (r^2\sin^2\theta)$
$= r^2\cos^2\theta + r^2\sin^2\theta$
$= r^2(\cos^2\theta + \sin^2\theta) = r^2$

$\Rightarrow \quad x^2 + y^2 + z^2 = r^2$

$\Rightarrow \quad m = 2, n = 2, p = 2$

Thus, the value of $(m + n + p - 4)^{(m+n+p+4)}$
$= 2^{10} = 1024$

21. Given, $x = \dfrac{2\sin\alpha}{1 + \cos\alpha + 3\sin\alpha}$

We have $\dfrac{\sin\alpha - 3\cos\alpha + 3}{2 - 2\cos\alpha}$

$= \dfrac{\sin\alpha + 3(1 - \cos\alpha)}{2(1 - \cos\alpha)}$

$= \dfrac{\sin\alpha}{2(1 - \cos\alpha)} + \dfrac{3}{2}$

$= \dfrac{\sin\alpha(1 + \cos\alpha)}{2(1 - \cos^2\alpha)} + \dfrac{3}{2}$

$= \dfrac{\sin\alpha(1 + \cos\alpha)}{2\sin^2\alpha} + \dfrac{3}{2}$

$= \dfrac{(1 + \cos\alpha)}{2\sin\alpha} + \dfrac{3}{2}$

$= \dfrac{(1 + \cos\alpha + 3\sin\alpha)}{2\sin\alpha}$

$= \dfrac{1}{x}$

22. We have

$$P = \sec^6\theta - \tan^6\theta - 3\sec^2\theta\tan^2\theta$$
$$= (\sec^2\theta - \tan^2\theta)^3 = 1$$
$$Q = \operatorname{cosec}^6\theta - \cot^6\theta - 3\operatorname{cosec}^2\theta\cot^2\theta$$
$$= (\operatorname{cosec}^2\theta - \cot^2\theta)^3 = 1$$

and $R = \sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$

$$= (\sin^2\theta + \cos^2\theta)^3 = 1$$

Hence, the value of $(P + Q + R)^{(P+Q+R)}$

$$= 3^3 = 27$$

23. We have $3\sin x + 4\cos x = 5$

Let $y = 3\cos x - 4\sin x$

Now, $y^2 + 5^2 = (3\cos x - 4\sin x)^2 + (3\sin x + 4\cos x)^2$

$\Rightarrow$ $y^2 + 25 = 9\cos^2 x + 16\sin^2 x - 24\sin x\cos x + 9\sin^2 x + 16\cos^2 x + 24\sin x\cos x$

$\Rightarrow$ $y^2 + 25 = 25(\cos^2 x + \sin^2 x) = 25$

$\Rightarrow$ $y^2 = 0$

$\Rightarrow$ $y = 0$

$\Rightarrow$ $3\cos x - 4\sin x = 0$

$\Rightarrow$ $3\cos x = 4\sin x$

$\Rightarrow$ $\tan x = 3/4$

Hence, the value of $2\sin x + \cos x + 4\tan x$

$$= 2\left(\frac{3}{5}\right) + \left(\frac{4}{5}\right) + 4\left(\frac{3}{4}\right) = 2 + 3 = 5$$

24. Given, $\sin A + \sin B + \sin C = -3$

$\Rightarrow$ $\sin A = -1, \sin B = -1, \sin C = -1$

$\Rightarrow$ $A = -\dfrac{\pi}{2}, B = -\dfrac{\pi}{2}, C = -\dfrac{\pi}{2}$

Hence, the value of $\cos A + \cos B + \cos C + 10$

$= 0 + 0 + 0 + 10 = 10.$

25. We have $(1 + \sin\theta)(1 + \cos\theta) = \dfrac{5}{4}$

$\Rightarrow$ $1 + \sin\theta + \cos\theta + \sin\theta\cos\theta = \dfrac{5}{4}$

$\Rightarrow$ $1 + t + \left(\dfrac{t^2 - 1}{2}\right) = \dfrac{5}{4}$ $(\sin\theta + \cos\theta = t$, say)

$\Rightarrow$ $t + \left(\dfrac{t^2 - 1}{2}\right) = \dfrac{1}{4}$

$\Rightarrow$ $t^2 + 2t - 1 = \dfrac{1}{2}$

$\Rightarrow$ $2t^2 + 4t - 3 = 0$

$\Rightarrow$ $t = \dfrac{-4 \pm \sqrt{16 + 24}}{4}$

$$= \frac{-4 \pm 2\sqrt{10}}{4} = -1 \pm \frac{1}{2}\sqrt{10}$$

$\Rightarrow$ $t = -1 + \dfrac{1}{2}\sqrt{10}$

$\Rightarrow$ $\sin\theta + \cos\theta = -1 + \dfrac{1}{2}\sqrt{10}$

Now, $(1 - \sin\theta)(1 - \cos\theta)$

$$= 1 - \sin\theta - \cos\theta + \sin\theta\cos\theta$$
$$= 1 - (\sin\theta + \cos\theta) + \sin\theta\cos\theta$$
$$= 1 - \left(-1 + \frac{\sqrt{10}}{2}\right) + \frac{1}{2}\left(\frac{10}{4} - \sqrt{10}\right)$$
$$= \left(2 + \frac{5}{4}\right) - \sqrt{10}$$
$$= \left(\frac{13}{4} - \sqrt{10}\right)$$

26. Given, $f(x) = \dfrac{9x^2\sin^2 x + 4}{x\sin x}$

$$= 9x\sin x + \frac{4}{x\sin x}$$

Applying AM $\geq$ GM we get,

$$\left(\frac{9x\sin x + \dfrac{4}{x\sin x}}{2}\right) \geq \sqrt{9x\sin x \times \frac{4}{x\sin x}}$$

$\Rightarrow$ $\left(\dfrac{9x\sin x + \dfrac{4}{x\sin x}}{2}\right) \geq 6$

$\Rightarrow$ $\left(9x\sin x + \dfrac{4}{x\sin x}\right) \geq 12$

Hence, the minimum value of $f(x)$ is 12.

27. We have, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

$\Rightarrow$ $\sin\theta = (\sqrt{2} - 1)\cos\theta$

$\Rightarrow$ $\cos\theta = \dfrac{\sin\theta}{(\sqrt{2} - 1)}$

$\Rightarrow$ $\cos\theta = (\sqrt{2} + 1)\sin\theta$

$\Rightarrow$ $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

28. We have, $\tan^2\theta = 1 - e^2$

$\Rightarrow$ $1 + \tan^2\theta = 1 + 1 - e^2 = 2 - e^2$

$\Rightarrow$ $\sec^2\theta = 2 - e^2$

$\Rightarrow$ $\sec\theta = \sqrt{2 - e^2}$

Now, $\sec\theta + \tan^3\theta \cdot \operatorname{cosec}\theta$

$$= \sec\theta + \frac{\sin^3\theta}{\cos^3\theta} \cdot \frac{1}{\sin\theta}$$
$$= \sec\theta + \frac{\sin^2\theta}{\cos^3\theta}$$
$$= \sec\theta + \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{1}{\cos\theta}$$
$$= \sec\theta + \tan^2\theta \cdot \sec\theta$$
$$= \sec\theta\,(1 + \tan^2\theta)$$
$$= \sec^3\theta$$
$$= (2 - e^2)^{3/2}$$

29. Given, $\sin\theta + \sin^2\theta + \sin^3\theta = 1$

$\Rightarrow (\sin\theta + \sin^3\theta) = 1 - \sin^2\theta = \cos^2\theta$

$\Rightarrow (\sin\theta + \sin^3\theta)^2 = (\cos^2\theta)^2$

$\Rightarrow (\sin\theta + \sin^3\theta)^2 = (\cos^2\theta)^2$

$\Rightarrow (1 - \cos^2\theta)(2 - \cos^2\theta)^2 = \cos^4\theta$

$\Rightarrow (1 - \cos^2\theta)(4 - 4\cos^2\theta + \cos^4\theta) = \cos^4\theta$

$\Rightarrow 4 - 4\cos^2\theta + \cos^4\theta - 4\cos^2\theta + 4\cos^4\theta - \cos^6\theta = \cos^4\theta$

$\Rightarrow \cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

30. Given, $x = \dfrac{2\sin\theta}{1 + \cos\theta + \sin\theta}$

$$= \frac{2\sin\theta}{(1+\sin\theta)+\cos\theta}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{\big((1+\sin\theta)+\cos\theta\big)\big((1+\sin\theta)-\cos\theta\big)}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{((1+\sin\theta)^2-\cos^2\theta)}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{(1+\sin^2\theta+2\sin\theta-\cos^2\theta)}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{(\sin^2\theta+2\sin\theta+(1-\cos^2\theta))}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{(2\sin\theta+2\sin^2\theta)}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{2\sin\theta(1+\sin\theta)}$$

$$= \frac{((1+\sin\theta)-\cos\theta)}{(1+\sin\theta)}$$

$$= \frac{(1-\cos\theta+\sin\theta)}{(1+\sin\theta)}$$

31. We have $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$

$= 3(\sin^4 x - 4\sin^3 x\cos x + 6\sin^2 x\cos^2 x - 4\sin x\cos^3 x + \cos^4 x)$
$+ 6(\sin^2 x + \cos^2 x + 2\sin x\cos x)$
$+ 4\{(\sin^2 x)^3 + (\cos^2 x)^3\}$

$= 3(\sin^4 x + \cos^4 x - 4\sin x\cos x (\sin^2 x + \cos^2 x) + 6\sin^2 x\cos^2 x)$
$+ 6(1 + 2\sin x\cos x)$
$+ 4(\sin^2 x + \cos^2 x)^2$
$- 12\sin^2 x\cos^2 x$

$= 3 - 6\sin^2 x\cos^2 x - 12\sin x\cos x$
$+ 18\sin^2 x\cos^2 x + 6 + 12\sin x\cos x$
$+ 4 - 12\sin^2 x\cos^2 x$

$= 3 + 6 + 4$

$= 13$

32. We have $\sin x + \sin^2 x = 1$

$\Rightarrow \sin x = 1 - \sin^2 x = \cos^2 x$

Now, $\cos^8 x + 2\cos^6 x + \cos^4 x$

$= (\cos^4 x)^2 + 2\cdot\cos^4 x\cdot\cos^2 x + (\cos^2 x)^2$

$= (\cos^4 x + \cos^2 x)^2$

$= (\sin^2 x + \sin x)^2$

$= (1)^2 = 1$

33. We have $81^{\sin^2\theta} + 81^{\cos^2\theta} = 30$

$\Rightarrow 81^{\sin^2\theta} + 81^{1-\sin^2\theta} = 30$

$\Rightarrow 81^{\sin^2\theta} + \dfrac{81}{81^{\sin^2\theta}} = 30$

$\Rightarrow a + \dfrac{81}{a} = 30,\ a = 81^{\sin^2\theta}$

$\Rightarrow a^2 - 30a + 81 = 0$

$\Rightarrow (a - 27)(a - 3) = 0$

$\Rightarrow a = 3, 27$

When $a = 3$

$\Rightarrow 81^{\sin^2\theta} = 3$

$\Rightarrow 3^{4\sin^2\theta} = 3$

$\Rightarrow 4\sin^2\theta = 1$

$\Rightarrow \sin^2\theta = \left(\dfrac{1}{2}\right)^2$

$\Rightarrow \sin^2\theta = \sin^2\left(\dfrac{\pi}{6}\right)$

$\Rightarrow \theta = \left(n\pi \pm \dfrac{\pi}{6}\right)$

$\Rightarrow \theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

When $a = 27$

$\Rightarrow 81^{\sin^2\theta} = 27$

$\Rightarrow 3^{4\sin^2\theta} = 3^3$

$\Rightarrow 4\sin^2\theta = 3$

$\Rightarrow \sin^2\theta = \left(\dfrac{\sqrt{3}}{2}\right)^2$

$\Rightarrow \sin^2\theta = \sin^2\left(\dfrac{\pi}{3}\right)$

$\Rightarrow \theta = \left(n\pi \pm \dfrac{\pi}{3}\right)$

$\Rightarrow \theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

Hence, the values of $\theta$ are $\dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{5\pi}{6}$.

34. We have $f_6(\theta) = \sin^6\theta + \cos^6\theta$

$= 1 - 3\sin^2\theta\cos^2\theta$

Also $f_4(\theta) = \sin^4\theta + \cos^4\theta$

$= 1 - 2\sin^2\theta\cos^2\theta$

Now, $\frac{1}{6}f_6(\theta)-\frac{1}{4}f_4(\theta)$

$=\frac{1}{6}(1-3\sin^2\theta\cos^2\theta)-\frac{1}{4}(1-2\sin^2\theta\cos^2\theta)$

$=\frac{1}{6}-\frac{1}{2}\sin^2\theta\cos^2\theta-\frac{1}{4}+\frac{1}{2}\sin^2\theta\cos^2\theta$

$=\frac{1}{6}-\frac{1}{4}$

$=-\frac{1}{12}$

35. Given, $x\sin\alpha=y\cos\alpha$

$\Rightarrow \quad \frac{x}{\cos\alpha}=\frac{y}{\sin\alpha}=k$

Also, $x\sin^3\alpha+y\cos^3\alpha=\sin\alpha\cos\alpha$

$\Rightarrow \quad k\sin^3\alpha\cos\alpha+k\sin\alpha\cos^3\alpha=\sin\alpha\cos\alpha$

$\Rightarrow \quad k\sin\alpha\cos\alpha(\sin^2\alpha+\cos^2\alpha)=\sin\alpha\cos\alpha$

$\Rightarrow \quad k\sin\alpha\cos\alpha=\sin\alpha\cos\alpha$

$\Rightarrow \quad k=1$

Thus, $x=k\cos\alpha=\cos\alpha$, $y=k\sin\alpha=\sin\alpha$

Now, $x^2+y^2=\cos^2\alpha+\sin^2\alpha=1$

36. Given, $\tan\theta+\sin\theta=m$, $\tan\theta-\sin\theta=n$,

$\Rightarrow \quad mn=\tan^2\theta-\sin^2\theta$

$=\sin^2\theta\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)=\frac{\sin^4\theta}{\cos^2\theta}$

Now, $m^2-n^2$

$=(\tan\theta+\sin\theta)^2-(\tan\theta-\sin\theta)^2$

$=4\tan\theta\sin\theta$

$=4\frac{\sin^2\theta}{\cos\theta}$

$=4\sqrt{mn}$

37. Given, $\frac{\cos^4 x}{\cos^2 y}+\frac{\sin^4 x}{\sin^2 y}=1$

$\Rightarrow \quad \frac{\cos^4 x}{\cos^2 y}+\frac{\sin^4 x}{\sin^2 y}=\cos^2 x+\sin^2 x$

$\Rightarrow \quad \frac{\cos^4 x}{\cos^2 y}-\cos^2 x=\sin^2 x-\frac{\sin^4 x}{\sin^2 y}$

$\Rightarrow \quad \cos^2 x\left(\frac{\cos^2 x}{\cos^2 y}-1\right)=\sin^2 x\left(1-\frac{\sin^2 x}{\sin^2 y}\right)$

$\Rightarrow \quad \frac{\cos^2 x}{\cos^2 y}(\cos^2 x-\cos^2 y)$

$=\frac{\sin^2 x}{\sin^2 y}(\sin^2 y-\sin^2 x)$

$=\frac{\sin^2 x}{\sin^2 y}(\cos^2 x-\cos^2 y)$

$\Rightarrow \quad (\cos^2 x-\cos^2 y)\left(\frac{\cos^2 x}{\cos^2 y}-\frac{\sin^2 x}{\sin^2 y}\right)=0$

$\Rightarrow \quad (\cos^2 x-\cos^2 y)=0, \left(\frac{\cos^2 x}{\cos^2 y}-\frac{\sin^2 x}{\sin^2 y}\right)=0$

$\Rightarrow \quad \cos^2 x=\cos^2 y$, $\sin^2 x=\sin^2 y$

Now, $\frac{\cos^4 y}{\cos^2 x}+\frac{\sin^4 y}{\sin^2 x}$

$=\frac{\cos^4 y}{\cos^2 y}+\frac{\sin^4 y}{\sin^2 y}$

$=\cos^2 y+\sin^2 y$

$=1$

38. We have $2f_6(\theta)-3f_4(\theta)+1$

$=2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1$

$=2(1-3\sin^2\theta\cos^2\theta)-(1-2\sin^2\theta\cos^2\theta)+1$

$=2-3+1$

$=0$

39. We have $\frac{\sin A}{\sin B}=p, \frac{\cos A}{\cos B}=q$

$\Rightarrow \quad \frac{\sin A}{\cos A}/\frac{\sin B}{\cos B}=\frac{p}{q}$

$\Rightarrow \quad \frac{\tan A}{\tan B}=\frac{p}{q}$

$\Rightarrow \quad \frac{\tan A}{p}=\frac{\tan B}{q}=\lambda$

Also, $\frac{\sin A}{\sin B}=p, \frac{\cos A}{\cos B}=q$

$\Rightarrow \quad \frac{\sin A\cos A}{\sin B\cos B}=pq$

$\Rightarrow \quad \frac{2\sin A\cos A}{2\sin B\cos B}=pq$

$\Rightarrow \quad \frac{\sin 2A}{\sin 2B}=pq$

$\Rightarrow \quad \frac{2\tan A}{1+\tan^2 A}/\frac{2\tan B}{1+\tan^2 B}=pq$

$\Rightarrow \quad \frac{2p\lambda}{1+p^2\lambda^2}/\frac{2q\lambda}{1+q^2\lambda^2}=pq$

$\Rightarrow \quad \frac{p}{(1+p^2\lambda^2)}\times\frac{(1+q^2\lambda^2)}{q}=pq$

$\Rightarrow \quad \frac{(1+q^2\lambda^2)}{(1+p^2\lambda^2)}=q^2$

$\Rightarrow \quad (1+q^2\lambda^2)=q^2(1+p^2\lambda^2)$

$\Rightarrow \quad \lambda^2(1-p^2)\, q^2 = q^2 - 1$

$$\Rightarrow \quad \lambda^2 = \frac{(q^2-1)}{(1-p^2)q^2}$$

$$\Rightarrow \quad \lambda = \pm\frac{1}{q}\sqrt{\frac{(q^2-1)}{(1-p^2)}}$$

Therefore, $\tan A = \pm\frac{p}{q}\sqrt{\frac{(q^2-1)}{(1-p^2)}}$

and $\tan B = \pm\sqrt{\frac{(q^2-1)}{(1-p^2)}}$

40. We have $\frac{\sin^4\alpha}{a} + \frac{\cos^4\alpha}{b} = \frac{1}{a+b}$

$$\Rightarrow \quad \left(\frac{a+b}{a}\right)\sin^4\alpha + \left(\frac{a+b}{b}\right)\cos^4\alpha = 1$$

$$\Rightarrow \quad \left(1+\frac{b}{a}\right)\sin^4\alpha + \left(1+\frac{a}{b}\right)\cos^4\alpha = 1$$

$$\Rightarrow \quad \left(\frac{b}{a}\sin^4\alpha + \frac{a}{b}\cos^4\alpha\right) + (\sin^4\alpha + \cos^4\alpha) = 1$$

$$\Rightarrow \quad \left(\frac{b}{a}\sin^4\alpha + \frac{a}{b}\cos^4\alpha\right) + (1 - 2\sin^2\alpha\cdot\cos^2\alpha) = 1$$

$$\Rightarrow \quad \left(\frac{b}{a}\sin^4\alpha + \frac{a}{b}\cos^4\alpha - 2\sin^2\alpha\cdot\cos^2\alpha\right) = 0$$

$$\Rightarrow \quad \left(\sqrt{\frac{b}{a}}\sin^2 a\right)^2 + \left(\sqrt{\frac{a}{b}}\cos^2\alpha\right)^2$$

$$-2\sqrt{\frac{b}{a}}\sin^2\alpha\cdot\sqrt{\frac{a}{b}}\cos^2\alpha = 0$$

$$\Rightarrow \quad \left(\sqrt{\frac{b}{a}}\sin^2\alpha - \sqrt{\frac{a}{b}}\cos^2\alpha\right)^2 = 0$$

$$\Rightarrow \quad \left(\sqrt{\frac{b}{a}}\sin^2\alpha - \sqrt{\frac{a}{b}}\cos^2\alpha\right) = 0$$

$$\Rightarrow \quad \sqrt{\frac{b}{a}}\sin^2\alpha = \sqrt{\frac{a}{b}}\cos^2\alpha$$

$$\Rightarrow \quad \frac{\sin^2\alpha}{a} = \frac{\cos^2\alpha}{b} = \frac{1}{a+b}$$

$$\Rightarrow \quad \sin^2\alpha = \frac{a}{a+b}, \cos^2\alpha = \frac{b}{a+b}$$

Now, $\frac{\sin^8\alpha}{a^3} + \frac{\cos^8\alpha}{b^3}$

$$= \frac{(\sin^2\alpha)^4}{a^3} + \frac{(\cos^2\alpha)^4}{b^3}$$

$$= \frac{\left(\frac{a}{a+b}\right)^4}{a^3} + \frac{\left(\frac{b}{a+b}\right)^4}{b^3}$$

$$= \frac{a^4}{a^3(a+b)^4} + \frac{b^4}{b^3(a+b)^4}$$

$$= \frac{a}{(a+b)^4} + \frac{b}{(a+b)^4}$$

$$= \frac{a+b}{(a+b)^4}$$

$$= \frac{1}{(a+b)^3}$$

41.

(i) $\sin(120°) = \sin(90\times 1 + 30°)$

$= \cos(30°) = \frac{\sqrt{3}}{2}$

(ii) $\sin(150°) = \sin(90\times 2 - 30°)$

$= \sin(30°) = \frac{1}{2}$

(iii) $\sin(210°) = \sin(90\times 2 + 30°)$

$= -\sin(30°) = -\frac{1}{2}$

(iv) $\sin(225°) = \sin(90\times 2 + 45°)$

$= -\sin(45°) = -\frac{1}{\sqrt{2}}$

(v) $\sin(300°) = \sin(90\times 3 + 30°)$

$= -\cos(30°) = -\frac{\sqrt{3}}{2}$

(vi) $\sin(330°) = \sin(90\times 3 + 60°)$

$= -\cos(60°) = -\frac{1}{2}$

(vii) $\sin(405°) = \sin(90\times 4 + 45°)$

$= \sin(45°) = \frac{1}{\sqrt{2}}$

(viii) $\sin(660°) = \sin(90\times 7 + 30°)$

$= \sin(30°) = \frac{1}{2}$

(ix) $\sin(1500°) = \sin(90\times 16 + 60°)$

$= \sin(60°) = \frac{\sqrt{3}}{2}$

(x) $\sin(2013°) = \sin(90 \times 22 + 33°)$
$= -\sin(33°)$

42. We have
$\cos(1°)\cos(2°)\cos(3°)\ldots\cos(189°)$.
$= \cos(1°)\cos(2°)\cos(3°)\ldots\cos(89°)$
$\cos(90°)\cos(91°)\ldots\cos(189°)$
$= \cos(1°)\cos(°)\cos(3°)\ldots\cos(89°)$
$\times 0 \times \cos(91°)\ldots\cos(189°)$
$= 0$

43. We have
$\tan(1°)\tan(2°)\tan(3°)\ldots(89°)$
$= \tan(1°)\tan(2°)\tan(3°)\ldots\tan(44°)$
$\tan(45°)\tan(46°)\ldots\tan(87°)\tan(88°)\tan(89°)$
$= \{\tan(1°) \times \tan(89°)\} \cdot \{\tan(2°) \times \tan(88°)$
$\ldots \{\tan(44°) \times \tan(46°)\}, \tan(45°)$
$= 1$

44. We have
$\tan 35° \cdot \tan 40° \cdot \tan 45° \cdot \tan 50° \cdot \tan 55°$
$= \{\tan 35° \times \tan 55°\}\ \{\tan 40° \times \tan 50°\} \cdot \tan 45°$
$= \{\tan 35° \times \cot 35°\} \cdot \{\tan 40° \times \cot 40°\} \times \tan 45°$
$= 1$

45. We have $\sin(10°) + \sin(20°) + \sin(30°)$
$+ \sin(40°) + \ldots + \sin(360°)$
$= \sin(10°) + \sin(20°) + \sin(30°)$
$+ \sin(40°) + \ldots + \sin(150°)$
$+ \sin(340°) + \sin(350°) + \sin(360°)$
$= \sin(10°) + \sin(20°) + \sin(30°)$
$+ \sin(40°) + \ldots + \sin(80°)$
$+ \sin(90°) + \sin(100°)$
$+ \sin(360° - 40°) + \sin(360° - 30°)$
$+ \sin(360° - 20) + \sin(360° - 10°)$
$+ \sin(360°)$
$= \sin(10°) + \sin(20°) + \sin(30°)$
$+ \sin(40°) + \ldots + \sin(80°)$
$+ \sin(90°) + \sin(100°)$
$-\sin(40°) - \sin(30°)$
$-\sin(20°) - \sin(10°) + \sin(180°)$
$= 0$

46. We have $\cos(10°) + \cos(20°) + \cos(30°)$
$+ \cos(40°) + \ldots + \cos(360°)$
$= \cos 20° + \cos 30° + \cos 40° + \ldots +$
$\cos 140° + \cos 150° + \cos 160° + \cos 170°$
$+ \cos 180° + (\cos 190° + \cos 200° +$
$\cos 210° + \cos 220° + \ldots + \cos 360°)$
$= \cos 10° + \cos 20° + \cos 30° + \cos 40°$
$+ \ldots - \cos 40° - \cos 50° - \cos 60°$
$- \cos 70° + \cos 180° + (\cos 190°$
$+ \cos 200° + \cos 210° + \cos 220°$
$+ \ldots + \cos 360°)$
$= \cos 180° + \cos 360°$
$= -1 + 1$
$= 0$

47. We have
$\sin^2 5° + \sin^2 10° + \sin^2 15° + \ldots + \sin^2 90°$
$= \sin^2 5° + \sin^2 10° + \sin^2 15° + \ldots + \sin^2 40 + \sin^2 45$
$+ \sin^2 50 + \sin^2 80 + \sin^2 85 + \sin^2 90°$
$= (\sin^2 5° + \sin^2 85°)$
$+ (\sin^2 10° + \sin^2 80°)$
$+ (\sin^2 15° + \sin^2 75°)$
$+ \ldots + (\sin^2 40° + \sin^2 50°)$
$+ (\sin^2 45° + \sin^2 90°)$
$= (\sin^2 5° + \cos^2 5°)$
$+ (\sin^2 10° + \cos^2 10°)$
$+ (\sin^2 15° + \cos^2 15°) + \ldots$
$\ldots + (\sin^2 40° + \cos^2 40°)$
$+ (\sin^2 45° + \sin^2 90°)$
$= (1 + 1 + \ldots 8 \text{ times}) + \left(\frac{1}{2} + 1\right)$
$= \left(8 + 1 + \frac{1}{2}\right)$
$= 9\frac{1}{2}$

48. We have $\sin^2\left(\frac{\pi}{18}\right) + \sin^2\left(\frac{\pi}{9}\right)$
$+ \sin^2\left(\frac{4\pi}{9}\right) + \sin^2\left(\frac{7\pi}{18}\right)$
$= \sin^2\left(\frac{\pi}{18}\right) + \sin^2\left(\frac{2\pi}{18}\right)$
$+ \sin^2\left(\frac{8\pi}{18}\right) + \sin^2\left(\frac{7\pi}{18}\right)$
$= \sin^2\left(\frac{\pi}{18}\right) + \sin^2\left(\frac{\pi}{2} - \frac{7\pi}{18}\right)$
$+ \sin^2\left(\frac{\pi}{2} - \frac{\pi}{18}\right) + \sin^2\left(\frac{7\pi}{18}\right)$
$= \sin^2\left(\frac{\pi}{18}\right) + \cos^2\left(\frac{7\pi}{18}\right)$
$+ \cos^2\left(\frac{\pi}{18}\right) + \sin^2\left(\frac{7\pi}{18}\right)$
$= \left\{\sin^2\left(\frac{\pi}{18}\right) + \cos^2\left(\frac{\pi}{18}\right)\right\}$
$+ \left\{\cos^2\left(\frac{7\pi}{18}\right) + \sin^2\left(\frac{7\pi}{18}\right)\right\}$
$= 1 + 1$
$= 2$

49. We have
$\tan(20°)\tan(25°)\tan(45°)\tan(65°)\tan(70°)$
$= \tan(20°)\tan(25°)\tan(45°)\tan(90° - 25°)\tan(90° - 20°)$
$= \tan(20°)\tan(25°)\tan(45°)\cot(25°)\cot(20°)$
$= \tan(45°)$
$= 1$

50. Given, $\sin(\theta_1) + \sin(\theta_2) + \sin(\theta_3) = 3$
It is possible only when each term of the above equation will provide the maximum value
Thus, $\sin(\theta_1) = 1, \sin(\theta_2) = 1, \sin(\theta_3) = 1$
$\Rightarrow \quad \theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}, \theta_3 = \frac{\pi}{2}$
Hence, the value of
$\cos(\theta_1) + \cos(\theta_2) + \cos(\theta_3)$
$= \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)$
$= 0$

51. We have $\sin^2 6° + \sin^2 12° + \ldots + \sin^2 90°$
$= (\sin^2 6° + \sin^2 84°) + (\sin^2 12° + \sin^2 78°) + \ldots + (\sin^2 42° + \sin^2 48°) + \sin^2 90°$
$= 7 \times 1 + 1$
$= 8$

52. We have $\sin^2 10° + \sin^2 20° + \ldots + \sin^2 90°$
$= (\sin^2 10° + \sin^2 80°) + (\sin^2 20° + \sin^2 70°) + \ldots + (\sin^2 40° + \sin^2 50°) + \sin^2 90°$
$= 4 \times 1 + 1$
$= 5$

53. We have $\sin^2 9° + \sin^2 18° + \ldots + \sin^2 90°$
$= (\sin^2 9° + \sin^2 81°) + (\sin^2 18° + \sin^2 72°) + \ldots + (\sin^2 36° + \sin^2 54°) + \sin^2 45° + \sin^2 90°$
$= 4 \times 1 + \frac{1}{2} + 1$
$= 5\frac{1}{2}$

54. We have $\tan 1° \cdot \tan 2° \cdot \tan 3° \ldots \tan 89°$
$= (\tan 1° \cdot \tan 89°)(\tan 2° \cdot \tan 88°)(\tan 3° \cdot \tan 87°) \ldots (\tan 44° \cdot \tan 46°)\tan 45°$
$= 1.1.1........1$
$= 1$

55. We have $\cos 1° \cdot \cos 2° \cdot \cos 3° \ldots \cos 189°$
$= \cos 1° \cdot \cos 2° \cdot \cos 3° \ldots \cos 89° \cos 90° \ldots \cos 180°$
$= 0$

56. Given equation is $2\sin^2\theta + 2\cos\theta = 0$
$\Rightarrow \quad 2 - 2\cos^2\theta + 3\cos\theta = 0$
$\Rightarrow \quad 2\cos^2\theta - 3\cos\theta - 2 = 0$
$\Rightarrow \quad 2\cos^2\theta - 4\cos\theta + \cos\theta - 2 = 0$
$\Rightarrow \quad 2\cos\theta(\cos\theta - 2) + 1(\cos\theta - 2) = 0$
$\Rightarrow \quad (2\cos\theta + 1)(\cos\theta - 2) = 0$
$\Rightarrow \quad \cos\theta = -\frac{1}{2}, 2$
$\Rightarrow \quad \theta = 120°, 240°$

57. Given equation is $\cos\theta + \sqrt{3}\sin\theta = 2$
$\Rightarrow \quad \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta = 1$
$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{3}\right) = 1$
$\Rightarrow \quad \left(\theta - \frac{\pi}{3}\right) = 0$
$\Rightarrow \quad \theta = \frac{\pi}{3}$

58. Given, $4n\alpha = \pi$
$2n\alpha = \frac{\pi}{2}$
Now, $\tan\alpha\tan\alpha \cdot \tan \cdot \tan 3\alpha \tan 3 \tan(2n - 1)\alpha$
$= \tan\alpha\tan\alpha(2n - 1)\alpha)(\tan 2\alpha\tan 2(2n - 2)\alpha) \cdot (\tan 3\alpha \cdot \tan(2n - 3)\alpha) \ldots$
$= (\tan\alpha \cdot \tan(2n\alpha - \alpha))(\tan 2\alpha \cdot \tan(2n\alpha - 2\alpha))(\tan 3\alpha \cdot \tan(2n\alpha - 3\alpha)) \ldots$
$= \left(\tan\alpha \cdot \tan\left(\frac{\pi}{2} - \alpha\right)\right) \cdot \left(\tan 2\alpha \cdot \tan\left(\frac{\pi}{2} - 2\alpha\right)\right) \cdot \left(\tan 3\alpha \cdot \tan\left(\frac{\pi}{2} - 3\alpha\right)\right) \ldots$
$= (\tan\alpha \times \cot\alpha) \cdot (\tan 2\alpha \cdot \cot 2\alpha) \cdot (\tan 3\alpha \cdot \cot 3\alpha)$
$= 1$

59. We have
$\cos(18°) + \cos(234°) + \cos(162°) + \cos(306°)$
$=$
$= \cos(18°) - \cos(54°) - \cos(18°) + \cos(54°)$
$= 0$

60. We have
$\cos(20°) + \cos(40°) + \cos(60°) + \ldots + \cos(180°)$
$= \cos(20°) + \cos(160°) + \cos(40°) + \cos(140°) + \cos(60°) + \cos(120°) + \cos(80°) + \cos(100°) + \cos(180°)$
$= \cos(180°) = -1$

61. We have
$\sin(20°) + \sin(40°) + \sin(60°) + \ldots + \sin(360°)$
$= \sin(20°) + \sin(340°) + \sin(40°) + \sin(320°) + \ldots + \sin(180°) + \sin(360°)$
$= 0$

63.
(i) Given equation is $\sin x = \frac{1}{2}$

As we know the period of $\sin x$ is $2\pi$
So, there is two solutions in $[0, 2\pi]$

(ii) Given equation is $\cos x = \dfrac{\sqrt{3}}{2}$

As we know the period of $\cos x$ is $2\pi$
For each $2\pi$. there is 2 solutions
So, it has 3 solutions

(iii) Given equation is $4 \sin^2 x - 1 = 0$

$\Rightarrow \quad \sin^2 x = \dfrac{1}{4}$

The period of $\sin^2 x$ is $\pi$.
For each $\pi$, there is two solutions.
So, it has 6 solutions.

(iv) Given equation is $\sin^2 x - 3 \sin x + 2 = 0$

$\Rightarrow$ $(\sin x - 1)(\sin x - 2) = 0$

$\Rightarrow$ $\sin x = 1, 2$

$\Rightarrow$ $\sin x = 1$

For each $2\pi$, there is 2 solutions.
So, it has 3 solutions.

(v) Given equation is

$\cos^2 x - \cos x - 2 = 0$

$\Rightarrow$ $\cos^2 x - 2 \cos x + \cos x - 2 = 0$

$\Rightarrow$ $\cos x(\cos x - 2) + 1(\cos x - 2) = 0$

$\Rightarrow$ $(\cos x + 1)(\cos x - 2) = 0$

$\Rightarrow$ $\cos x = -1, 2$

$\Rightarrow$ $\cos x = -1$

For each $2\pi$, there are 2 solutions.
So, it has 3 solutions.

64. We have,

(i) $\sin (15°) = \sin (45° - 30°)$

$= \sin (45°) \cos (30°) - \cos (45°) \sin (30°)$

$= \dfrac{1}{2} \cdot \dfrac{\sqrt{3}}{2} - \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{2}}$

$= \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$

(ii) $\cos (15°) = \cos (45° - 30°)$

$= \cos (45°) \cos (30°) + \sin (45°) \sin (30°)$

$= \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2}$

$= \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$

(iii) $\tan (15°) = \tan (45° - 30°)$

$= \dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \cdot \tan 30°}$

$= \dfrac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$

$= \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$

$= \dfrac{(\sqrt{3} - 1)^2}{3 - 1}$

$= \dfrac{3 + 1 - 2\sqrt{3}}{2}$

$= \dfrac{4 - 2\sqrt{3}}{2}$

$= 2 - \sqrt{3}$

> ***Note:***
> (i) $\cot (15°) = \dfrac{1}{\tan (15°)} = \dfrac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}$
> (ii) $\tan (105°) = -\cot (15°) = -(2 + \sqrt{3})$
> (iii) $\cot (105°) = -\tan (15°) = -(2 - \sqrt{3})$
> $= \sqrt{3} - 2$

65. We have $\tan (75°) + \cot (75°)$

$= \cot (15°) + \tan (15°)$

$= (2 + \sqrt{3}) + (2 - \sqrt{3})$

$= 4$

66. We have $\cos (9°) + \sin (9°)$

$= \sqrt{2}\left(\dfrac{1}{\sqrt{2}} \cos (9°) + \dfrac{1}{\sqrt{2}} \sin (9°)\right)$

$= \sqrt{2}\,(\sin (45°) \cos (9°) + \cos (45°) \sin (9°))$

$= \sqrt{2}\,(\sin (45° + 9°))$

$= \sqrt{2} \sin (54°)$

67. We have $\tan (70°) = \tan (50° + 20°)$

$\Rightarrow \quad \tan (70°) = \dfrac{\tan 50° + \tan 20°}{1 - \tan 50° \tan 20°}$

$\Rightarrow$ $\tan (70°) - \tan (70°) \cdot \tan (50°) \cdot \tan (20°)$
$= \tan 50° + \tan 20°$

$\Rightarrow$ $\tan (70°) - \tan (70°) \tan (50°) \cdot \cot (70°)$
$= \tan 50° + \tan 20°$

$\Rightarrow$ $\tan (70°) - \tan (50°) = \tan 50° + \tan 20°$

$\Rightarrow$ $\tan (70°) = 2 \tan 50° + \tan 20°$

68. We have $\dfrac{\cos 20° - \sin 20°}{\cos 20° + \sin 20°}$

$= \dfrac{1 - \tan 20°}{1 + \tan 20°}$

$= \dfrac{\tan 45° - \tan 20°}{1 + \tan 45° \tan 20°}$

$= \tan (45° - 20°)$

$= \tan (25°)$

69. We have $\dfrac{\cos 7° + \sin 7°}{\cos 7° - \sin 7°} = \dfrac{1 + \tan (7°)}{1 - \tan (7°)}$

$= \tan (45° - 7°)$

$= \tan (52°)$

70. We have tan (45°)

$\Rightarrow$ tan (25° + 20°) = 1

$$\Rightarrow \frac{\tan 25° + \tan 20°}{1 - \tan 25° \tan 20°} = 1$$

$\Rightarrow$ tan 25° + tan 20° = 1 – tan 25° tan 20

$\Rightarrow$ tan 25° + tan 20° + tan 25° tan 20° = 1

71. We have $A + B = 45°$

$\Rightarrow$ tan $(A + B)$ = tan (45°)

$\Rightarrow$ tan $(A + B)$ = 1

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = 1$$

$\Rightarrow$ tan $A$ + tan $B$ = 1 = tan $A$ · tan $B$

$\Rightarrow$ tan $A$ + tan $B$ + tan $A$ · tan $B$ = 1

$\Rightarrow$ 1 + tan $A$ + tan $B$ + tan $A$ · tan $B$ = 1 + 1 = 2

$\Rightarrow$ (1 + tan $A$) + tan $B$ (1 + tan $A$ = 2)

$\Rightarrow$ (1 + tan $A$) (1 + tan $B$ = 2

72. We have (1 + tan 245°) (1 + tan 250°)

(1 + tan 260°) (1 – tan 200°)

(1 – tan 205°) (1 – tan 215°)

= {(1 + tan 245°) (1 + tan (–200°))}

{(1 + tan 250°) (1 + tan (–205°))}

{(1 + tan 260°) (1 + tan (–215°))}

$= 2 \times 2 \times 2$

$= 8$

73. Now, tan 13 $A$ = tan (9$A$ + 4$A$)

$$= \frac{\tan 9A + \tan 4A}{1 - \tan 9A \tan 4A}$$

tan 13$A$ – tan 4$A$ tan 9$A$ tran 13$A$

= tan 9$A$ + tan 4$A$

tan 13$A$ – tan 9$A$ – tan 4 $A$

= tan 4$A$ · tan 9A · tan 13A

74. Do yourself.

75. We have $\tan(\alpha + \beta)$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$$

$$= \frac{\frac{m}{m+1} + \frac{1}{2m+1}}{1 - \frac{m}{m+1} \cdot \frac{1}{2m+1}}$$

$$= \frac{\frac{2m^2 + m + m + 1}{(m+1)(2m+1)}}{\frac{2m^2 + 3m + 1 - m}{(m+1)(2m+1)}}$$

$$= \frac{2m^2 + 2m + 1}{2m^2 + 2m + 1} = 1$$

$\tan(\alpha + \beta) = 1$

$$(\alpha + \beta) = \frac{\pi}{4}$$

76. We have

$\sin^2 A + \sin^2 (A - B) - 2\sin A \cos B \sin (A - B)$

$= \sin^2 A + \sin (A - B) (\sin (A - B) - 2 \sin A \cos B)$

$= \sin^2 A - \sin (A - B) \sin (A + B)$

$= \sin^2 A - (\sin^2 A - \sin^2 B)$

$= \sin^2 B$

77. We have

$\cos 2x \cos 2y + \cos^2 (x + y) - \cos^2 (x - y)$

$$= \frac{1}{2}[2\cos 2x \cos 2y + 2\cos^2(x+y) - 2\cos^2(x-y)]$$

$$\frac{1}{2}[\cos(2x+2y) + \cos(2x-2y) + 1$$

$$+ \cos(2x+2y) - 1 - \cos(2x-2y)]$$

$$= \frac{1}{2}[\cos(2x+2y) + \cos(2x+2y)]$$

$$= \frac{1}{2}[2\cos(2x+2y)]$$

$= \cos(2x + 2y)$

78. Given, $\sin\theta = \frac{x - y}{x + y}$

Applying componendo and dividendo, we get,

$$\frac{\sin\theta + 1}{\sin\theta - 1} = \frac{x - y + x + y}{x - y - x - y}$$

$$\Rightarrow \frac{\sin\theta + 1}{\sin\theta - 1} = -\frac{x}{y}$$

$$\Rightarrow \frac{1 + \sin\theta}{1 - \sin\theta} = \frac{x}{y}$$

$$\Rightarrow \left(\frac{\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}\right)^2 = \frac{x}{y}$$

$$\Rightarrow \left(\frac{1 + \tan\left(\frac{\theta}{2}\right)}{1 - \tan\left(\frac{\theta}{2}\right)}\right)^2 = \frac{x}{y}$$

$$\Rightarrow \left(\frac{1 + \tan\left(\frac{\theta}{2}\right)}{1 - \tan\left(\frac{\theta}{2}\right)}\right) = \pm\sqrt{\frac{x}{y}}$$

$$\Rightarrow \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \pm\sqrt{\frac{x}{y}}$$

79. If $\tan\alpha = \frac{Q\sin\beta}{P + Q\cos\beta}$,

prove that $\tan(\beta - \alpha) = \frac{P\sin\alpha}{Q + P\cos\alpha}$

80. We have

$$\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=-\frac{3}{2}$$

$\Rightarrow$ $2(\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha))+3=0$

$\Rightarrow$ $2(\cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha)$
$+2(\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha)+3=0$

$\Rightarrow$ $(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)$
$+(\cos^2\gamma+\sin^2\gamma)+$
$2(\cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha)$
$+2(\sin a\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha)=0$

$\Rightarrow$ $\cos^2\alpha+\cos^2\beta+\cos^2\gamma$
$+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma$
$+2\cos\gamma\cos\alpha+\sin^2\alpha+\sin^2\beta+\sin^2\gamma$
$+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma$
$+2\sin\gamma\sin\alpha=0$

$\Rightarrow$ $(\cos\alpha+\cos\beta+\cos\gamma)^2$
$+(\sin\alpha+\sin\beta+\sin\gamma)^2=0$

$\Rightarrow$ $(\cos\alpha+\cos\beta+\cos\gamma)=0$

and $(\sin\alpha+\sin\beta+\sin\gamma)=0$

81. We have $\tan(\alpha+\theta)=n\tan(\alpha-\theta)$

$$\Rightarrow \frac{\tan(\alpha+\theta)}{\tan(\alpha-\theta)}=\frac{n}{1}$$

$$\Rightarrow \frac{\tan(\alpha+\theta)+\tan(\alpha-\theta)}{\tan(\alpha+\theta)-\tan(\alpha-\theta)}=\frac{n+1}{n-1}$$

$$\Rightarrow \frac{\sin(\alpha+\theta+\alpha-\theta)}{\sin(\alpha+\theta-\alpha+\theta)}=\frac{n+1}{n-1}$$

$$\Rightarrow \frac{\sin(2\alpha)}{\sin(2\theta)}=\frac{n+1}{n-1}$$

$$\Rightarrow \frac{\sin(2\theta)}{\sin(2\alpha)}=\frac{n-1}{n+1}$$

Hence, the result.

82. Given $\sin\alpha+\sin\beta=a$ …(i)

and $\cos\alpha+\cos\beta=b$ …(ii)

(i) divides by (ii), we get,

$$\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=\frac{a}{b}$$

$$\Rightarrow \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a}{b}$$

$$\Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right)=\frac{a}{b}$$

(i) Now, $\cos(\alpha+\beta)$

$$=\frac{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)}$$

$$=\frac{1-\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}}=\frac{b^2-a^2}{a^2+b^2}$$

(ii) $\sin(\alpha+\beta)$

$$=\frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)}$$

$$=\frac{2(b/a)}{1+(b/a)^2}=\frac{2ab}{a^2+b^2}$$

83. Given equation is $a\cos\theta+b\sin\theta=c$

$$a\left(\frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}\right)+b\left(\frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}\right)=c$$

$$a(1-\tan^2(\theta/2))+2b\tan(\theta/2)=c(1+\tan^2(\theta/2))$$

$$(a+c)\tan^2(\theta/2)-2b\tan(\theta/2)+(c-a)=0$$

Let its roots be $\tan\left(\frac{\alpha}{2}\right), \tan\left(\frac{\beta}{2}\right)$

Thus, $\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=\frac{2b}{a+c}$

$$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)=\frac{c-a}{a+c}$$

$$\tan\left(\frac{\alpha+\beta}{2}\right)=\frac{\tan(\alpha/2)+\tan(\beta/2)}{1-\tan(\alpha/2)\tan(\beta/2)}$$

$$=\frac{\frac{2b}{a+c}}{1-\frac{c-a}{c+a}}=\frac{2b}{2a}=\frac{b}{a}$$

(i) Now, $\cos(\alpha+\beta)$

$$=\frac{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)}$$

$$=\frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}}=\frac{a^2-b^2}{a^2+b^2}$$

(ii) Now, $\tan\left(\frac{\alpha-\beta}{2}\right)$

$$= \frac{\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right)}{1 + \tan\left(\frac{\alpha}{2}\right) \cdot \tan\left(\frac{\beta}{2}\right)}$$

$$= \frac{\sqrt{\left(\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)\right)^2 - 4\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}}{1 + \tan\left(\frac{\alpha}{2}\right) \cdot \tan\left(\frac{\beta}{2}\right)}$$

$$= \frac{\sqrt{\left(\frac{2b}{a+c}\right)^2 - 4\left(\frac{c-a}{c+a}\right)}}{1 + \frac{c-a}{c+a}}$$

$$= \frac{\sqrt{4b^2 - 4(c^2 - a^2)}}{2c}$$

$$= \frac{\sqrt{a^2 + b^2 - c^2}}{c}$$

Now,

$$\cos(\alpha - \beta) = \frac{1 - \tan^2\left(\frac{\alpha - \beta}{2}\right)}{1 + \tan^2\left(\frac{\alpha + \beta}{2}\right)}$$

$$= \frac{1 - \left(\frac{a^2 + b^2 - c^2}{c^2}\right)}{1 + \left(\frac{a^2 + b^2 - c^2}{c^2}\right)}$$

$$= \frac{2c^2 - (a^2 + b^2)}{a^2 + b^2}$$

84. Given, $a \tan\theta + b \sec\theta = c$

$\Rightarrow \quad (a \tan\theta - c)^2 = (-b \sec\theta)^2$

$\Rightarrow \quad a^2 \tan^2\theta - 2ac \tan\theta + c^2 = b^2 \sec^2\theta$

$\Rightarrow \quad a^2 \tan^2\theta - 2ac \tan\theta + c^2 = b^2 + b^2 \tan^2\theta$

$\Rightarrow \quad (a^2 - b^2)\tan^2\theta - 2ac \tan\theta + (c^2 + b^2) = 0$

Let its roots be $\tan\alpha$, $\tan\beta$

So, $\quad \tan\alpha + \tan\beta = \frac{2ac}{a^2 - b^2}$

and $\quad \tan\alpha \cdot \tan\beta = \frac{c^2 - b^2}{a^2 - b^2}$

Now, $\tan(\alpha + \beta)$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\frac{2ac}{a^2 - b^2}}{1 - \frac{c^2 - b^2}{a^2 - b^2}}$$

$$= \frac{2ac}{a^2 - b^2 - c^2 + b^2} = \frac{2ac}{a^2 - c^2}$$

85. Given, $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$

$\Rightarrow \quad \tan(\pi\cos\theta) = \tan\left(\pm\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow \quad (\pi\cos\theta) = \left(\pm\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow \quad \cos\theta + \sin\theta = \pm\frac{1}{2}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \pm\frac{1}{2\sqrt{2}}$

$\Rightarrow \quad \cos\left(\frac{\pi}{4} - \theta\right) = \pm\frac{1}{2\sqrt{2}}$

86. We have $\tan\theta = \frac{x\sin\varphi}{1 - x\cos\varphi}$

$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{x\sin\varphi}{1 - x\cos\varphi}$

$\Rightarrow \quad \sin\theta - x\sin\theta\cos\varphi = x\cos\theta\sin\varphi$

$\Rightarrow \quad x\sin(\theta + \varphi) = \sin\theta$

$\Rightarrow \quad x = \frac{\sin\theta}{\sin(\theta + \varphi)}$

Similarly, $y = \frac{\sin\varphi}{\sin(\theta + \varphi)}$

Dividing the above relations, we get,

$$\frac{x}{y} = \frac{\sin\theta}{\sin\varphi}$$

$\Rightarrow \quad x\sin\varphi = y\sin\theta$

Hence, the result.

87. We have $\tan(\alpha + \beta) = c$

$\Rightarrow \quad \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = c$

$\Rightarrow \quad \frac{a}{1 - \tan\alpha\tan\beta} = c$

$\Rightarrow \quad \tan\alpha\tan\beta = \frac{a}{c} - 1 = \frac{a - c}{c} \quad$ …(i)

Now, $\cot\alpha + \cot\beta = b$

$\Rightarrow \quad \frac{1}{\tan\alpha} + \frac{1}{\tan\beta} = b$

$\Rightarrow \quad \frac{\tan\alpha + \tan\beta}{\tan\alpha \cdot \tan\beta} = b$

$\Rightarrow \quad \tan\alpha \cdot \tan\beta = \frac{a}{b} \quad$ …(ii)

From (i) and (ii), we get, $\frac{a - c}{c} = \frac{a}{b}$

$\Rightarrow \quad ac + bc = ab$

Which is the required relation.

88. We have,

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$$

$$=\frac{\dfrac{\sin\alpha}{\cos\alpha}-\left(\dfrac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}\right)}{1+\dfrac{\sin\alpha}{\cos\alpha}\times\left(\dfrac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}\right)}$$

$$=\frac{\sin\alpha(1-n\sin^2\alpha)-n\sin\alpha\cos^2\alpha}{\cos\alpha(1-n\sin^2\alpha)+n\sin^2\alpha\cdot\cos\alpha}$$

$$=\frac{\sin\alpha-n\sin\alpha(\sin^2\alpha+\cos^2\alpha)}{\cos\alpha-n\sin^2\alpha\cdot\cos\alpha+n\sin^2\alpha\cdot\cos\alpha}$$

$$=\frac{\sin\alpha-n\sin\alpha}{\cos\alpha}$$

$$=\frac{(1-n)\sin\alpha}{\cos\alpha}$$

$$=(1-n)\tan\alpha$$

89. Let $A=x+y-z$, $B=y+z-x$, $C=x+y-z$

Then, $A+B+C=(x+y-z)+(y+z-x)+(x+y-z)$
$=(x+y+z)=0$

$\Rightarrow\quad A+B=-C$

$\Rightarrow\quad \cot(A+B)=\cot(-C)$

$$\Rightarrow\quad \frac{\cot A\cot B-1}{\cot A+\cot B}=\cot(-C)$$

$$\Rightarrow\quad \frac{\cot A\cot B-1}{\cot A+\cot B}=-\cot C$$

$\Rightarrow\quad \cot A\cot B-1=-\cot A\cot C-\cot B\cot C$

$\Rightarrow\quad \cot A\cot B+\cot A\cot C+\cot B\cot C=1$

$\Rightarrow\quad \cot(x+y-z)\cot(y+z-x)$
$+\cot(y+z-x)\cot(zx+x-y)$
$+\cot(z+x-y)\cot(x+y-z)=1$

90. We have $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$

$$=\frac{\frac{3}{2}\tan\beta-\tan\beta}{1+\frac{3}{2}\tan\beta\tan\beta}$$

$$=\frac{\tan\beta}{2+3\tan^2\beta}$$

$$=\frac{\dfrac{\sin\beta}{\cos\beta}}{2+3\dfrac{\sin^2\beta}{\cos^2\beta}}$$

$$=\frac{\sin\beta\cos\beta}{2\cos^2\beta+3\sin^2\beta}$$

$$=\frac{\sin\beta\cos\beta}{2+\sin^2\beta}$$

$$=\frac{2\sin\beta\cos\beta}{4+2\sin^2\beta}$$

$$=\frac{\sin 2\beta}{4+1-\cos 2\beta}$$

$$=\frac{\sin 2\beta}{5-\cos 2\beta}$$

91. We have

$$\frac{\sin 5A-\sin 3A}{\cos 5A+\cos 3A}=\frac{2\cos 4A\sin A}{2\cos 4A\cos A}$$
$$=\tan A$$

92. We have

$$\frac{\sin A+\sin 3A}{\cos A+\cos 3A}=\frac{2\sin 2A\cos A}{2\cos 2A\cos A}$$
$$=\tan 2A$$

93. We have

$$\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)}{2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)}$$

$$=\tan\left(\frac{A+B}{2}\right)$$

94. We have $\sin 38°+\sin 22°$

$=2\sin(30°)\cos(8°)$

$=2\times\dfrac{1}{2}\times\cos(90°-82°)$

$=\sin(82°)$

95. We have $\sin 105°+\cos 105°$

$=\sin(105°)-\sin(15°)$

$=2\cos(60°)\sin(45°)$

$=2\times\dfrac{1}{2}\times\sin(45°)$

$=\sin(45°)$

$=\cos(45°)$

96. We have $\cos 55°+\cos 65°+\cos 175°$

$=\cos(65°)+\cos(55°)-\cos(5°)$

$=2\cos(60°)\cos(5°)-\cos(5°)$

$=2\times\dfrac{1}{2}\times\cos(5°)-\cos(5°)$

$=\cos(5°)-\cos(5°)$

$=0$

97. We have $\cos 20°+\cos 100°+\cos 140°$

$=\cos(100°)+\cos(20°)-\text{co}(40°)$

$=2\cos(60°)\cos(40°)-\cos(40°)$

$=2\times\dfrac{1}{2}\times\cos(40°)-\cos(40°)$

$=\cos(40°)-\cos(40°)$

$=0.$

98. We have

$\sin 50° - \sin 70 + \sin 10°$

$= \sin(50°) + \sin(10°) - \sin(70°)$

$= 2\sin(30°)\cos(20°) - \sin(70°)$

$= 2 \times \frac{1}{2} \times \cos(20°) - \sin(90° - 20°)$

$= \cos(20°) - \cos(20°)$

$= 0.$

99. We have

$\sin(47°) + \cos(77°) = \cos(17°)$

$= \sin(47°) + \sin(13°)$

$= 2\sin\left(\frac{47° + 13°}{2}\right)\cos\left(\frac{47° - 13°}{2}\right)$

$= 2\sin(30°)\cos(17°)$

$= 2 \times \frac{1}{2} \times \cos(17°)$

$= \cos(17°)$

100. We have

$\cos(80°) + \cos(40°) - \cos(20°)$

$= 2\cos\left(\frac{80° + 40°}{2}\right)\cos\left(\frac{80° - 40°}{2}\right) - \cos(20°)$

$= 2\cos(60°)\cos(20°) - \cos(20°)$

$= 2 \times \frac{1}{2} \times \cos(20°) - \cos(20°)$

$= \cos(20°) - \cos(20°)$

$= 0.$

101. We have

$\sin(10°) + \sin(20°) + \sin(40°)$
$+ \sin(50°) - \sin(70°) - \sin(80°)$

$= \{\sin(50°) + \sin(10°)\} + \{\sin(40°) + \sin(20°)\}$
$- \sin(70°) - \sin(80°)$

$= 2\sin\left(\frac{50° + 10°}{2}\right)\cos\left(\frac{50° - 10°}{2}\right)$
$+ 2\sin\left(\frac{40° + 20°}{2}\right)\cos\left(\frac{40° - 20°}{2}\right)$
$- \sin(70°) - \sin(80°)$

$= 2\sin(30°)\cos(20°) + 2\sin(30°)\cos(10°)$
$- \sin(70°) - \sin(80°)$

$= \cos(20°) + \cos(10°) - \sin(70°) - \sin(80°)$

$= \cos(20°) + \cos(10°) - \cos(20°) - \cos(10°)$

$= 0$

102. We have

$(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$= 2 + 2\cos(\alpha - \beta)$

$= 2 + (1 + \cos(\alpha - \beta))$

$= 2 \times 2\cos^2\left(\frac{\alpha - \beta}{2}\right)$

$= 4\cos^2\left(\frac{\alpha - \beta}{2}\right)$

103. We have $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$= 2 - 2\cos(\alpha - \beta)$

$= 2(1 - \cos(\alpha - \beta))$

$= 2 \times 2\sin^2\left(\frac{\alpha - \beta}{2}\right)$

$= 4\sin^2\left(\frac{\alpha - \beta}{2}\right)$

104. Do yourself.

105. We have $\cos(20°)\cos(40°)\cos(80°)$

$= \cos(40°)\cos(20°)\cos(80°)$

$= \frac{1}{4}[4\cos(60° - 20°)\cos(20°)\cos(60° + 20°)]$

$= \frac{1}{4} \times \cos(20° \times 3)$

$= \frac{1}{4} \times \cos(60°)$

$= \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

106. We have $\cos 25° \cos 35° \cos 65°$

$= \cos 25° \cos 35° \cos 65°$

$= \frac{1}{4}[4\cos(60° - 25)\cos 25° \cos(60° + 25)]$

$= \frac{1}{4} \times \cos(25° \times 3)$

$= \frac{1}{4} \times \cos(75°)$

$= \frac{1}{4} \times \sin(15°)$

$= \frac{\sqrt{3} - 1}{8\sqrt{2}}$

107. We have $\sin(20°)\sin(40°)\sin(80°)$

$= \sin(20°)\sin(40°)\sin(80°)$

$= \frac{1}{4}[4\sin(60° - 20°)\sin(20°)\sin(60° + 20°)]$

$= \frac{1}{4} \times \sin(60°)$

$= \frac{\sqrt{3}}{8}$

108. We have sin (10°) sin (50°) sin (60°) sin (70°)

$$= \frac{\sqrt{3}}{2}[\sin(50°)\sin(10°)\sin(70°)]$$

$$= \frac{\sqrt{3}}{2}[\sin(60° - 10°)\sin(10°)\sin(60° + 10°)]$$

$$= \frac{\sqrt{3}}{8}[4\sin(60° - 10°)\sin(10°)\sin(60° + 10°)]$$

$$= \frac{\sqrt{3}}{8} \times \sin(30°)$$

$$= \frac{\sqrt{3}}{8} \times \frac{1}{2} = \frac{\sqrt{3}}{16}$$

109. We have cos (10°) cos (30°) cos (50°) cos (70°)

$$= \frac{\sqrt{3}}{2}[\cos(50°)\cos(10°)\cos(70°)]$$

$$= \frac{\sqrt{3}}{2}[\cos(60° - 10°)\cos(10°)\cos(60° + 10°)]$$

$$= \frac{\sqrt{3}}{8}[4\cos(60° - 10°)\cos(10°)\cos(60° + 10°)]$$

$$= \frac{\sqrt{3}}{8} \times \cos(30°)$$

$$= \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2}$$

$$= \frac{3}{16}$$

110. We have $\dfrac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A}$

$$= \frac{\sin 5A + \sin 3A + \sin A}{\cos 5A + \cos 3A + \cos A}$$

$$= \frac{(\sin 5A + \sin A) + \sin 3A}{(\cos 5A + \cos A) + \cos 3A}$$

$$= \frac{2\sin 3A\cos 2A + \sin 3A}{2\cos 3A\cos 2A + \cos 3A}$$

$$= \frac{\sin 3A(2\cos 2A + 1)}{\cos 3A(2\cos 2A + 1)}$$

$$= \tan 3A$$

111. We have $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x}$

$$= \frac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x}$$

$$= \frac{2\cos 3x\cos x + \cos 3x}{2\sin 3x\cos x + \sin 3x}$$

$$= \frac{\cos 3x(2\cos x + 1)}{\sin 3x(2\cos x + 1)}$$

$$= \cot 3x$$

112. We have $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A}$

$$= \frac{(\sin 7A + \sin A) + (\sin 5A + \sin 3A)}{(\cos 7A + \cos A) + (\cos 5A + \cos 3A)}$$

$$= \frac{2\sin 4A\cos 3A + 2\sin 4A\cos A}{2\cos 4A\cos 3A + 2\cos 4A\cos A}$$

$$= \frac{2\sin 4A(\cos 3A + \cos A)}{2\cos 4A(\cos 3A + \cos A)}$$

$$= \tan 4A$$

113. We have $\dfrac{\sin A + \sin 2A + \sin 4A + \sin 5A}{\cos A + \cos 2A + \cos 4A + \cos 5A}$

$$= \frac{(\sin 5A + \sin A) + (\sin 4A + \sin 2A)}{(\cos 5A + \cos A) + (\cos 4A + \cos 2A)}$$

$$= \frac{2\sin 3A\cos 2A + 2\sin 3A\cos A}{2\cos 3A\cos 2A + 2\cos 3A\cos A}$$

$$= \frac{\sin 3A(\cos 2A + \cos A)}{\cos 3A(\cos 2A + \cos A)}$$

$$= \frac{\sin 3A}{\cos 3A}$$

$$= \tan 3A$$

Hence, the result.

114. Given $\sin A - \sin B = \dfrac{1}{2}$ …(i)

and $\cos A - \cos B = \dfrac{1}{3}$ …(ii)

Dividing (i) by (ii) we get,

$$\frac{\sin A - \sin B}{\cos A - \cos B} = \frac{1/2}{1/3} = \frac{3}{2}$$

$$\Rightarrow \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)} = \frac{3}{2}$$

$$\Rightarrow \frac{\cos\left(\frac{A+B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)} = -\frac{3}{2}$$

$$\Rightarrow \cot\left(\frac{A+B}{2}\right) = -\frac{3}{2}$$

$$\Rightarrow \tan\left(\frac{A+B}{2}\right) = -\frac{2}{3}.$$

115. Given, $\sin A + \sin B = \dfrac{1}{4}$,

and $\cos A + \cos B = \dfrac{1}{2}$

Dividing, the above relations, we get,

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{1}{2}$$

$$\Rightarrow \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)} = \frac{1}{2}$$

$$\Rightarrow \tan\left(\frac{A+B}{2}\right) = \frac{1}{2}$$

Hence, the result.

116. Given,

$$\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$$

$$\Rightarrow \operatorname{cosec} A + \operatorname{cosec} B = \sec B + \sec A$$

$$\Rightarrow \frac{1}{\sin A} - \frac{1}{\sin B} = \frac{1}{\cos B} - \frac{1}{\cos A}$$

$$\Rightarrow \frac{\sin B - \sin A}{\sin A \sin B} = \frac{\cos A - \cos B}{\cos A \cos B}$$

$$\Rightarrow \frac{\sin A \sin B}{\cos A \cos B} = \frac{\sin B - \sin A}{\cos A - \cos B}$$

$$\Rightarrow \tan A \tan B = \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$\Rightarrow \tan A \tan B = \cot\left(\frac{A+B}{2}\right)$$

Hence, the result.

117. Given,

$$\sin 2A = \lambda \sin 2B$$

$$\Rightarrow \frac{\sin 2A}{\sin 2B} = \frac{\lambda}{1}$$

$$\Rightarrow \frac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \frac{\lambda + 1}{\lambda - 1}$$

$$\Rightarrow \frac{2\sin(A+B)\cos(A-B)}{2\cos(A+B)\sin(A-B)} = \frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \tan(A+B)\cot(A-B) = \frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$$

118. We have

$$\sqrt{3}\cot(20°) - 4\cos(20°)$$

$$= \frac{\sqrt{3}\cos(20°) - 4\sin(20°)\cos(20°)}{\sin(20°)}$$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos(20°) - 2\sin(20°)\cos(20°)\right)}{\sin(20°)}$$

$$= \frac{2(\sin(60°)\cos(20°) - \sin(40°))}{\sin(20°)}$$

$$= \frac{(2\sin(60°)\cos(20°) - 2\sin(40°))}{\sin(20°)}$$

$$= \frac{(\sin(80°) + \sin(40°) - 2\sin(40°))}{\sin(20°)}$$

$$= \frac{(\sin(80°) - \sin(40°))}{\sin(20°)}$$

$$= \frac{2\cos(60°)\sin(20°)}{\sin(20°)}$$

$$= 1$$

119. Given,

$\sin\alpha + \sin\beta = a$ …(i)

and $\cos\alpha + \cos\beta = b$ …(ii)

(i) divides by (ii), we get,

$$\frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \frac{a}{b}$$

$$\Rightarrow \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)} = \frac{a}{b}$$

$$\Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{a}{b}$$

Now,

$$\cos(\alpha+\beta) = \frac{1 - \tan^2\left(\frac{\alpha+\beta}{2}\right)}{1 + \tan^2\left(\frac{\alpha+\beta}{2}\right)}$$

$$= \frac{1 - \frac{a^2}{b^2}}{1 + \frac{a^2}{b^2}} = \frac{b^2 - a^2}{a^2 + b^2}$$

120. We have

$$\sin A = \sqrt{1 - \frac{1}{4}\left(x + \frac{1}{x}\right)^2}$$

$$= i\sqrt{\frac{1}{4}\left(x + \frac{1}{x}\right)^2 - 1}$$

$$= i\sqrt{\frac{1}{4}\left(x - \frac{1}{x}\right)^2}$$

$$= \frac{i}{2}\left(x - \frac{1}{x}\right)$$

Similarly, $\sin B = \frac{i}{2}\left(y - \frac{1}{y}\right)$

Now, $\cos(A - B)$

$= \cos A \cos B + \sin A \sin B$

$= \frac{1}{4}\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right) + \frac{1}{4}\left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right)$

$= \frac{1}{4}\left(2xy + \frac{2}{xy}\right)$

$= \frac{1}{2}\left(xy + \frac{1}{xy}\right)$

121. We have

$\sin(47°) + \sin(61°) - \sin(11°) - \sin(25°)$

$= (\sin(61°) + \sin(47°)) - (\sin(25°) + \sin(11°))$

$= 2\sin(54°)\cos(7°) - 2\sin(18°)\cos(7°)$

$= 2\cos(7°)[\sin(54°) - \sin(18°)]$

$= 2\cos(7°)[\cos(36°) - \sin(18°)]$

$= 2\cos(7°)\left[\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right]$

$= 2\cos(7°) \times \frac{1}{2}$

$= \cos(7°)$

122. Do yourself.

123. We have

$-\sqrt{7^2 + 5^2} \le 2k + 1 \le \sqrt{7^2 + 5^2}$

$-\sqrt{74} \le 2k + 1 \le \sqrt{74}$

$\frac{-\sqrt{74} - 1}{2} \le k \le \frac{\sqrt{74} - 1}{2}$

$k = -4, -3, -2, -1, 0, 1, 2, 3$

Hence, the number of integral values of

124. We have

$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma)$

$= (\cos\alpha + \cos\beta) + (\cos(\alpha + \beta + \gamma) + \cos\gamma)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$+ 2\cos\left(\frac{\alpha+\beta+\gamma+\gamma}{2}\right)\cos\left(\frac{\alpha+\beta+\gamma-\gamma}{2}\right)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$+ 2\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\times$

$\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\}$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\times\cos\left(\frac{\frac{\alpha-\beta}{2} + \frac{\alpha+\beta+2\gamma}{2}}{2}\right)$

$\cos\left(\frac{\frac{\alpha-\beta}{2} - \frac{\alpha+\beta+2\gamma}{2}}{2}\right)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\times\left\{\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\}$

125. We have

$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$

$= \frac{\sin\theta}{\cos\theta} = \tan\theta$

126. We have

$\frac{1+\cos 2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{2\sin\theta\cos\theta}$

$= \frac{\cos\theta}{\sin\theta} = \cot\theta$

127. We have $(\cot\theta - \tan\theta)$

$= \left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right)$

$= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$

$= \frac{2\cos 2\theta}{2\sin\theta\cos\theta}$

$= \frac{2\cos 2\theta}{\sin 2\theta}$

$= 2\cot 2\theta$

128. We have

$\tan\theta + 2\tan(2\theta) + 4\tan(4\theta) + 8\cot 8\theta$

$= \cot\theta(\cot\theta - \tan\theta) + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta$

$= \cot\theta - 2\cot 2\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta$

$= \cot\theta - 2(\cot 2\theta - \tan 2\theta) + 4\tan 4\theta + 8\cot 8\theta$

$= \cot\theta - 4\cot 4\theta + 4\tan 4\theta + 8\cot 8\theta$

$= \cot\theta - 4(\cot 4\theta \tan 4\theta) + 8\cot 8\theta$

$= \cot\theta - 8\cot 8\theta + 8\cot 8\theta$

$= \cot\theta$

129. We have

$a\cos(2\theta) + b\sin(2\theta)$

$= a\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + b\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= a\left(\frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}}\right) + b\left(\frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}}\right)$

$$= a\left(\frac{a^2 - b^2}{a^2 + b^2}\right) + b\left(\frac{2ab}{a^2 + b^2}\right)$$

$$= \frac{a(a^2 - b^2 + 2b^2)}{a^2 + b^2}$$

$$= \frac{a(a^2 + b^2)}{a^2 + b^2}$$

$$= a$$

130. We have

$\sqrt{3}$ cosec (20°) – sec (20°)

$$= \frac{\sqrt{3}}{\sin(20°)} - \frac{1}{\cos(20°)}$$

$$= \frac{\sqrt{3}\cos(20°) - \sin(20°)}{\sin(20°)\cos(20°)}$$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos(20°) - \frac{1}{2}\sin(20°)\right)}{2\sin(20°)\cos(20°)}$$

$$= \frac{4(\sin(20°)\cos(20°) - \cos(60°)\sin(20°))}{2\sin(20°)\cos(20°)}$$

$$= \frac{4(\sin(60° - 20°))}{\sin(40°)}$$

$$= 4$$

131. We have

tan (9°) + tan (27°) – tan (63°) + tan (81°)

= {tan (9°) + tan (81°)} – {tan (27°) + tan (63°)}

= {tan (9°) + cot (9°)} – {tan (27°) + cot (27°)}

$$= \left\{\frac{\sin(9°)}{\cos(9°)} + \frac{\cos(9°)}{\sin(9°)}\right\} - \left\{\frac{\sin(27°)}{\cos(27°)} + \frac{\cos(27°)}{\sin(27°)}\right\}$$

$$= \left\{\frac{\sin^2(9°) + \cos^2(9°)}{\sin(9°)\cos(9°)}\right\} - \left\{\frac{\sin^2(27°) + \cos^2(27°)}{\sin(27°)\cos(27°)}\right\}$$

$$= \left\{\frac{2}{2\sin(9°)\cos(9°)}\right\} - \left\{\frac{2}{2\sin(27°)\cos(27°)}\right\}$$

$$= \left\{\frac{2}{\sin(18°)}\right\} - \left\{\frac{2}{\sin(54°)}\right\}$$

$$= \left\{\frac{2}{\left(\frac{\sqrt{5} - 1}{4}\right)}\right\} - \left\{\frac{2}{\frac{\sqrt{5} + 1}{4}}\right\}$$

$$= \left\{\frac{8}{\sqrt{5} - 1}\right\} - \left\{\frac{8}{\sqrt{5} + 1}\right\}$$

$$= \left\{\frac{8(\sqrt{5} + 1 - \sqrt{5} + 1)}{5 - 1}\right\}$$

$$= 4$$

132. We have $\left(\frac{\sec 8A - 1}{\sec 4A - 1}\right)$

$$= \frac{\frac{1}{\cos 8A} - 1}{\frac{1}{\cos 4A} - 1}$$

$$= \frac{1 - \cos 8A}{1 - \cos 4A} \times \frac{\cos 4A}{\cos 8A}$$

$$= \frac{2\sin^2 4A}{2\sin^2 2A} \times \frac{\cos 4A}{\cos 8A}$$

$$= \frac{2\sin 4A\cos 4A}{\cos 8A} \times \frac{\sin 4A}{2\sin^2 2A}$$

$$= \frac{\sin 8A}{\cos 8A} \times \frac{2\sin 2A\cos 2A}{2\sin^2 2A}$$

$$= \frac{\sin 8A}{\cos 8A} \times \frac{\cos 2A}{\sin 2A}$$

$$= \tan 8A \times \cot 2A$$

$$= \frac{\tan 8A}{\tan 2A}$$

133. We have

$$\cos^2(\theta) + \cos^2\left(\frac{2\pi}{3} - \theta\right) + \cos^2\left(\frac{2\pi}{3} + \theta\right)$$

$$= \frac{1}{2}\left(2\cos^2(\theta) + 2\cos^2\left(\frac{2\pi}{3} - \theta\right) + 2\cos^2\left(\frac{2\pi}{3} + \theta\right)\right)$$

$$= \frac{1}{2}(1 + \cos(2\theta)) + \frac{1}{2}\left(1 + \cos\left(\frac{4\pi}{3} - 2\theta\right)\right)$$

$$+ \frac{1}{2}\left(1 + \cos\left(\frac{4\pi}{3} + 2\theta\right)\right)$$

$$= \frac{1}{2}\left(3 + \left(\cos 2\theta + \cos\left(\frac{4\pi}{3} - 2\theta\right) + \cos\left(\frac{4\pi}{3} + 2\theta\right)\right)\right)$$

$$= \frac{1}{2}\left(3 + \left(\cos 2\theta + 2\cos\left(\frac{4\pi}{3}\right)\cos(2\theta)\right)\right)$$

$$= \frac{1}{2}\left(3 + \left(\cos 2\theta + 2\left(-\frac{1}{2}\right)\cos(2\theta)\right)\right)$$

$$= \frac{3}{2}$$

134. We have

$\sin^2\theta + \sin^2(120° + \theta) + \sin^2(240° + \theta)$

$$= \frac{1}{2}\left(2\sin^2\theta + 2\sin^2\left(\frac{2\pi}{3} + \theta\right)\right)$$

$$+ \frac{1}{2}\left(2\sin^2\left(\frac{4\pi}{3} + \theta\right)\right)$$

$$= \frac{1}{2}\left((1 - \cos 2\theta) + \left(1 - \cos\left(\frac{4\pi}{3} + 2\theta\right)\right)\right)$$

$$+ \frac{1}{2}\left(\left(1 - \cos\left(\frac{8\pi}{3} + 2\theta\right)\right)\right)$$

$$= \frac{3}{2} - \frac{1}{2}\cos 2\theta - \frac{1}{2}\left[\cos\left(\frac{4\pi}{3} + 2\theta\right) + \cos\left(\frac{8\pi}{3} + 2\theta\right)\right]$$

$$= \frac{3}{2} - \frac{1}{2}\cos 2\theta - \frac{1}{2}(2\cos(120°)\cos 2\theta)$$

$$= \frac{3}{2} - \frac{1}{2}\cos 2\theta + \frac{1}{2}\cos(2\theta)$$

$$= \frac{3}{2}$$

135. We have

$$4\sin(\theta)\sin\left(\frac{\pi}{3} - \theta\right)\sin\left(\frac{\pi}{3} + \theta\right)$$

$$= 4\sin(\theta) \times \left(\sin\left(\frac{\pi}{3} - \theta\right)\sin\left(\frac{\pi}{3} + \theta\right)\right)$$

$$= 4\sin(\theta) \times \left(\sin^2\left(\frac{\pi}{3}\right) - \sin^2\theta\right)$$

$$= 4\sin(\theta) \times \left(\frac{3}{4} - \sin^2\theta\right)$$

$$= 4 \times \left(\frac{3}{4}\sin\theta - \sin^3\theta\right)$$

$$= (3\sin\theta - 4\sin^3\theta)$$

$$= \sin(3\theta)$$

136. We have

$$\sin(20°)\sin(40°)\sin(80°)$$

$$= \frac{1}{4}(\sin(3.20°))$$

$$= \frac{1}{4}(\sin(60°))$$

$$= \frac{\sqrt{3}}{8}$$

137. We have,

$$4\cos(\theta)\cdot\cos(60° - \theta)\cdot\cos(60° + \theta)$$

$$= 4\cos(\theta)\cdot(\cos(60° - \theta)\cdot\cos(60° + \theta))$$

$$= 4\cos(\theta)\cdot(\cos^2(60°) - \sin^2\theta)$$

$$= 4\cos(\theta)\cdot\left(\frac{1}{4} - 1 + \cos^2\theta\right)$$

$$= 4\cos(\theta)\cdot\left(-\frac{3}{4} + \cos^2\theta\right)$$

$$= \cos(\theta)\cdot(-3 + 4\cos^2\theta)$$

$$= (4\cos^3\theta - 3\cos(\theta))$$

$$= \cos(3\theta)$$

138. We have

$$\cos(10°)\cdot\cos(50°)\cdot\cos(70°)$$

$$= \cos(10°)\cdot\cos(60° - 10°)\cdot\cos(60° + 10°)$$

$$= \frac{1}{4}(4\cos(10°)\cdot\cos(60° - 10°)\cdot\cos(60° + 10°))$$

$$= \frac{1}{4}(\cos(3\cdot 10°))$$

$$= \frac{\sqrt{3}}{8}$$

139. We have

$$\tan(\theta) - \tan(60° - \theta) + \tan(60° + \theta)$$

$$= \tan(\theta) - \frac{\tan(60°) - \tan(\theta)}{1 + \tan(60°)\cdot\tan(\theta)}$$

$$+ \frac{\tan(60°) + \tan(\theta)}{1 - \tan(60°)\cdot\tan(\theta)}$$

$$= \tan(\theta) - \frac{\sqrt{3} - \tan(\theta)}{1 + \sqrt{3}\cdot\tan(\theta)} + \frac{\sqrt{3} + \tan(\theta)}{1 - \sqrt{3}\cdot\tan(\theta)}$$

$$= \frac{-\sqrt{3} + 3\tan(\theta) + \tan(\theta) - \sqrt{3}\tan^2(\theta) + \sqrt{3} + 3\tan(\theta) + \tan(\theta) + \sqrt{3}\tan^2(\theta)}{(1 - 3\tan^2\theta)}$$

$$= \tan(\theta) + \frac{8\tan(\theta)}{1 - 3\tan^2(\theta)}$$

$$= \frac{\tan(\theta) - 3\tan^3(\theta) + 8\tan(\theta)}{1 - 3\tan^2(\theta)}$$

$$= 3 \times \left(\frac{3\tan(\theta) - \tan^3(\theta)}{1 - 3\tan^2(\theta)}\right)$$

$$= 3\tan(3\theta)$$

140. We have

$$\cos(\theta)\cos(2\theta)\cdot\cos(2^2\theta)\cdot\cos(2^3\theta)\ldots\cos(2^{n-1}\theta)$$

$$= \frac{1}{2\sin\theta}(2\sin\theta\cos\theta)(\cos 2\theta\cdot\cos(2^2\theta)\ldots\cos(2^{2n-1}\theta))$$

$$= \frac{1}{2^2\sin\theta}(2\sin 2\theta\cos 2\theta)(\cos(2^2\theta)\ldots\cos(2^{2n-1}\theta))$$

$$= \frac{1}{2^3\sin\theta}(2\sin 4\theta\cos 4\theta)(\cos(2^3\theta)\ldots\cos(2^{2n-1}\theta))$$

$$= \frac{1}{2^4\sin\theta}(2\sin 2^3\theta\cos 2^3\theta)(\cos(2^4\theta)\ldots\cos(2^{2n-1}\theta))$$

$$\vdots$$

$$= \frac{1}{2^n\sin\theta}(2\sin 2^{n-1}\theta\cos 2^{n-1}\theta)$$

$$= \frac{1}{2^n\sin\theta}(\sin 2^n\theta)$$

$$= \frac{\sin(2^n\theta)}{2^n\sin\theta}$$

141. We have $\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{8\pi}{7}\right)$

$$= \cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\pi + \frac{\pi}{7}\right)$$

$$= -\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2^2\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2^3\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{4\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\right)$$

$$= -\frac{\sin\left(\frac{8\pi}{7}\right)}{2^3\sin\left(\frac{\pi}{7}\right)}$$

$$= -\frac{\sin\left(\pi + \frac{\pi}{7}\right)}{2^3\sin\left(\frac{\pi}{7}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{7}\right)}{8\sin\left(\frac{\pi}{7}\right)}$$

$$= \frac{1}{8}$$

142. Let $z = \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)$

$$\Rightarrow \quad 2z\sin\left(\frac{\pi}{7}\right)$$

$$= 2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right) + 2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$+ 2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right)$$

$$= \sin\left(\frac{3\pi}{7}\right) - \sin\left(\frac{\pi}{7}\right) + \sin\left(\frac{5\pi}{7}\right) - \sin\left(\frac{3\pi}{7}\right)$$

$$+ \sin\left(\frac{7\pi}{7}\right) - \sin\left(\frac{5\pi}{7}\right)$$

$$= -\sin\left(\frac{\pi}{7}\right)$$

Thus, $2z\sin\left(\frac{\pi}{7}\right) = -\sin\left(\frac{\pi}{7}\right)$

$$\Rightarrow \quad z = -\frac{1}{2}$$

$$\Rightarrow \quad \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$$

143. Given

$$M = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$\Rightarrow \quad M^2 = a^2\cos^2\theta + b^2\sin^2\theta + a^2\sin^2\theta + b^2\cos^2\theta$$
$$+ 2\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)(a^2\sin^2\theta + b^2\cos^2\theta)}$$

$$\Rightarrow \quad M^2 = a^2 + b^2 + 2\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)}$$
$$\sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}$$

$$\Rightarrow \quad M^2 = a^2 + b^2 + 2[(a^4 + b^4)\sin^2\theta\cos^2\theta$$
$$+ a^2b^2(\sin^4\theta + \cos^4\theta)]^{1/2}$$

$$\Rightarrow \quad M^2 = a^2 + b^2 + 2[(a^4 + b^4)\sin^2\theta\cos^2\theta$$
$$+ a^2b^2(1 - 2\sin^2\theta + \cos^2\theta)]^{1/2}$$

$$\Rightarrow \quad M^2 = a^2 + b^2 + 2((a^4 + b^4 - 2a^2b^2)\sin^2\theta\cos^2\theta$$
$$+ a^2b^2)^{1/2}$$

$$\Rightarrow \quad M^2 = a^2 + b^2 + \sqrt{\begin{matrix}(4(a^4 + b^4 - 2a^2b^2)\\ \sin^2\theta\cos^2\theta + 4a^2b^2)\end{matrix}}$$

$$\Rightarrow \quad M^2 = a^2 + b^2 + \sqrt{\begin{matrix}((a^2 - b^2)^2 + (\sin(2\theta))^2\\ + 4a^2b^2)\end{matrix}}$$

Thus, maximum $(M^2) = a^2 + b^2 + (a^2 + b^2)$
$= 2(a^2 + b^2)$

and minimum $(M^2) = a^2 + b^2 + 2ab = (a + b)^2$

Hence, the value of $m_1 - m_2$
= maximum $(M^2)$ – minmum $(M^2)$
$= 2(a^2 + b^2) - (a + b)^2$

144. We have

$\tan(4\theta) = \tan(3\theta + \theta)$

$$= \frac{\tan 3\theta + \tan\theta}{1 - \tan 3\theta\tan\theta}$$

$$= \frac{\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} + \tan\theta}{1 - \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\tan\theta}$$

$$= \frac{\frac{3\tan\theta - \tan^3\theta + \tan\theta - 3\tan^3\theta}{1 - 3\tan^2\theta}}{\frac{1 - 3\tan^2\theta - 3\tan^2\theta + \tan^4\theta}{1 - 3\tan^2\theta}}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

145. We have

$$\left(\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin 9x}{\cos 27x}\right)$$

$$=\frac{1}{2}\left(\frac{2\sin x\cos x}{\cos 3x\cos x}+\frac{2\sin 3x\cos 3x}{\cos 3x\cos 9x}+\frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x}\right)$$

$$=\frac{1}{2}\left(\frac{\sin 2x}{\cos 3x\cos x}+\frac{\sin 6x}{\cos 3x\cos 9x}+\frac{\sin 18x}{\cos 9x\cos 27x}\right)$$

$$=\frac{1}{2}\left(\frac{\sin(3x-x)}{\cos 3x\cos x}+\frac{\sin(9x-3x)}{\cos 3x\cos 9x}+\frac{\sin(27x-9x)}{\cos 9x\cos 27x}\right)$$

$$=\frac{1}{2}\left(\frac{\sin 3x\cos x-\cos 3x\sin x}{\cos 3x\cos x}+\frac{\sin 9x\cos 3x-\cos 9x\sin 3x}{\cos 3x\cos 9x}+\frac{\sin 27x\cos 9x-\cos 27x\sin 9x}{\cos 9x\cos 27x}\right)$$

$$=\frac{1}{2}(\tan 3x-\tan x+\tan 9x-\tan 3x+\tan 27x-\tan 9x)$$

$$=\frac{1}{2}(\tan 27x-\tan x)$$

146. We have

$$\tan\left(\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)(1+\sec 2^2\theta)\ldots\sec(2^n\theta)$$

Now, $\tan\left(\frac{\theta}{2}\right)(1+\sec\theta)$

$$=\frac{\sin(\theta/2)}{\cos(\theta/2)}\times\frac{2\cos^2(\theta/2)}{\cos\theta}$$

$$=\frac{2\sin(\theta/2)\cos(\theta/2)}{\cos\theta}$$

$$=\frac{\sin\theta}{\cos\theta}=\tan\theta$$

Also, $\tan\theta\times(1+\sec 2\theta)$

$$=\frac{\sin\theta}{\cos\theta}\times\left(\frac{1+\cos 2\theta}{\cos 2\theta}\right)$$

$$=\frac{\sin\theta}{\cos\theta}\times\left(\frac{2\cos^2\theta}{\cos 2\theta}\right)$$

$$=\frac{2\sin\theta\cos\theta}{\cos 2\theta}$$

$$=\frac{\sin 2\theta}{\cos 2\theta}=\tan 2\theta$$

Thus, $\tan\left(\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)(1+\sec 2^2\theta)\ldots\sec(2^n\theta)$

$$=\tan(2^n\theta)$$

147.

(i) Here, $a = 3$, $b = 4$ and $c = 10$

Thus, the minimum values of $f(x)$

$$=-\sqrt{a^2+b^2}+c=-5+10=5$$

and the maximum values of

$$f(x)=\sqrt{a^2+b^2}+c=5+10=15.$$

(ii) Maximum value $=\sqrt{3^3+4^2}+10=15$

Minimum value $=-\sqrt{3^2+4^2}+10=5$

(iii) Maximum value = 3 + 4 = 7

Minimum value = –3 + 4 = 1

(iv) Maximum value = 2 + 5 = 7

Minimum value = –2 + 5 = 3

(v) Maximum value $=\sqrt{2}$

Minimum value $=-\sqrt{2}$

(vi) Maximum value $=\sqrt{2}$

Minimum value $=-\sqrt{2}$

(vii) Maximum value = sin 1

Minimum value = –sin 1

(viii) Maximum value = cos 1

Minimum value = 0

(ix) Maximum value $=\sqrt{2}$

Minimum value $=-\sqrt{2}$

(x) Given $f(x) = \cos(\sin x) + \sin(\cos x)$.

$= \cos(\sin x) + \sin(\sin x) + \sin(\cos x) - \sin(\sin x)$

Maximum value $=\sqrt{2}+\sin 1$

Minimum value $=\sqrt{2}-\sin 1$

148. $R_f$ = [minimum $f(x)$, maximum $f(x)$]

$$=[-\sqrt{2}+3,\sqrt{2}+3]$$

149. Given $2\sin^2\theta+3\cos^2\theta$

$$=2(\sin^2\theta+\cos^2\theta)+\cos^2\theta$$

$$=2+\cos^2\theta$$

Maximum value = 2 + 1 = 3

Minimum value = 2 + 0 = 2

150. Let $f(\theta)=5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)+3$

$$=5\cos\theta+\frac{3}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3$$

$$=\frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3$$

Maximum value $=\sqrt{\frac{169}{4}+\frac{27}{4}}+3=\sqrt{\frac{196}{4}}+3=10$

Minimum value $= -\sqrt{\frac{169}{4}+\frac{27}{4}}+3=-7+3=-4$

151. Let $f(\theta) = \cos^2\theta + 3\sin^2\theta - 3\sin 2\theta + 2$
$= 1 + 2\sin^2\theta - 3\sin 2\theta + 2$
$= 3 + 1 - \cos 2\theta - 3\sin 2\theta$
$= 4 - (\cos 2\theta + 3\sin 2\theta)$
Maximum value $= 4+\sqrt{10}$
Minimum value $= 4-\sqrt{10}$

152. Let $f(x) = \operatorname{cosec}^2 x + 25\sec^2 x$
$= 1 + \cot^2 x + 25 + 25\tan^2 x$
$= 26 + \cot^2 x + 25\tan^2 x$
$\geq 26 + 10 = 36$
Hence, the minimum value is 26.

153. Given expression is $2 - \cos x + \sin^2 x$
$= 2 - \cos x + 1 - \cos^2 x$
$= 3 - \cos x - \cos^2 x$
$= -(\cos^2 x + \cos x - 3)$
$= -\left(\left(\cos x+\frac{1}{2}\right)^2-3-\frac{1}{4}\right)$
$=\frac{13}{4}-\left(\cos x+\frac{1}{2}\right)^2$
Maximum value $= \frac{13}{4}$
Minimum value $= \frac{13}{4}-\frac{1}{4}=3$

154. Given $y = 4\sin^2\theta - \cos 2\theta$
$= 2(2\sin^2\theta) - \cos 2\theta$
$= 2(1 - \cos 2\theta) - \cos 2\theta$
$= 2 - 3\cos 2\theta$
$= 2 + 3(-\cos 2\theta)$
Maximum value $= 2 + 3 = 5$
Minimum value $= 2 - 3 = -1$
Hence $y$ lies in $[-1, 5]$

155. Here, $m = -3 + 5 = 2$
and $n = 3 + 2 = 5$
Hence, the value of $(m + n + 2) = 9$

156. Given $f(x) = \sin^2 x + \cos^4 x$
$=\frac{1}{2}(2\sin^2 x)+\frac{1}{4}(2\cos^2 x)^2$
$=\frac{1}{2}(1-\cos 2x)+\frac{1}{4}(1+\cos 2x)^2$
$=\frac{1}{2}(1-\cos 2x)+\frac{1}{4}(1+2\cos 2x+\cos^2 2x)$
$=\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\cos^2 2x$
$=\frac{3}{4}+\frac{1}{4}\cos^2 2x$
Maximum value $= \frac{3}{4}+\frac{1}{4}\cdot 1=\frac{3}{4}+\frac{1}{4}=\frac{4}{4}=1$
Minimum value $= \frac{3}{4}+\frac{1}{4}\cdot 0=\frac{3}{4}$

157. Given $f(x) = \cos^2 x + \sin^4 x$
$=\frac{1}{2}(2\cos^2 x)+\frac{1}{4}(2\sin^2 x)^2$
$=\frac{1}{2}(1+\cos 2x)+\frac{1}{4}(1-\cos 2x)^2$
$=\frac{1}{2}(1+\cos 2x)+\frac{1}{4}(1-2\cos 2x+\cos^2 2x)$
$=\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\cos^2 x$
$=\frac{3}{4}+\frac{1}{4}\cos^2 2x$
Maximum value $= \frac{3}{4}+\frac{1}{4}\cdot 1=\frac{3}{4}+\frac{1}{4}=\frac{4}{4}=1$
Minimum value $= \frac{3}{4}+\frac{1}{4}\cdot 0=\frac{3}{4}$

158. Given $f(x) = \sin^4 x + \cos^4 x$
$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$
$= 1-\frac{1}{2}\sin^2 2x$
Maximum value $= 1 + 0 = 1$
Minimum value $= 1-\frac{1}{2}=\frac{1}{2}$

159. We have, $f(\theta) = \sin^6\theta + \cos^6\theta$
$= (\sin^2\theta)^3 + (\cos^2\theta)^3$
$= (\sin^2\theta + \cos^2\theta)^2 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)$
$= 1 - 3\sin^2\theta\cos^2\theta$
$=1-\frac{3}{4}(4\sin^2\theta\cos^2\theta)$
$=1-\frac{3}{4}(\sin^2 2\theta)$
$=1+\frac{3}{4}(-\sin^2 2\theta)$
As we know, $-1 \leq (-\sin^2 2\theta) \leq 0$
$\Rightarrow \quad -\frac{3}{4}\leq\frac{3(-\sin^2 2\theta)}{4}\leq 0$
$\Rightarrow \quad 1-\frac{3}{4}\leq 1+\frac{3(-\sin^2 2\theta)}{4}\leq 1$
$\Rightarrow \quad \frac{1}{4}\leq f(\theta)\leq 1$
Hence, the maximum value = 1 and the minimum value $=\frac{1}{4}$

160. Now, $A$

$= \cos^2\theta + \sin^4\theta$

$$= \frac{1}{2}(2\sin^2\theta) + \frac{1}{4}(4\sin^4\theta)$$

$$= \frac{1}{2}(1 - \cos(2\theta)) + \frac{1}{4}(1 + \cos(2\theta))^2$$

$$= \frac{1}{2}(1 + \cos(2\theta)) + \frac{1}{4}(1 - 2\cos(2\theta) + \cos^2(2\theta))$$

$$= \frac{1}{2} + \frac{1}{2}\cos(2\theta) + \frac{1}{4} - \frac{1}{2}\cos(2\theta) + \frac{1}{4}(\cos^2(2\theta))$$

$$= \frac{3}{4} + \frac{1}{4}(\cos^2(2\theta))$$

Maximum value of $A = m_1 = \frac{3}{4} + \frac{1}{4}\cdot 1 = 1$

Also, $B$

$= \sin^2\theta + \cos^4\theta$

$$= \frac{1}{2}(2\sin^2\theta) + \frac{1}{4}(4\sin^4\theta)$$

$$= \frac{1}{2}(2\sin^2\theta) + \frac{1}{4}(2\cos^2\theta)^2$$

$$= \frac{1}{2}(1 - \cos(2\theta)) + \frac{1}{4}(1 + \cos(2\theta))^2$$

$$= \frac{1}{2}(1 - \cos(2\theta)) + \frac{1}{4}(1 + 2\cos(2\theta) + \cos^2(2\theta))$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4}\cos^2(2\theta)$$

$$= \frac{3}{4} + \frac{1}{4}\cos^2(2\theta)$$

Thus, the minimum value of $B = m = \frac{3}{4} + \frac{1}{4}\cdot 0 = \frac{3}{4}$

Now, the value of $m_1^2 + m_2^2 + m_1 m_2$

$$= 1 + \frac{9}{16} + \frac{3}{4}$$

$$= \frac{37}{16}$$

161. Given $f(x) = (\sin x + \cos x + \text{cosec } 2x)^3$

As we know that,

AM $\geq$ GM

$$\left(\frac{\sin x + \cos x + \text{cosec } 2x}{3}\right) \geq \sqrt[3]{(\sin x \cdot \cos x \cdot \text{cosec } 2x)}$$

$$\left(\frac{\sin x + \cos x + \text{cosec } 2x}{3}\right) \geq \sqrt[3]{\frac{1}{2}}$$

$$\left(\frac{\sin x + \cos x + \text{cosec } 2x}{3}\right)^3 \geq \frac{1}{2}$$

$$(\sin x + \cos x + \text{cosec } 2x)^3 \geq \frac{27}{2}$$

Hence, the minimum value of $f(x)$ is $\frac{27}{2}$

162. Find the maximum and minimum values of

$$f(x) = \frac{5}{\sin^2\theta - 6\sin\theta\cos\theta + 3\cos^2\theta}$$

163. Given $f(x) = \frac{a^2}{\cos^2 x} + \frac{b^2}{\sin^2 x}$

$= a^2\sec^2 x + b^2\text{cosec}^2 x$

$= a^2 + a^2\tan^2 x + b^2 + b^2\cot^2 x$

$= a^2 + b^2 + (a^2\tan^2 x + b^2\cot^2 x)$

$\geq a^2 + b^2 + 2ab = (a + b)^2$

Hence, the minimum value of $f(x)$ is $(a + b)^2$.

164. We have, $f(x) = \frac{x^2\sin^2 x + 4}{x\sin x}$

$$= x\sin x + \frac{4}{x\sin x} \geq 4$$

Hence, the minimum values of $f(x)$ is 4

165. Given $f(x) = \log_x y + \log_y y$

As we know,

AM $\geq$ GM

$$\frac{\log_x y + \log_y x}{2} \geq \sqrt{\log_x y \cdot \log_y x} = 1$$

$$\frac{\log_x y + \log_y x}{2} \geq 1$$

$$\log_x y + \log_y x \geq 2$$

Hence, the minimum value of $f(x)$ is 2.

166. Given $f(x) = 2\log_{10} x - \log_x(0.01)$, $x > 1$

$= 2\log_{10} x - \log_x(10)^{-2}$

$= 2\log_{10} x + 2\log_x(10)$

$= 2(\log_{10} x + \log_x(10))$

$\geq 2.\,2 = 4$

Hence, the minimum value of $f(x)$ is 4

167. Given $f(x, y, z) = \frac{(x^2+1)(y^2+1)(z^2+1)}{xyz}$

$$= \left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right)\left(\frac{z^2+1}{z}\right)$$

$$= \left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right)\left(z + \frac{1}{z}\right)$$

$\geq 2.2.2 = 8$

Hence, the minimum value is 2.

168. Given $f(x, y, z) = \frac{(x^3+2)(y^3+2)(z^3+2)}{xyz}$

$$= \left(\frac{x^3+2}{x}\right)\left(\frac{y^3+2}{y}\right)\left(\frac{z^3+2}{z}\right)$$
$$= \left(x^2+\frac{2}{x}\right)\left(y^2+\frac{2}{y}\right)\left(z^2+\frac{2}{z}\right)$$
$$\geq 3.3.3 = 27$$

Hence, the minimum value is 27.

169. We have $f(a, b, c, d)$

$$= \frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd}$$
$$= \frac{(a^2+1)}{a} \times \frac{(b^2+1)}{b} \times \frac{(c^2+1)}{c} \times \frac{(d^2+1)}{d}$$
$$= \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\left(d+\frac{1}{d}\right)$$
$$\geq 2.2.2.2 = 16$$

Hence, the minimum value is 16.

170. Do yourself.

171. We have $\sin^2(24°) - \sin^2(6°)$

$$= \sin(24° + 6°)\ \sin(24° - 6°)$$
$$= \sin(30°) \times \sin(18°)$$
$$= \frac{1}{2} \times \sin(18°)$$
$$= \frac{1}{2} \times \frac{\sqrt{5}-1}{4}$$
$$= \frac{(\sqrt{5}-1)}{8}$$

172. We have $\sin^2(48°) - \cos^2(12°)$

$$= \cos(48° + 12°) \times \cos(48° - 12°)$$
$$= \cos(60°) \times \cos(36°)$$
$$= \frac{1}{2} \times \frac{\sqrt{5}+1}{4}$$
$$= \frac{\sqrt{5}+1}{8}$$

173. We have $\sin(12°) \cdot \sin(48°) \cdot \sin(54°)$

$$= \frac{1}{\sin(72°)}(\sin(12°) \cdot \sin(48°) \cdot \sin(72°))(\sin(54°))$$
$$= \frac{1}{4\sin(72°)}$$
$$(4\sin(60° - 12°) \cdot \sin(12°) \cdot \sin(60° + 12°)) \times (\cos(36°))$$
$$= \frac{1}{4\sin(72°)}(\sin(36°) \cdot \cos(36°))$$
$$= \frac{1}{8\sin(72°)}(2\sin(36°) \cdot \cos(36°))$$
$$= \frac{1}{8\sin(72°)}(\sin(72°))$$
$$= \frac{1}{8}$$

174. We have

$\sin(6°) \cdot \sin(42°) \cdot \sin(66°)\sin(78°)$

$$= \frac{1}{4\sin(54°)}$$
$$(4\sin(6°) \cdot \sin(60° - 6°) \cdot \sin(60° + 6°)) \times (\sin(78°) \cdot \sin(42°))$$
$$= \frac{1}{4\cos(36°)}(\sin(18°)\sin(72°) \cdot \sin(42°))$$
$$= \frac{1}{16\cos(36°)}$$
$$(4\sin(18°) \cdot \sin(60° + 18°) \cdot \sin(60° - 18°))$$
$$= \frac{1}{16\cos(36°)}(\sin(54°))$$
$$= \frac{1}{16\cos(36°)}(\cos(36°))$$
$$= \frac{1}{16}$$

175. We have $4(\sin(24°) + \cos(6°))$

$$= 4(\sin(24°) + \sin(84°))$$
$$= 4\left(2\sin\left(\frac{24°+84°}{2}\right)\cos\left(\frac{24°-84°}{2}\right)\right)$$
$$= 8(\sin(54°)\cos(30°))$$
$$= 8(\sin(36°)\cos(30°))$$
$$= 8\left(\frac{\sqrt{5}+1}{4} \times \frac{1}{2}\right)$$
$$= \left(\sqrt{5}+1\right)$$

176. We have

$\tan(6°) \cdot \tan(42°) \cdot \tan(66°)\tan(78°)$

$$= (\tan(6°) \cdot \tan(66°)) \times (\tan(42°) \cdot \tan(78°))$$
$$= \frac{1}{\tan(54°)} \times (\tan(6°) \cdot \tan(54°) \cdot \tan(66°)) \times (\tan(42°) \cdot \tan(78°))$$
$$= \frac{1}{\tan(54°)} \times (\tan(6°) \cdot \tan(60° - 6°) \cdot \tan(60° + 6°)) \times (\tan(42°) \cdot \tan(78°))$$
$$= \frac{1}{\tan(54°)}(\tan(18°)) \times (\tan(42°) \cdot \tan(78°))$$
$$= \frac{1}{\tan(54°)}(\tan(18°) \cdot \tan(42°) \cdot \tan(78°))$$
$$= \frac{1}{\tan(54°)}$$
$$(\tan(60° - 18°) \cdot \tan(18°) \cdot \tan(60° + 18°))$$
$$= \frac{1}{\tan(54°)} \times (\cdot \tan(54°))$$
$$= 1$$

177. We have $\left(1+\cos\left(\frac{\pi}{8}\right)\right)\cdot\left(1+\cos\left(\frac{3\pi}{8}\right)\right)$

$$\cdot\left(1+\cos\left(\frac{5\pi}{8}\right)\right)\cdot\left(1+\cos\left(\frac{7\pi}{8}\right)\right)$$

$$=\left(1+\cos\left(\frac{\pi}{8}\right)\right)\cdot\left(1+\cos\left(\frac{7\pi}{8}\right)\right)$$

$$\cdot\left(1+\cos\left(\frac{3\pi}{8}\right)\right)\cdot\left(1+\cos\left(\frac{5\pi}{8}\right)\right)$$

$$=\left(1+\cos\left(\frac{\pi}{8}\right)\right)\cdot\left(1-\cos\left(\frac{\pi}{8}\right)\right)$$

$$\cdot\left(1+\cos\left(\frac{3\pi}{8}\right)\right)\cdot\left(1-\cos\left(\frac{3\pi}{8}\right)\right)$$

$$=\left(1-\cos^2\left(\frac{\pi}{8}\right)\right)\cdot\left(1-\cos^2\left(\frac{3\pi}{8}\right)\right)$$

$$=\left(\sin^2\left(\frac{\pi}{8}\right)\right)\cdot\left(\sin^2\left(\frac{3\pi}{8}\right)\right)$$

$$=\frac{1}{4}\left(2\sin^2\left(\frac{\pi}{8}\right)\right)\cdot\left(2\sin^2\left(\frac{3\pi}{8}\right)\right)$$

$$=\frac{1}{4}\left(1-\cos\left(\frac{\pi}{4}\right)\right)\cdot\left(1-\cos\left(\frac{3\pi}{4}\right)\right)$$

$$=\frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)\cdot\left(1+\frac{1}{\sqrt{2}}\right)$$

$$=\frac{1}{4}\left(1-\frac{1}{2}\right).$$

$$=\frac{1}{8}.$$

178. We have

$$\sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{5\pi}{8}\right)+\sin^4\left(\frac{7\pi}{8}\right)$$

$$=\sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{\pi}{8}\right)$$

$$=2\left(\sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)\right)$$

$$=2\left(\sin^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)\right)$$

$$=2\left(1-2\sin^2\left(\frac{\pi}{8}\right)\cdot\cos^2\left(\frac{\pi}{8}\right)\right)$$

$$=\left(2-2\sin^2\left(\frac{\pi}{8}\right)\cdot 2\cos^2\left(\frac{\pi}{8}\right)\right)$$

$$=\left(2-\left(2\sin^2\left(\frac{\pi}{8}\right)\right)\cdot\left(2\cos^2\left(\frac{\pi}{8}\right)\right)\right)$$

$$=\left(2-\left(1-\cos\left(\frac{\pi}{4}\right)\right)\cdot\left(1+\cos\left(\frac{\pi}{4}\right)\right)\right)$$

$$=\left(2-\left(1-\frac{1}{\sqrt{2}}\right)\cdot\left(1+\frac{1}{\sqrt{2}}\right)\right)$$

$$=\left(2-\left(1-\frac{1}{2}\right)\right)$$

$$=\frac{3}{2}$$

179. We have

$$\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{5\pi}{8}\right)+\cos^4\left(\frac{7\pi}{8}\right)$$

$$=\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)$$

$$+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)$$

$$=2\left(\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)\right)$$

$$=2\left(\cos^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{\pi}{8}\right)\right)$$

$$=2\left(1-2\cos^2\left(\frac{\pi}{8}\right)\cdot\sin^2\left(\frac{\pi}{8}\right)\right)$$

$$=\left(2-\left(2\cos^2\left(\frac{\pi}{8}\right)\right)\cdot\left(2\sin^2\left(\frac{\pi}{8}\right)\right)\right)$$

$$=\left(2-\left(1+\cos\left(\frac{\pi}{4}\right)\right)\cdot\left(1+\cos\left(\frac{\pi}{4}\right)\right)\right)$$

$$=\left(2-\left(1+\frac{1}{\sqrt{2}}\right)\cdot\left(1-\frac{1}{\sqrt{2}}\right)\right)$$

$$=\left(2-\left(1-\frac{1}{2}\right)\right)$$

$$=\frac{3}{2}$$

180. We have tan (20°) tan (80°)

$$=\frac{1}{\tan(40°)}[\tan(40°)\tan(20°)\tan(80°)]$$

$$=\frac{1}{\tan(40°)}\times\tan(60°)$$

$$=\frac{\sqrt{3}}{\tan(40°)}$$

$$=\sqrt{3}\cot(40°)$$

$$=\sqrt{3}\tan(60°)$$

181. We have tan (10°) tan (70°)

$$= \frac{1}{\tan(50°)}[\tan(50°)\tan(10°)\tan(70°)]$$

$$= \frac{1}{\tan(50°)} \times \tan(30°)$$

$$= \frac{1}{\sqrt{3}} \times \cot(50°)$$

$$= \frac{1}{\sqrt{3}} \times \tan(40°)$$

182. We have sin 55° – sin 19° + sin 53° – sin 17°

= (sin (55°) + sin (53°)) – (sin (19°) + sin (17°))

= 2 sin (54°) cos (1°) – 2 sin (18°) cos (1°)

= 2 cos (1°)[sin (54°) – sin (18°)]

$$= 2\cos(1°)\left[\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right]$$

$$= 2\cos(1°) \times \frac{1}{2} = \cos(1°)$$

183. We have

$$\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right)$$

$$= -\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2\sin\left(\frac{\pi}{7}\right)} \times \left[2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)\right]\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2^2\sin\left(\frac{\pi}{7}\right)} \times \left[2\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\right]\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2^3\sin\left(\frac{\pi}{7}\right)} \times \left[2\sin\left(\frac{4\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\right]$$

$$= -\frac{1}{2^3\sin\left(\frac{\pi}{7}\right)} \times \sin\left(\frac{8\pi}{7}\right)$$

$$= -\frac{1}{2^3\sin\left(\frac{\pi}{7}\right)} \times \sin\left(\pi + \frac{\pi}{7}\right)$$

$$= -\frac{1}{2^3\sin\left(\frac{\pi}{7}\right)} \times -\sin\left(\frac{\pi}{7}\right)$$

$$= \frac{1}{2^3} = \frac{1}{8}$$

184. Let $S = \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)$

$$\Rightarrow 2\sin\left(\frac{\pi}{7}\right)S = 2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right) + 2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right) + 2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right)$$

$$\Rightarrow 2\sin\left(\frac{\pi}{7}\right)S = \left(\sin\left(\frac{3\pi}{7}\right) - \sin\left(\frac{\pi}{7}\right)\right) + \left(\sin\left(\frac{5\pi}{7}\right) - \sin\left(\frac{3\pi}{7}\right)\right) + \left(\sin\left(\frac{7\pi}{7}\right) - \sin\left(\frac{5\pi}{7}\right)\right)$$

$$\Rightarrow 2\sin\left(\frac{\pi}{7}\right)S = -\sin\left(\frac{\pi}{7}\right)$$

$$\Rightarrow S = -\frac{1}{2}$$

Hence, the result.

185. We have $\tan\left(7\frac{1}{2}°\right) + \cot\left(7\frac{1}{2}°\right)$

$$= \frac{\sin\left(7\frac{1}{2}°\right)}{\cos\left(7\frac{1}{2}°\right)} + \frac{\cos\left(7\frac{1}{2}°\right)}{\sin\left(7\frac{1}{2}°\right)}$$

$$= \frac{\sin^2\left(7\frac{1}{2}°\right) + \cos^2\left(7\frac{1}{2}°\right)}{\sin\left(7\frac{1}{2}°\right)\cos\left(7\frac{1}{2}°\right)}$$

$$= \frac{2}{2\sin\left(7\frac{1}{2}°\right)\cos\left(7\frac{1}{2}°\right)}$$

$$= \frac{2}{\sin(15°)}$$

$$= \frac{2}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \frac{4\sqrt{2}}{\sqrt{3}-1}$$

$$= \frac{4\sqrt{2}(\sqrt{3}+1)}{2} = 2\sqrt{2}(\sqrt{3}+1)$$

186. Let $y = \cos\left(\frac{x}{2}\right) - \sqrt{3}\sin\left(\frac{x}{2}\right)$

$$= 2\left(\frac{1}{2}\cos\left(\frac{x}{2}\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{x}{2}\right)\right)$$

$$= 2\cos\left(\frac{x}{2} + \frac{\pi}{3}\right)$$

Now, $\frac{dy}{dx} = -\sin\left(\frac{x}{2} + \frac{\pi}{3}\right)$

For maximum or minimum, $\frac{dy}{dx} = 0$

$\Rightarrow \quad \sin\left(\frac{x}{2} + \frac{\pi}{3}\right) = 0$

$\Rightarrow \quad \left(\frac{x}{2} + \frac{\pi}{3}\right) = \pi$

$\Rightarrow \quad \frac{x}{2} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\Rightarrow \quad x = \frac{4\pi}{3}$

Hence, the value of $x$ is $\frac{4\pi}{3}$

187. If $\alpha$ and $\beta$ be two different roots $a \cos\theta + b \sin\theta = c$, then prove that

$\sin(\alpha + \beta) = \frac{2ab}{a^2 + b^2}$.

188. We have, $\sin 2A + \sin 2B + \sin 2C$

$= (\sin 2A + \sin 2B) + \sin 2C$
$= 2(\sin(A + B)\cos(A - B)) + \sin 2C$
$= 2(\sin(\pi - C)\cdot\cos(A - B)) + 2\sin C\cos C$
$= 2(\sin C\cdot\cos(A - B)) + 2\sin C\cos C$
$= 2(\sin C(\cos(A - B)) + \cos C)$
$= 2\sin C(\cos(A - B) + \cos(\pi - (A + B)))$
$= 2\sin C(\cos(A - B) - \cos(A + B))$
$= 2\sin C(2\sin A\sin B)$
$= 4\sin A\cdot\sin B\cdot\sin C$

189. We have $\cos 2A + \cos 2B + \cos 2C$

$= (\cos 2A + \cos 2B) + \cos 2C$
$= 2\cos(A + B)\cos(A - B) + \cos 2C$
$= 2\cos\{\pi - C\}\cos(A - B) + \cos 2C$
$= -2\cos C\cos(A + B) + 2\cos^2 C - 1$
$= -1 - 2\cos C(\cos(A - B) - \cos C)$
$= -1 - 2\cos C(\cos(A - B) + \cos(A + B))$
$= -1 - 2\cos C(2\cos A\cdot\cos B)$
$= -1 - 4\cos A\cdot\cos B\cdot\cos C$

190. We have $\sin^2 A + \sin^2 B - \sin^2 C$

$= \sin^2 A + (\sin^2 B - \sin^2 C)$
$= \sin^2 A + \sin(B + C)\sin(B - C)$
$= \sin^2 A + \sin(\pi - A)\sin(B - C)$
$= \sin^2 A + \sin A\sin(B - C)$
$= \sin A(\sin A + \sin(B - C))$
$= \sin A(\sin(B + C) + \sin(B - C))$
$= \sin A(2\sin B\cos C)$
$= 2\sin A\cdot\sin B\cdot\cos C$

191.(i) We have $\sin^2 A + \sin^2 B + \sin^2 C$

$= 1 - \cos^2 A + \sin^2 B + \sin^2 C$
$= 1 - (\cos^2 A - \sin^2 B) + (1 - \cos^2 C)$
$= 2 - (\cos^2 A - \sin^2 B) - \cos^2 C$
$= 2 - (\cos(A + B)\cdot\cos(A - B)) - \cos^2 C$
$= 2 - (\cos(\pi - C)\cdot\cos(A - B)) - \cos^2 C$
$= 2 + \cos C(\cos(A - B) - \cos C)$
$= 2 + \cos C(\cos(A - B) + \cos(A + B))$
$= 2 + \cos C(2\cos A\cdot\cos B)$
$= 2 + 2\cos A\cdot\cos B\cdot\cos C$

192. We have $\sin 2A + \sin 2B + \sin 2C$

$= 4\sin A\sin B\sin C$

Also, $\cos A + \cos B + \cos C - 1$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin^2\left(\frac{C}{2}\right)$$
$$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin^2\left(\frac{C}{2}\right)$$
$$= 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin^2\left(\frac{C}{2}\right)$$
$$= 2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-\mathrm{B}}{2}\right) - \sin\left(\frac{C}{2}\right)\right)$$
$$= 2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right)$$
$$= 2\sin\left(\frac{C}{2}\right)2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)$$
$$= 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

Thus, $\frac{\sin 2A + \sin 2B + \sin 2C}{\cos A + \cos B + \cos C - 1}$

$$= \frac{4\sin A\sin B\sin C}{4\sin(A/2)\sin(B/2)\sin(B/2)}$$
$$= 8\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$$

193.

(i) We have $A + B + C = \pi$

$\Rightarrow \quad A + B = \pi - C$

$\Rightarrow \quad \tan(A + B) = \tan(\pi - C)$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A\cdot\tan B} = -\tan C$

$\Rightarrow \quad \tan A + \tan B = -\tan C(1 - \tan A\cdot\tan B)$

$\Rightarrow \quad \tan A + \tan B = -\tan C + \tan A\cdot\tan B\cdot\tan C$

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A\cdot\tan B\cdot\tan C$

(ii) As we know that,

$\tan A + \tan B + \tan C = \tan A\cdot\tan B\cdot\tan C$

Dividing both the sides by

'$\tan A\cdot\tan B\cdot\tan C$', we get,

$$= \frac{\tan A}{\tan A\cdot\tan B\cdot\tan C} + \frac{\tan B}{\tan A\cdot\tan B\cdot\tan C}$$
$$+ \frac{\tan C}{\tan A\cdot\tan B\cdot\tan C} = 1$$

$\Rightarrow \quad \dfrac{1}{\tan B\cdot\tan C}+\dfrac{1}{\tan A\cdot\tan C}+\dfrac{1}{\tan A\cdot\tan B}=1$

$\Rightarrow \quad \cot B\cdot\cot C+\cot A\cdot\cot C+\cot A\cdot\cot B=1$

194. We have $A+B+C=2\pi$

$\Rightarrow \quad A+B+C=2\pi$

$\Rightarrow \quad A+B=2\pi-(C+D)$

$\Rightarrow \quad \tan(A+B)=\tan\{2\pi-(C+D)\}$

$\Rightarrow \quad \tan(A+B)=-\tan(C+D)$

$\Rightarrow \quad \dfrac{\tan A+\tan B}{1-\tan A\cdot\tan B}=-\dfrac{\tan C+\tan D}{1-\tan C\cdot\tan D}$

$\Rightarrow \quad (\tan A+\tan B)(1-\tan C\cdot\tan D)$
$=-(\tan C+\tan D)(1-\tan A\cdot\tan B)$

$\Rightarrow \quad \tan A+\tan B-\tan A\cdot\tan C\cdot\tan D$
$-\tan B\cdot\tan C\cdot\tan D$
$=-\tan C-\tan D+\tan A\cdot\tan B\cdot\tan C+\tan A\cdot\tan B\cdot\tan D$

$\Rightarrow \quad \tan A+\tan B+\tan C+\tan D$
$=\tan A\cdot\tan C\cdot\tan D+\tan B\cdot\tan C\cdot\tan D+\tan A\cdot\tan B\cdot\tan C+\tan A\cdot\tan B\cdot\tan D$

$$\Rightarrow \quad \frac{\tan A+\tan B+\tan C+\tan D}{\tan A\cdot\tan B\cdot\tan C\cdot\tan D}$$
$$=\frac{\tan A\cdot\tan B\cdot\tan C}{\tan A\cdot\tan B\cdot\tan C\cdot\tan D}+\frac{\tan A\cdot\tan C\cdot\tan D}{\tan A\cdot\tan B\cdot\tan C\cdot\tan D}+\frac{\tan A\cdot\tan B\cdot\tan D}{\tan A\cdot\tan B\cdot\tan C\cdot\tan D}+\frac{\tan B\cdot\tan C\cdot\tan D}{\tan A\cdot\tan B\cdot\tan C\cdot\tan D}$$

$$\Rightarrow \quad \frac{\tan A+\tan B+\tan C+\tan D}{\tan A\cdot\tan B\cdot\tan C\cdot\tan D}=\frac{1}{\tan A}+\frac{1}{\tan B}+\frac{1}{\tan C}+\frac{1}{\tan D}$$

$$\Rightarrow \quad \frac{\tan A+\tan B+\tan C+\tan D}{\tan A\cdot\tan B\cdot\tan C\cdot\tan D}=\cot A+\cot B+\cot C+\cot D$$

$$\Rightarrow \quad \frac{\tan A+\tan B+\tan C+\tan D}{\cot A+\cot B+\cot C+\cot D}=\tan A\cdot\tan B\cdot\tan C\cdot\tan D$$

194. We have $(\cot A+\cot B)=\dfrac{\cos A}{\sin A}+\dfrac{\cos B}{\sin B}$

$$=\frac{\cos A\sin B+\sin A\cdot\cos B}{\sin A\sin B}$$

$$=\frac{\sin(A+B)}{\sin A\sin B}=\frac{\sin C}{\sin A\sin B}$$

Similarly, $(\cot B+\cot C)=\dfrac{\sin A}{\sin B\sin C}$

and $(\cot C+\cot A)=\dfrac{\sin B}{\sin A\sin C}$

Thus, $(\cot A+\cot B)(\cot B+\cot C)$
$(\cot C+\cot A)$

$$=\frac{\sin C}{\sin A\sin B}\times\frac{\sin B}{\sin A\sin C}\times\frac{\sin A}{\sin B\sin C}$$

$$=\frac{1}{\sin A\sin B\sin C}$$

$=\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C$

195. Put $x=\tan A$, $y=\tan B$ and $z=\tan C$

Given, $xy+yz+zx=1$

$\Rightarrow \quad \tan A\cdot\tan B+\tan B\cdot\tan C+\tan C\cdot\tan A=1$

$\Rightarrow \quad \tan B\cdot\tan C+\tan C\cdot\tan A=1-\tan A\cdot\tan B$

$\Rightarrow \quad \tan C(\tan B+\tan A)=1-\tan A\cdot\tan B$

$\Rightarrow \quad \dfrac{\tan A+\tan B}{1-\tan A\cdot\tan B}=\dfrac{1}{\tan C}$

$\Rightarrow \quad \tan(A+B)=\cot C=\tan\left(\dfrac{\pi}{2}-C\right)$

$\Rightarrow \quad (A+B)=\left(\dfrac{\pi}{2}-C\right)$

$\Rightarrow \quad (A+B+C)=\dfrac{\pi}{2}$

Now,

$$\text{LHS}=\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}$$

$$=\frac{\tan A}{1-\tan^2 A}+\frac{\tan B}{1-\tan^2 B}+\frac{\tan C}{1-\tan^2 C}$$

$$=\frac{1}{2}\left(\frac{2\tan A}{1-\tan^2 A}+\frac{2\tan B}{1-\tan^2 B}+\frac{2\tan C}{1-\tan^2 C}\right)$$

$$=\frac{1}{2}(\tan 2A+\tan 2B+\tan 2C)$$

$$=\frac{1}{2}(\tan 2A\cdot\tan 2B\cdot\tan 2C)$$

$$=\frac{1}{2}\left(\frac{2\tan A}{1-\tan^2 A}\cdot\frac{2\tan B}{1-\tan^2 B}\cdot\frac{2\tan C}{1-\tan^2 C}\right)$$

$$=\left(\frac{4\tan A\cdot\tan B\cdot\tan C}{(1-\tan^2 A)(1-\tan^2 B)(1-\tan^2 C)}\right)$$

$$=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Hence, the result.

196. Put $x=\tan A$, $y=\tan B$ and $z=\tan C$

Given, $xy+yz+zx=1$

$\Rightarrow \quad \tan A\cdot\tan B+\tan B\cdot\tan C+\tan C\cdot\tan A=1$

$\Rightarrow \quad A + B + C = \dfrac{\pi}{2}$

Now, LHS

$$= \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2}$$

$$= \frac{1}{2}\left(\frac{2x}{1+x^2} + \frac{2y}{1+y^2} + \frac{2z}{1+z^2}\right)$$

$$= \frac{1}{2}\left(\frac{2\tan A}{1+\tan^2 A} + \frac{2\tan B}{1+\tan^2 B} + \frac{2\tan C}{1+\tan^2 C}\right)$$

$$= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{1}{2}(4\cos A \cdot \cos B \cdot \cos C)$$

$$= 2\cos A \cdot \cos B \cdot \cos C$$

$$= \frac{2}{\sec A \cdot \sec B \cdot \sec C}$$

$$= \frac{2}{\sqrt{(1+\tan^2 A)(1+\tan^2 B)(1+\tan^2 C)}}$$

$$= \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}$$

Hence, the result.

197. Let $A = \pi - \beta$, $B = \beta - \gamma$, $C = \gamma - \alpha$

Now, $A + B + C = 0$

$\Rightarrow \quad A + B = -C$

$\Rightarrow \quad \tan(A + B) = \tan(-C) = -\tan C$

$\Rightarrow \quad \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\Rightarrow \quad \tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\Rightarrow \quad \tan A + \tan B + \tan C$

$\quad = \tan A \tan B \tan C$

$\Rightarrow \quad \tan(\alpha - \beta) + \tan(\beta - \gamma) + \tan(\gamma - \alpha)$

$\quad = \tan(\alpha - \beta)\tan(\beta - \gamma)\tan(\gamma - \alpha)$

198. We have $\cot A + \cot B + \cot C = \sqrt{3}$

$\Rightarrow \quad (\cot A + \cot B + \cot C)^2 = 3$

$\Rightarrow \quad \cot^2 A + \cot^2 B + \cot^2 C$

$\quad + 2(\cot A \cot B + \cot B \cot C + \cot C \cot A) = 3$

$\Rightarrow \quad \cot^2 A + \cot^2 B + \cot^2 C + 2 = 3$

$\Rightarrow \quad \cot^2 A + \cot^2 B + \cot^2 C = 1$

$\Rightarrow \quad \cot^2 A + \cot^2 B + \cot^2 C$

$\quad = (\cot A \cot B + \cot B \cot C + \cot C \cot A)$

$\Rightarrow \quad \dfrac{1}{2}[(\cot A - \cot B)^2 + (\cot A - \cot B)^2 + (\cot A - \cot B)^2] = 0$

$\Rightarrow \quad (\cot A - \cot B)^2 = 0$, $(\cot B - \cot C)^2 = 0$, $(\cot C - \cot A)^2 = 0$

$\Rightarrow \quad \cot A = \cot B$, $\cot B = \cot C$, $\cot C = \cot A$

$\Rightarrow \quad \cot A = \cot B = \cot C$

So, $\Delta$ is an equilateral.

199. Given expression is

$x + y + z = xyz$

Put $x = \tan A$, $y = \tan B$ and $Z = \tan C$

So, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\tan A + \tan B = -\tan C(1 - \tan A \tan B)$

$$\frac{\tan A + \tan B}{(1 - \tan A \tan B)} = \tan(\pi - C)$$

$\tan(A + B) = \tan(\pi - C)$

$(A + B) = (\pi - C)$

$(A + B + C) = \pi$

$(3A + 3B + 3C) = 3\pi$

$(3A + 3B) = 3\pi - 3C$

$\tan(3A + 3B) = \tan(3\pi - 3C)$

$$\frac{\tan 3A + \tan 3B}{1 - \tan 3A \tan 3B} = -\tan 3C$$

$\tan 3A + \tan 3B + \tan 3C$

$= \tan 3A \cdot \tan 3B \cdot \tan 3C$

$$\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} + \frac{3\tan B - \tan^3 B}{1 - 3\tan^2 B} + \frac{3\tan C - \tan^3 C}{1 - 3\tan^2 C}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \cdot \frac{3\tan B - \tan^3 B}{1 - 3\tan^2 B} \cdot \frac{3\tan C - \tan^3 C}{1 - 3\tan^2 C}$$

$$\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2}$$

$$= \frac{3x - x^3}{1 - 3x^2} \cdot \frac{3y - y^3}{1 - 3y^2} \cdot \frac{3z - z^3}{1 - 3z^2}$$

Hence, the result.

200. We have $1 + \cos 56° + \cos 58° - \cos 66°$

$= (1 - \cos(66°)) + (\cos(58°)) + \cos(56°))$

$= 2\sin^2(33°) + 2\cos(57°)\cos(1°)$

$= 2\sin^2(33°) + 2\sin(33°)\cos(1°)$

$= 2\sin(33°)(\sin(33°) + \cos(1°))$

$= 2\sin(33°)(\cos(57°) + \cos(1°))$

$= 2\sin(33°)(2\cos(29°)\cos(28°))$

$= 4\cos(29°)\sin(28°)\sin(33°)$

Hence, the result.

201. Let $S = \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta)$

$+ \sin(\alpha + 3\beta) + \ldots + \sin(\alpha + (n-1)\alpha)$

Now,

$$2\sin\alpha\sin\left(\frac{\beta}{2}\right) = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)$$

$$2\sin(\alpha + \beta)\cdot\sin\left(\frac{\beta}{2}\right) = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)$$

$$2\sin(\alpha + 2\beta)\cdot\sin\left(\frac{\beta}{2}\right) = \cos\left(\alpha + \frac{3\beta}{2}\right) - \cos\left(\alpha + \frac{5\beta}{2}\right)$$

$$\vdots$$

Adding,we get

$$2\sin(\alpha+(n-1)\beta)\cdot\sin\left(\frac{\beta}{2}\right)$$
$$=\cos\left(\alpha+\frac{(2n-3)\beta}{2}\right)-\cos\left(\alpha+\frac{(2n-1)\beta}{2}\right)$$
$$=2\sin\left(\alpha+\frac{(n-1)\beta}{2}\right)\times\sin\left(\frac{\beta}{2}\right)$$

Thus $S=\dfrac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\times\sin\left(\alpha+\dfrac{(n-1)\beta}{2}\right)$

$$\Rightarrow\quad S=\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\times 2\cos\left(\alpha+\frac{n-1}{2}\beta\right)$$

202. Do yourself.
203. Do yourself.
204. Let $S=\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+\beta)$ $+\cos(\alpha+\beta)+\ldots+\cos(\alpha+(n-1)\alpha)$

Now, $2\cos\alpha\sin\left(\frac{\beta}{2}\right)$

$$=\sin\left(\alpha+\frac{\beta}{2}\right)-\sin\left(\alpha-\frac{\beta}{2}\right)$$

$$2\cos(\alpha+\beta)\sin\left(\frac{\beta}{2}\right)$$
$$=\sin\left(\alpha+\frac{3\beta}{2}\right)-\sin\left(\alpha+\frac{\beta}{2}\right)$$

$$2\cos(\alpha+3\beta)\sin\left(\frac{\beta}{2}\right)$$
$$=\sin\left(\alpha+\frac{5\beta}{2}\right)-\sin\left(\alpha+\frac{3\beta}{2}\right)$$

$$\vdots$$

$$2\cos(\alpha+(n-1)\beta)\sin\left(\frac{\beta}{2}\right)$$
$$=\sin\left(\alpha+\frac{(2n-1)\beta}{2}\right)-\sin\left(\alpha+\frac{(2n-3)\beta}{2}\right)$$

Adding all we get,

$$2\sin\left(\frac{\beta}{2}\right)\times S=\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin(\alpha-\beta)$$

$$\Rightarrow\quad 2\sin\left(\frac{\beta}{2}\right)\times S$$
$$=2\cos\left(\alpha+\frac{n-1}{2}\beta\right)\times\sin\left(\frac{n\beta}{2}\right)$$

205. Do yourself

206. Let $t_n=\dfrac{\sin x}{\sin(n+1)x\cdot\sin(n+2)x}$

$$\Rightarrow\quad t_n=\frac{\sin[(n+2)x-(n+1)x]}{\sin(n+1)x\cdot\sin(n+2)x}$$
$$=\frac{\sin(n+2)x\cos(n+1)x}{\sin(n+1)x\cdot\sin(n+2)x}$$
$$-\frac{\cos(n+2)x\sin(n+1)x}{\sin(n+1)x\cdot\sin(n+2)x}$$
$$=\cot(n+1)x-\cot(n+2)x$$

Thus, $t_1=\cot 2x-\cot 3x$

$t_2=\cot 3x-\cot 4x$

$t_3=\cot 4x-\cot 5x$

$\vdots$

$t_n=\cot(n+1)x-\cot(n+2)x$

Adding all we get,

$S=\cot 2x-\cot(n+2)x$

## Level III

1. We have

$\tan\alpha+2\tan 2\alpha+4\tan 4\alpha+8\cot 8\alpha$

$=\cot\alpha-(\cot\alpha-\tan\alpha)+2\tan 2\alpha$
$\quad+4\tan 4\alpha+8\cot 8\alpha$

$=\cot\alpha-2(\cot 2\alpha-\tan 2\alpha)$
$\quad+4\tan 4\alpha+8\cot 8\alpha$

$=\cot\alpha-2.2\cot 4\alpha+4\tan 4\alpha+8\cot 8\alpha$

$=\cot\alpha-4(\cot 4\alpha-\tan 4\alpha)+8\tan 8\alpha$

$=\cot\alpha-4.2\cot 8\alpha+8\cot 8\alpha$

$=\cot\alpha-8\cot 8\alpha+8\cot 8\alpha$

$=\cot\alpha$

2. We have

$\tan 9^\circ-\tan 27^\circ-\tan 63^\circ+\tan 81^\circ$

$=(\tan 9^\circ+\tan 81^\circ)-(\tan 27^\circ+\tan 63^\circ)$

$=(\tan 9^\circ+\cot 9^\circ)-(\tan 27^\circ+\cot 27^\circ)$

$$=\left(\frac{1}{\sin 9^\circ\cos 9^\circ}\right)-\left(\frac{1}{\sin 27^\circ\cos 27^\circ}\right)$$
$$=\left(\frac{2}{2\sin 9^\circ\cos 9^\circ}\right)-\left(\frac{2}{2\sin 27^\circ\cos 27^\circ}\right)$$
$$=\left(\frac{2}{\sin 18^\circ}-\frac{2}{\sin 54^\circ}\right)$$
$$=\left(\frac{2}{\sin 18^\circ}-\frac{2}{\cos 36^\circ}\right)$$
$$=\left(\frac{8}{\sqrt{5}-1}-\frac{8}{\sqrt{5}+1}\right)$$
$$=\left(\frac{8(\sqrt{5}+1-\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}\right)$$
$$=\left(\frac{8\times 2}{4}\right)=4$$

3. We have

$$\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin 9x}{\cos 27x}$$

$$=\frac{1}{2}\left(\frac{2\sin x\cos x}{\cos 3x\cos x}+\frac{2\sin 3x\cos 3x}{\cos 3x\cos 9x}+\frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x}\right)$$

$$=\frac{1}{2}\left(\frac{\sin 2x}{\cos 3x\cos x}+\frac{\sin 6x}{\cos 3x\cos 9x}+\frac{\sin 18x}{\cos 9x\cos 27x}\right)$$

$$=\frac{1}{2}\left(\frac{\sin(3x-x)}{\cos 3x\cos x}+\frac{\sin(9x-3x)}{\cos 3x\cos 9x}+\frac{\sin(27x-9x)}{\cos 9x\cos 27x}\right)$$

$$=\frac{1}{2}\left(\frac{\sin 3x\cos x-\cos 3x\sin x}{\cos 3x\cos x}+\frac{\sin 9x\cos 3x-\cos 9x\sin 3x}{\cos 3x\cos 9x}+\frac{\sin 27x\cos 9x-\cos 27x\sin 9x}{\cos 9x\cos 27x}\right)$$

$$=\frac{1}{2}(\tan 3x-\tan x+\tan 9x-\tan 3x+\tan 27x-\tan 9x)$$

$$=\frac{1}{2}(\tan 27x-\tan x)$$

4. We have $\frac{\sin x}{\sin y}=\frac{1}{2}, \frac{\cos x}{\cos y}=\frac{3}{2}$

$$\frac{\tan x}{\tan y}=\frac{1}{3}$$

Now $\tan(x+y)$

$$=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$=\frac{\tan x+3\tan x}{1-\tan x\cdot 3\tan x}$$

$$=\frac{4\tan x}{1-3\tan^2 x} \qquad \ldots(i)$$

Also, $\sin y=2\sin x, \cos y=\frac{2}{3}\cos x$

$\sin^2 y+\cos^2 y$

$$=4\sin^2 x+\frac{4}{9}\cos^2 x$$

$$=\frac{36\sin^2 x+4\cos^2 x}{9}$$

$$=\frac{32\sin^2 x+4}{9}$$

$$\Rightarrow \quad \frac{32\sin^2 x+4}{9}=1$$

$$\Rightarrow \quad 32\sin^2 x+4=9$$

$$\Rightarrow \quad 32\sin^2 x=5$$

$$\Rightarrow \quad \sin^2 x=\frac{5}{32}$$

$$\Rightarrow \quad \sin x=\frac{\sqrt{5}}{4\sqrt{2}}$$

$$\Rightarrow \quad \tan x=\frac{\sqrt{5}}{3\sqrt{3}} \qquad \ldots(ii)$$

From (i) and (ii) we get,

$$\tan(x+y)=\frac{\frac{4\sqrt{5}}{3\sqrt{3}}}{1-\frac{15}{27}}=\frac{4\sqrt{5}\times 27}{12\times 3\sqrt{3}}=\sqrt{15}$$

5. We have

$$\sin^4\left(\frac{\pi}{16}\right)+\sin^4\left(\frac{3\pi}{16}\right)+\sin^4\left(\frac{5\pi}{16}\right)+\sin^4\left(\frac{7\pi}{16}\right)$$

$$=\sin^4\left(\frac{\pi}{16}\right)+\sin^4\left(\frac{3\pi}{16}\right)+\sin^4\left(\frac{\pi}{2}-\frac{3\pi}{16}\right)+\sin^4\left(\frac{\pi}{2}-\frac{\pi}{16}\right)$$

$$=\sin^4\left(\frac{\pi}{16}\right)+\sin^4\left(\frac{3\pi}{16}\right)+\cos^4\left(\frac{3\pi}{16}\right)+\cos^4\left(\frac{\pi}{16}\right)$$

$$=\left(\sin^4\left(\frac{\pi}{16}\right)+\cos^4\left(\frac{\pi}{16}\right)\right)+\left(\sin^4\left(\frac{3\pi}{16}\right)+\cos^4\left(\frac{3\pi}{16}\right)\right)$$

$$=2-2\sin^2\left(\frac{\pi}{16}\right)\cdot\cos^2\left(\frac{\pi}{16}\right)-2\sin^2\left(\frac{3\pi}{16}\right)\cdot\cos^2\left(\frac{3\pi}{16}\right)$$

$$=2-\frac{1}{2}\left(\left(2\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)\right)^2+\left(2\sin\left(\frac{3\pi}{16}\right)\cos\left(\frac{3\pi}{16}\right)\right)^2\right)$$

$$=2-\frac{1}{2}\left(\sin^2\left(\frac{\pi}{8}\right)+\sin^2\left(\frac{3\pi}{8}\right)\right)$$

$$=2-\frac{1}{2}\left(\sin^2\left(\frac{\pi}{8}\right)+\sin^2\left(\frac{\pi}{2}-\frac{\pi}{8}\right)\right)$$

$$=2-\frac{1}{2}\left(\sin^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{\pi}{8}\right)\right)$$

$$=2-\frac{1}{2}$$

$$=\frac{3}{2}$$

6. We have

$$\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=-\frac{3}{2}$$

$\Rightarrow$ $2(\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)+3=0$

$\Rightarrow$ $2(\cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha)$ $+2(\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha)+3=0$

$\Rightarrow$ $(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\alpha)$ $+(\cos^2\gamma+\sin^2\gamma)$

$2(\cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha)$ $+2(\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha)=0$

$\Rightarrow$ $\cos^2\alpha+\cos^2\beta+\cos^2\gamma$ $+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma$ $+2\cos\gamma\cos\alpha+\sin^2\alpha+\sin^2\beta+\sin^2\gamma$

$+ 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma$
$+ 2\sin\gamma\sin\alpha = 0$

$\Rightarrow (\cos\alpha + \cos\beta + \cos\gamma)^2$
$+ (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$

$\Rightarrow (\cos\alpha + \cos\beta + \cos\gamma) = 0$
and $(\sin\alpha + \sin\beta + \sin\gamma) = 0$

7. We have $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$

$\Rightarrow \cos\alpha\cos\beta\sin\alpha\sin\beta = 1$

$\Rightarrow \cos(\alpha + \beta) = 1$

Therefore,

$$\sin(\alpha + \beta) = \sqrt{1 - \cos^2(\alpha + \beta)} = \sqrt{1 - 1} = 0$$

Now, $1 + \cot\alpha \cdot \tan\beta$

$$= 1 + \frac{\cos\alpha \cdot \sin\beta}{\sin\alpha \cdot \cos\beta}$$

$$= \frac{\sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta}{\sin\alpha \cdot \cos\beta}$$

$$= \frac{\sin(\alpha + \beta)}{\sin\alpha \cdot \cos\beta}$$

$$= 0$$

8. Given, $\beta + \gamma = \alpha$

$\Rightarrow \tan(\beta + \gamma) = \tan\alpha$

$\Rightarrow \tan(\beta + \gamma) = \tan(90° - \alpha)$

$\Rightarrow \tan(\beta + \gamma) = \cot\alpha$

$$\Rightarrow \frac{\tan\beta + \tan\gamma}{1 - \tan\beta\tan\gamma} = \cot\alpha$$

$$\Rightarrow \frac{\tan\beta + \tan\gamma}{1 - \tan\beta\tan\gamma} = \cot\beta$$

$\Rightarrow \tan\beta + \tan\gamma = \cot\beta - \tan\beta \cdot \cot\beta\ \tan\gamma$

$\Rightarrow \tan\beta + \tan\gamma = \cot\beta - \tan\gamma$

$\Rightarrow \tan\beta + \tan\gamma = \cot(90° - \alpha) - \tan\gamma$

$\Rightarrow \tan\beta + \tan\gamma = \tan\alpha - \tan\gamma$

$\Rightarrow \tan\alpha = \tan\beta + 2\tan\gamma$

9. We have $\tan\left(\frac{\pi}{24}\right) = \frac{\sin(\pi/24)}{\cos(\pi/24)}$

$$= \frac{2\sin(\pi/24)\cos(\pi/24)}{2\cos^2(\pi/24)}$$

$$= \frac{\sin(\pi/12)}{1 + \cos(\pi/12)}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2} + \sqrt{3} + 1}$$

$$= \frac{\sqrt{3} - 1}{(2\sqrt{2} + (\sqrt{3} + 1))} \times \frac{(2\sqrt{2} - (\sqrt{3} + 1))}{(2\sqrt{2} - (\sqrt{3} + 1))}$$

$$= \frac{(\sqrt{3} - 1)(2\sqrt{2} - (\sqrt{3} + 1))}{(8 - (\sqrt{3} + 1)^2)}$$

$$= \frac{2\sqrt{6} - 3 - \sqrt{3} - 2\sqrt{2} + \sqrt{3} + 1}{(4 - 2\sqrt{3})}$$

$$= \frac{2\sqrt{6} - 2 - 2\sqrt{2}}{(4 - 2\sqrt{3})}$$

$$= \frac{\sqrt{6} - 1 - \sqrt{2}}{(2 - \sqrt{3})}$$

$$= (\sqrt{6} - \sqrt{2} - 1)(2 + \sqrt{3})$$

$$= 2\sqrt{6} - 2\sqrt{2} - 2 + \sqrt{18} - \sqrt{6} - \sqrt{3}$$

$$= \sqrt{6} + \sqrt{2} - \sqrt{4} - \sqrt{3}$$

$$= \sqrt{6} - 2 - \sqrt{3} + \sqrt{2}$$

$$= (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)$$

Thus, $a = 3$, $b = 2$, $c = 2$ and $d = 1$

Hence, the value of $(a + b + c + d + 2)$
$= 3 + 2 + 2 + 1 + 2$
$= 10$

10. Given,

$$\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta + \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta - \frac{2\pi}{3}\right)} = m(\text{say})$$

Now, $x + y + z$

$$= m \times \left(\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta - \frac{2\pi}{3}\right)\right)$$

$$= m \times \left(\cos\theta + 2\cos\theta \cdot \cos\left(\frac{2\pi}{3}\right)\right)$$

$$= m \times \left(\cos\theta + 2\cos\theta \cdot \left(-\frac{1}{2}\right)\right)$$

$= m \times (\cos\theta - \cos\theta)$
$= 0$

Hence, the value of $x + y + z$ is zero.

11. We have $\sin(25°)\sin(35°)\sin(85°)$

$= \sin(25°)\sin(35°)\sin(85°)$

$$= \frac{1}{4}(4\sin(60° - 25)\sin(25°)\sin(60° + 25°))$$

$$= \frac{1}{4} \times \sin(3 \times 25°)$$

$$= \frac{1}{4} \times \sin(75°)$$

$$= \frac{1}{4} \times \cos(15°)$$

$$= \frac{1}{4} \times \left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right)$$

$$= \left(\frac{\sqrt{3} + 1}{8\sqrt{2}}\right)$$

$$= \left(\frac{\sqrt{3} + 1}{\sqrt{128}}\right)$$

Thus, $a = 3$, $b = 1$ and $c = 128$
Hence, the value of $(a + b + c + 2)$
$= 3 + 1 + 128 - 2$
$= 130$

12. We have
$\sqrt{3}\cot(20°) - 4\cos(20°)$

$$= \frac{\sqrt{3}\cos(20°) - 4\sin(20°)\cos(20°)}{\sin(20°)}$$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos(20°) - 2\sin(20°)\cos(20°)\right)}{\sin(20°)}$$

$$= \frac{2(\sin(60°)\cos(20°) - \sin(40°))}{\sin(20°)}$$

$$= \frac{(2\sin(60°)\cos(20°) - 2\sin(40°))}{\sin(20°)}$$

$$= \frac{(\sin(80°) + \sin(40°) - 2\sin(40°))}{\sin(20°)}$$

$$= \frac{(\sin(80°) - \sin(40°))}{\sin(20°)}$$

$$= \frac{2\cos(60°)\sin(20°)}{\sin(20°)}$$

$$= 1$$

13. We have
$\sin(2°) + \sin(4°) + \sin(6°) + \sin(8°) + \ldots + \sin(180°)$
$= \sin(2°) + \sin(2° + 2°) + \sin(2° + 2.2°)$
$+ \sin(2° + 3.2°) + \ldots + (2° + (90 - 1)2°)$

$$= \frac{\sin\left(\frac{90° \cdot 2°}{2}\right)}{\sin\left(\frac{2°}{2}\right)} \times \sin\left(2° + (90° - 1)\frac{2°}{2}\right)$$

$$= \frac{1}{\sin(1°)} \times \sin(91°)$$

$$= \frac{\cos(1°)}{\sin(1°)}$$

$= \cot(1°)$

14. We have

$$\sin\left(\frac{\pi}{2013}\right) + \sin\left(\frac{\pi}{2013} + \frac{2\pi}{2013}\right) + \sin\left(\frac{\pi}{2013} + \frac{2.2\pi}{2013}\right) + \ldots\ldots + \sin\left(\frac{\pi}{2013} + \frac{1006.2\pi}{2013}\right)$$

$$= \frac{\sin\left(2013 \cdot \left(\frac{\frac{2\pi}{2013}}{2}\right)\right)}{\sin\left(\frac{\frac{2\pi}{2013}}{2}\right)} \times \sin\left(\frac{\pi}{2013} + (2012)\left(\frac{\frac{2\pi}{2013}}{2}\right)\right)$$

$= 0$

15. Put $y = \sin(4\theta)$

Then $\tan(y) = \dfrac{1 + \sqrt{1 + \sin(4\theta)}}{1 + \sqrt{1 - \sin(4\theta)}}$

$$\Rightarrow \quad \tan(y) = \frac{1 + \sqrt{(\cos(2\theta) + \sin(2\theta))^2}}{1 + \sqrt{(\cos(2\theta) - \sin(2\theta))^2}}$$

$$\Rightarrow \quad \tan(y) = \frac{1 + \cos(2\theta) + \sin(2\theta)}{1 + (\cos(2\theta) - \sin(2\theta))}$$

$$\Rightarrow \quad \tan(y) = \frac{2\cos^2(\theta) + \sin(2\theta)}{2\cos^2(\theta) - \sin(2\theta)}$$

$$\Rightarrow \quad \tan(y) = \frac{2\cos^2(\theta) + 2\sin(\theta)\cos(\theta)}{2\cos^2(\theta) - 2\sin(\theta)\cos(\theta)}$$

$$\Rightarrow \quad \tan(y) = \frac{\cos(\theta) + \sin(\theta)}{\cos(\theta) - \sin(\theta)}$$

$$\Rightarrow \quad \tan(y) = \tan\left(\frac{\pi}{4} - \theta\right)$$

$$\Rightarrow \quad y = \frac{\pi}{4} - \theta$$

$\Rightarrow \quad 4y = \pi - 4\theta$
$\Rightarrow \quad \sin(4y) = \sin(\pi - 4\theta) = \sin(4\theta) = y$

Hence, the result.

16. We have

$$\tan\left(\frac{\pi}{4} + \alpha\right) = \frac{1 + \tan\alpha}{1 - \tan\alpha} = \frac{\cos\alpha + \sin\alpha}{\cos\alpha - \sin\alpha}$$

$$\Rightarrow \quad \tan^2\left(\frac{\pi}{4} + \alpha\right) = \frac{1 + \sin 2\alpha}{1 - \sin 2\alpha}$$

Thus, $\tan\left(\dfrac{\pi}{4} + \dfrac{y}{2}\right) = \tan^3\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$,

$$\Rightarrow \quad \frac{1 + \sin y}{1 - \sin y} = \frac{(1 + \sin x)^3}{(1 - \sin x)^3}$$

Applying componendo and dividendo, we get,

$$\Rightarrow \quad \frac{2\sin y}{2} = \frac{2(3\sin x + \sin^3 x)}{2(1 + 3\sin^2 x)}$$

$$\Rightarrow \quad \sin y = \frac{(3\sin x + \sin^3 x)}{(1 + 3\sin^2 x)}$$

Hence, the result.

17. Given, $\tan\beta = \dfrac{\tan\alpha + \tan\gamma}{1 + \tan\alpha\tan\gamma}$

$$= \frac{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}{\cos\alpha\cos\gamma + \sin\alpha\sin\gamma}$$

$$= \frac{\sin(\alpha + \gamma)}{\cos(\alpha - y)}$$

Thus,

$$\sin(2\beta) = \frac{2\tan\beta}{1+\tan^2\beta}$$

$$= \frac{2\left(\frac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}\right)}{1+\frac{\sin^2(\alpha+\gamma)}{\cos^2(\alpha-\gamma)}}$$

$$= \frac{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)}$$

$$= \frac{\sin((\alpha+\gamma)+(\alpha-\gamma)) + \sin((\alpha+\gamma)-(\alpha-\gamma))}{1+\sin^2(\alpha+\gamma)-\sin^2(\alpha-\gamma)}$$

$$= \frac{\sin(2\alpha)+\sin(2\gamma)}{1+\sin(\alpha+\gamma+\alpha-\gamma)\sin(\alpha+\gamma-\alpha+\gamma)}$$

$$= \frac{\sin(2\alpha)+\sin(2\gamma)}{1+\sin(2\alpha)\sin(2\gamma)}$$

Hence, the result.

18. We have

$16\sin^2(27°) = 8\times 2\sin^2(27°)$

$= 8\times(1-\cos(54°))$

$= 8\times\left(1-\frac{\sqrt{10-2\sqrt{5}}}{4}\right)$

$= (8-2\sqrt{10-2\sqrt{5}})$

$= (8-2\sqrt{(5+\sqrt{5})(3-\sqrt{5})})$

$= ((5+\sqrt{5})+(3-\sqrt{5})$

$\quad -2\sqrt{(5+\sqrt{5})}\sqrt{(3-\sqrt{5})})$

$= (\sqrt{(5+\sqrt{5})}-\sqrt{(3-\sqrt{5})})^2$

$\Rightarrow 4\sin(27°) = (\sqrt{(5+\sqrt{5})}-\sqrt{(3-\sqrt{5})})$

Thus, $a = 5$, $b = 5$, $c = 3$ and $d = 5$

Now, $a+b+c+d+2$

$= 5+5+3+5+2$

$= 20.$

19. We have $(1+\sin\theta)(1+\cos\theta) = \frac{5}{4}$

$\Rightarrow \quad 1+\sin\theta+\cos\theta+\sin\theta\cos\theta = \frac{5}{4}$

$\Rightarrow \quad 1+t+\left(\frac{t^2-1}{2}\right) = \frac{5}{4}$

$(\sin\theta+\cos\theta = t$, say)

$\Rightarrow \quad t+\left(\frac{t^2-1}{2}\right) = \frac{1}{4}$

$\Rightarrow \quad t^2+2t-1 = \frac{1}{2}$

$\Rightarrow \quad 2t^2+4t-3 = 0$

$t = \frac{-4\pm\sqrt{16+24}}{4}$

$\Rightarrow \quad = \frac{-4\pm 2\sqrt{10}}{4} = -1\pm\frac{1}{2}\sqrt{10}$

$\Rightarrow \quad t = -1+\frac{1}{2}\sqrt{10}$

$\Rightarrow \quad \sin\theta+\cos\theta = -1+\frac{1}{2}\sqrt{10}$

Now, $(1-\sin\theta)(1-\cos\theta)$

$= 1-\sin\theta-\cos\theta+\sin\theta\cos\theta$

$= 1-(\sin\theta+\cos\theta)+\sin\theta\cos\theta$

$= 1-\left(-1+\frac{\sqrt{10}}{2}\right)+\frac{1}{2}\left(\frac{10}{4}-\sqrt{10}\right)$

$= \left(2+\frac{5}{4}\right)-\sqrt{10}$

$= \left(\frac{13}{4}-\sqrt{10}\right)$

20. We have $3\sin x+4\cos x = 5$

Let $y = 3\cos x-4\sin x$

Now, $y^2+5^2$

$= (3\cos x-4\sin x)^2+(3\sin x+4\cos x)^2$

$= 9\cos^2 x+16\sin^2 x-24\sin x\cos x$

$\quad +9\sin^2 x+16\cos^2 x+24\sin x\cos x$

$\Rightarrow \quad y^2+25 = 25$

$\Rightarrow \quad y^2 = 0$

$\Rightarrow \quad y = 0$

$\Rightarrow \quad 3\cos x-4\sin x = 0$

$\Rightarrow \quad 3\cos x = 4\sin x$

$\Rightarrow \quad \tan x = 3/4$

Hence, the value of $2\sin x+\cos x+4\tan x$

$= 2\left(\frac{3}{5}\right)+\left(\frac{4}{5}\right)+4\left(\frac{3}{4}\right) = 2+3 = 5$

21. We have $\cos A = \tan B$

$\Rightarrow \quad \cos^2 A = \tan^2 B$

$\Rightarrow \quad \cos^2 A = \sec^2 B-1$

$\Rightarrow \quad 1+\cos^2 A = \sec^2 B$

$\Rightarrow \quad 1+\cos^2 A = \sec^2 B = \cot^2 C$

$\Rightarrow \quad 1+\cos^2 A = \cot^2 C$

$\Rightarrow \quad 2-\sin^2 A = \frac{\cos^2 C}{\sin^2 C} = \frac{\cos^2 C}{1-\cos^2 C}$

$\Rightarrow \quad 2-\sin^2 A = \frac{\tan^2 A}{1-\tan^2 A}$

$\Rightarrow \quad 2-\sin^2 A = \frac{\sin^2 A}{\cos^2 A-\sin^2 A}$

$\Rightarrow \quad 2-\sin^2 A = \frac{\sin^2 A}{1-2\sin^2 A}$

$\Rightarrow \quad 2 - 4\sin^2 A - \sin^2 A + 2\sin^4 A = \sin^2 A$

$\Rightarrow \quad 2\sin^4 A - 6\sin^2 A + 2 = 0$

$\Rightarrow \quad \sin^4 A - 3\sin^2 A + 1 = 0$

$$\Rightarrow \quad \sin^2 A = \frac{3 \pm \sqrt{9-4}}{2}$$

$$\Rightarrow \quad \sin^2 A = \frac{3 \pm \sqrt{5}}{2}$$

$$\Rightarrow \quad \sin^2 A = \frac{6 \pm 2\sqrt{5}}{4}$$

$$\Rightarrow \quad \sin^2 A = \left(\frac{\sqrt{5}-1}{2}\right)^2$$

$$\Rightarrow \quad \sin A = \left(\frac{\sqrt{5}-1}{2}\right)$$

$$\Rightarrow \quad \sin A = 2\left(\frac{\sqrt{5}-1}{4}\right) = 2\sin(18°)$$

Similarly, we can prove that,

$\sin B = 2\sin(18°) = \sin C$

22. We have

$$\tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{3\pi}{16}\right) + \tan^2\left(\frac{5\pi}{16}\right) + \tan^2\left(\frac{7\pi}{16}\right)$$

$$= \tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{3\pi}{16}\right) + \tan^2\left(\frac{\pi}{2} - \frac{3\pi}{16}\right) + \tan^2\left(\frac{\pi}{2} - \frac{\pi}{16}\right)$$

$$= \tan^2\left(\frac{\pi}{16}\right) + \cot^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{3\pi}{16}\right) + \cot^2\left(\frac{3\pi}{16}\right)$$

$$= \left(\tan\left(\frac{\pi}{16}\right) + \cot\left(\frac{\pi}{16}\right)\right)^2 + \left(\tan\left(\frac{3\pi}{16}\right) + \cot\left(\frac{3\pi}{16}\right)\right)^2 - 4$$

$$= \frac{1}{\sin^2\left(\frac{\pi}{16}\right)\cos^2\left(\frac{\pi}{16}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{16}\right)\cos^2\left(\frac{3\pi}{16}\right)} - 4$$

$$= \frac{4}{\left(2\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)\right)^2} + \frac{4}{\left(2\sin\left(\frac{3\pi}{16}\right)\cos\left(\frac{3\pi}{16}\right)\right)^2} - 4$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\sin^2\left(\frac{3\pi}{8}\right)} - 4$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\sin^2\left(\frac{\pi}{2} - \frac{\pi}{8}\right)} - 4$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\cos^2\left(\frac{\pi}{8}\right)} - 4$$

$$= 4\left(\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\cos^2\left(\frac{\pi}{8}\right)}\right) - 4$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)} - 4$$

$$= \frac{8}{\left(2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right)^2} - 4$$

$$= \frac{8}{\sin^2\left(\frac{\pi}{4}\right)} - 4$$

$$= \frac{8}{\frac{1}{2}} - 4 = 16 - 4 = 12$$

23. We have $\sin(1°) \cdot \sin(2°) \cdot \sin(3°) \ldots \sin(89°)$

$= \sin(1°) \cdot \sin(2°) \cdot \sin(3°) \ldots \sin(44°) \cdot \sin(45°)$
$\sin(46°) \cdot \sin(47°) \cdot \sin(48°) \ldots \sin(89°)$

$= \sin(1°) \cdot \sin(2°) \cdot \sin(3°) \ldots \sin(44°) \cdot \sin(45°)$
$\cos(44°) \cdot \cos(43°) \cdot \cos(42°) \ldots \cos(1°)$

$= \sin(1°) \cdot \sin(2°) \cdot \sin(3°) \ldots \sin(44°) \cdot \sin(45°)$
$\cos(1°) \cdot \cos(2°) \cdot \cos(3°) \ldots \cos(44°)$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{2^{44}}(\sin(2°)\sin(4°)\sin(6°)\ldots\sin(88°))$$

$$= \frac{1}{2^{89/2}}(\sin(2°)\sin(4°)\sin(6°)\ldots\sin(88°))$$

Thus, $\sin(1°) \cdot \sin(3°) \cdot \sin(5°) \ldots \sin(89°)$

$$6 = \frac{1}{2^{89/2}}$$

Therefore, $n = \frac{89}{2}$

24. We have $(1 + \tan(1°))(1 + \tan(2°))(1 + \tan(3°)) \ldots (1 + \tan(45°)) = 2^n$

$\Rightarrow \quad (1 + \tan(1°))(1 + \tan(44°))(1 + \tan(2°))(1 + \tan(43°)) \ldots (1 + \tan(22°))(1 + \tan(23°)) \times (1 + \tan(45°)) = 2^n$

$\Rightarrow \quad 2^{22} \times (1 + 1) = 2^n$

$\Rightarrow \quad 2^n = 2^3$

$\Rightarrow \quad n = 23$

Hence, the value of $n$ is 23.

25. We have

$$\cos\left(\frac{7\pi}{10}\right) = \cos\left(\pi - \frac{3\pi}{10}\right) = -\cos\left(\frac{3\pi}{10}\right)$$

$$\cos\left(\frac{9\pi}{10}\right) = \cos\left(\pi - \frac{\pi}{10}\right) = -\cos\left(\frac{\pi}{10}\right)$$

Therefore,

$$\left(1+\cos\left(\frac{\pi}{10}\right)\right)\left(1+\cos\left(\frac{3\pi}{10}\right)\right)\left(1+\cos\left(\frac{7\pi}{10}\right)\right)\left(1+\cos\left(\frac{9\pi}{10}\right)\right)$$

$$= \left(1+\cos\left(\frac{\pi}{10}\right)\right)\left(1+\cos\left(\frac{3\pi}{10}\right)\right)\left(1-\cos\left(\frac{3\pi}{10}\right)\right)\left(1-\cos\left(\frac{\pi}{10}\right)\right)$$

$$= \left(1-\cos^2\left(\frac{\pi}{10}\right)\right)\left(1-\cos^2\left(\frac{3\pi}{10}\right)\right)$$

$$= \sin^2\left(\frac{\pi}{10}\right)\sin^2\left(\frac{3\pi}{10}\right)$$

$$= \sin^2(18°)\sin^2(45°)$$

$$= \left(\frac{\sqrt{5}-1}{4}\right)^2\left(\frac{\sqrt{5}+1}{4}\right)^2$$

$$= \left(\frac{5-1}{16}\right)^2 = \frac{1}{16}$$

26. We have

$\cos(60°)\cos(36°)\cos(42°)\cos(78°)$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}\right)\cdot\frac{1}{2}(2\cos 78°\cos 42°)$$

$$= \frac{1}{4}\left(\frac{\sqrt{5}+1}{4}\right)(\cos 120° + \cos 36°)$$

$$= \frac{1}{4}\left(\frac{\sqrt{5}+1}{4}\right)\left(-\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right)$$

$$= \frac{1}{4}\left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)$$

$$= \frac{(5-1)}{64} = \frac{4}{64} = \frac{1}{16}$$

27. We have

$$f_6(\theta) = \sin^6\theta + \cos^6\theta$$
$$= 1 - 3\sin^2\theta\cos^2\theta$$

Also, $f_4(\theta) = \sin^4\theta + \cos^4\theta$

$$= 1 - 2\sin^2\theta\cos^2\theta$$

Now, $\frac{1}{6}f_6(\theta) - \frac{1}{4}f_4(\theta)$

$$= \frac{1}{6}(1-3\sin^2\theta\cos^2\theta) - \frac{1}{4}(1-2\sin^2\theta\cos^2\theta)$$

$$= \frac{1}{6} - \frac{1}{2}\sin^2\theta\cos^2\theta - \frac{1}{4} + \frac{1}{2}\sin^2\theta\cos^2\theta$$

$$= \frac{1}{6} - \frac{1}{4}$$

$$= -\frac{1}{12}$$

28. We have $\sin^2(\sin\theta) + \cos^2(\cos\theta)$

$= \sin^2(\cos\theta) + \cos^2(\cos\theta) + \sin^2(\sin\theta) - \sin^2(\cos\theta)$

$= (\sin^2(\cos\theta) + \cos^2(\cos\theta)) + \sin^2(\sin\theta) - \sin^2(\cos\theta)$

$= 1 + (\sin^2(\sin\theta) - \sin^2(\cos\theta))$

Maximum value of $f(\theta)$

$$= 1 + \left(\sin^2\left(\sin\left(\frac{\pi}{2}\right)\right) - \sin^2\left(\cos\left(\frac{\pi}{2}\right)\right)\right)$$

$= 1 + \sin^2(1)$

Minimum value of $f(\theta)$

$= 1 + (\sin^2(\sin(\theta)) - \sin^2(\cos(\theta)))$

$= 1 - \sin^2(1)$

29. We have $f(\theta) = (3\sin(\theta) - 4\cos(\theta) - 10)(3\sin(\theta) + 4\cos(\theta) - 10)$

$= (9\sin^2(\theta) - 16\cos^2(\theta)) - 10(3\sin\theta + 4\cos\theta) - 10(3\sin\theta - 4\cos\theta)$

$= (9\sin^2(\theta) - 16\cos^2(\theta)) - 10(3\sin\theta + 4\cos\theta) + 3\sin\theta - 4\cos\theta)$

$= (9\sin^2(\theta) - 16\cos^2(\theta)) - 60\sin(\theta)$

$= 25\sin^2\theta - 60\sin(\theta) - 16$

$= (5\sin\theta - 6)^2 - 36 - 16$

$= (5\sin\theta - 6)^2 - 52$

Hence, the minimum value of

$f(\theta) = 121 - 52 = 69$

30. Now, $0 < \sin^{2010}\theta \le \sin^2\theta$ …(i)

and $\quad 0 < \cos^{2014}\theta \le \cos^2\theta$ …(ii)

Adding (i) and (ii), we get,

$0 < \sin^{2010}\theta + \cos^{2014}\theta \le \sin^2\theta + \cos^2\theta$

$\Rightarrow \quad 0 < A \le 1$

Thus, the range of $A = (0, 1]$

31. We have $\frac{\sin A}{\sin B} = p, \frac{\cos A}{\cos B} = q$

$$\Rightarrow \quad \frac{\sin A}{\cos A}/\frac{\sin B}{\cos B} = \frac{p}{q}$$

$$\Rightarrow \quad \frac{\tan A}{\tan B} = \frac{p}{q}$$

$$\Rightarrow \quad \frac{\tan A}{p} = \frac{\tan B}{q} = \lambda$$

Also, $\dfrac{\sin A}{\sin B} = p, \dfrac{\cos A}{\cos B} = q$

$\Rightarrow \quad \dfrac{\sin A \cos A}{\sin B \cos B} = pq$

$\Rightarrow \quad \dfrac{2 \sin A \cos A}{2 \sin B \cos B} = pq$

$\Rightarrow \quad \dfrac{\sin 2A}{\sin 2B} = pq$

$\Rightarrow \quad \dfrac{2 \tan A}{1 + \tan^2 A} / \dfrac{2 \tan B}{1 + \tan^2 B} = pq$

$\Rightarrow \quad \dfrac{2p\lambda}{1 + p^2\lambda^2} / \dfrac{2q\lambda}{1 + q^2\lambda^2} = pq$

$\Rightarrow \quad \dfrac{p}{(1 + p^2\lambda^2)} \times \dfrac{(1 + q^2\lambda^2)}{q} = pq$

$\Rightarrow \quad \dfrac{(1 + q^2\lambda^2)}{(1 + p^2\lambda^2)} = q^2$

$\Rightarrow \quad (1 + q^2\lambda^2) = q^2(1 + p^2\lambda^2)$

$\Rightarrow \quad \lambda^2(1 - p^2)q^2 = q^2 - 1$

$\Rightarrow \quad \lambda^2 = \dfrac{(q^2 - 1)}{(1 - p^2)q^2}$

$\Rightarrow \quad \lambda = \pm \dfrac{1}{q}\sqrt{\dfrac{(q^2 - 1)}{(1 - p^2)}}$

Therefore, $\tan A = \pm \dfrac{p}{q}\sqrt{\dfrac{(q^2 - 1)}{(1 - p^2)}}$

and $\tan B = \pm \sqrt{\dfrac{(q^2 - 1)}{(1 - p^2)}}$

32. We have

$$\frac{\tan(\alpha - \beta)}{\tan \alpha} + \frac{\sin^2\gamma}{\sin^2\alpha} = 1$$

$\Rightarrow \quad \dfrac{\sin^2\gamma}{\sin^2\alpha} = 1 - \dfrac{\tan(\alpha - \beta)}{\tan \alpha}$

$\Rightarrow \quad \dfrac{\sin^2\gamma}{\sin^2\alpha} = \dfrac{1}{\tan \alpha}(\tan \alpha - \tan(\alpha - \beta))$

$\Rightarrow \quad \sin^2\gamma = \dfrac{\sin^2\alpha}{\tan \alpha}\left(\tan \alpha - \dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}\right)$

$\Rightarrow \quad \sin^2\gamma = \dfrac{\sin^2\alpha}{\tan \alpha}\left(\dfrac{\tan \beta(1 + \tan^2\alpha)}{(1 + \tan \alpha \tan \beta)}\right)$

$\Rightarrow \quad \sin^2\gamma(1 + \tan \alpha \tan \beta) = \dfrac{\sin^2\alpha}{\tan \alpha}\left(\dfrac{\tan \beta}{\cos^2\alpha}\right)$

$\Rightarrow \quad \sin^2\gamma(1 + \tan \alpha \tan \beta) = \dfrac{\sin^2\alpha}{\cos^2\alpha}\left(\dfrac{\tan \beta}{\tan \alpha}\right)$

$\Rightarrow \quad \sin^2\gamma(1 + \tan \alpha \tan \beta) = \tan^2\alpha\left(\dfrac{\tan \beta}{\tan \alpha}\right)$

$\Rightarrow \quad \sin^2\gamma(1 + \tan \alpha \tan \beta) = \tan \alpha \tan \beta$

$\Rightarrow \quad \sin^2\gamma = \tan \alpha \tan \beta(1 - \sin^2\gamma)$

$\Rightarrow \quad \dfrac{\sin^2\gamma}{\cos^2\gamma} = \tan \alpha \tan \beta$

$\Rightarrow \quad \tan^2\gamma = \tan \alpha \tan \beta$

Hence, the result.

33. We have $\tan\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{1 - e}{1 + e}} \tan\left(\dfrac{\varphi}{2}\right)$

$\Rightarrow \quad \tan^2\left(\dfrac{\theta}{2}\right) = \left(\dfrac{1 - e}{1 + e}\right)\tan^2\left(\dfrac{\varphi}{2}\right)$

$\Rightarrow \quad \dfrac{1}{\tan^2\left(\dfrac{\varphi}{2}\right)} = \left(\dfrac{1 - e}{1 + \mathrm{e}}\right)\dfrac{1}{\tan^2\left(\dfrac{\theta}{2}\right)}$

$\Rightarrow \quad \dfrac{1 - \tan^2\left(\dfrac{\varphi}{2}\right)}{1 + \tan^2\left(\dfrac{\varphi}{2}\right)} = \dfrac{1 - e - \tan^2\left(\dfrac{\theta}{2}\right) - e\tan^2\left(\dfrac{\theta}{2}\right)}{1 - e + \tan^2\left(\dfrac{\theta}{2}\right) + e\tan^2\left(\dfrac{\theta}{2}\right)}$

$\Rightarrow \quad \cos\varphi = \dfrac{\left(1 - \tan^2\left(\dfrac{\theta}{2}\right)\right) - e\left(1 + \tan^2\left(\dfrac{\theta}{2}\right)\right)}{\left(1 + \tan^2\left(\dfrac{\theta}{2}\right)\right) - e\left(1 - \tan^2\left(\dfrac{\theta}{2}\right)\right)}$

$\Rightarrow \quad \cos\varphi = \dfrac{\dfrac{\left(1 - \tan^2\left(\dfrac{\theta}{2}\right)\right)}{\left(1 + \tan^2\left(\dfrac{\theta}{2}\right)\right)} - e}{1 - e\dfrac{\left(1 - \tan^2\left(\dfrac{\theta}{2}\right)\right)}{\left(1 + \tan^2\left(\dfrac{\theta}{2}\right)\right)}}$

$\Rightarrow \quad \cos\varphi = \dfrac{\cos\theta - e}{1 + e\cos\theta}$

Hence, the result.

34. We have $\cos\theta = \dfrac{a\cos\varphi + b}{a + b\cos\varphi}$

$\Rightarrow \quad \dfrac{1 - \tan^2\left(\dfrac{\theta}{2}\right)}{1 + \tan^2\left(\dfrac{\theta}{2}\right)} = \dfrac{a\left(\dfrac{1 - \tan^2(\varphi/2)}{1 + \tan^2(\varphi/2)}\right) + b}{a + b\left(\dfrac{1 - \tan^2(\varphi/2)}{1 + \tan^2(\varphi/2)}\right)}$

$\Rightarrow \quad \dfrac{1 - \tan^2\left(\dfrac{\theta}{2}\right)}{1 + \tan^2\left(\dfrac{\theta}{2}\right)} = \dfrac{(a + b) - (a - b)\tan^2(\varphi/2)}{(a + b) + (a - b)\tan^2(\varphi/2)}$

$$\Rightarrow \quad \frac{2}{-2\tan^2\left(\frac{\theta}{2}\right)} = \frac{2(a+b)}{-2(a-b)\tan^2(\varphi/2)}$$

$$\Rightarrow \quad \frac{1}{\tan^2\left(\frac{\theta}{2}\right)} = \frac{(a+b)}{(a-b)\tan^2(\varphi/2)}$$

$$\Rightarrow \quad \frac{\tan^2(\varphi/2)}{\tan^2(\theta/2)} = \frac{(a+b)}{(a-b)}$$

$$\Rightarrow \quad \tan^2(\theta/2) = \frac{(a+b)}{(a-b)}\tan^2(\varphi/2)$$

$$\Rightarrow \quad \tan(\theta/2) = \sqrt{\frac{(a+b)}{(a-b)}}\tan(\varphi/2)$$

Hence, the result.

35. We have

$$a^2 + b^2 = (\sin x + \sin y)^2 + (\cos x + \cos y)^2$$
$$= 2 + 2(\cos x \cos y + \sin x \sin y)$$
$$= 2 + 2\cos(x - y)$$

$\Rightarrow \cos(x-y) = (a^2 + b^2 - 2)/2$

As we know

$$\tan^2\left(\frac{A}{2}\right) = \frac{1-\cos A}{1+\cos A}$$

$$\Rightarrow \quad \tan^2\left(\frac{x-y}{2}\right) = \frac{1-\cos(x-y)}{1+\cos(x-y)}$$

$$\Rightarrow \quad \tan^2\left(\frac{x-y}{2}\right) = \frac{1-(a^2+b^2-2)/2}{1+(a^2+b^2-2)/2}$$

$$\Rightarrow \quad \tan^2\left(\frac{x-y}{2}\right) = \frac{2-(a^2+b^2-2)}{2+(a^2+b^2-2)}$$

$$\Rightarrow \quad \tan^2\left(\frac{x-y}{2}\right) = \frac{4-a^2-b^2}{a^2+b^2}$$

$$\Rightarrow \quad \tan\left(\frac{x-y}{2}\right) = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

Hence, the result.

36. As we know

$$\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta}$$

Put $\theta = \left(22\frac{1}{2}\right)^\circ$

$$\Rightarrow \quad \tan\left(22\frac{1}{2}^\circ\right) = \frac{1-\cos 45^\circ}{\sin 45^\circ}$$

$$= \frac{1-1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2}-1$$

Let $A = 11\frac{1^\circ}{4}$

$$\Rightarrow \quad 2A = 22\frac{1^\circ}{2}$$

$$\Rightarrow \quad \tan(2A) = \tan\left(22\frac{1^\circ}{2}\right) = \sqrt{2}-1$$

$$\Rightarrow \quad \frac{2\tan A}{1-\tan^2 A} = \sqrt{2}-1$$

$$\Rightarrow \quad \frac{2y}{1-y^2} = \sqrt{2}-1$$

$$\Rightarrow \quad 1-y^2 = \frac{2}{(\sqrt{2}-1)}y$$

$$\Rightarrow \quad 1-y^2 = 2(\sqrt{2}+1)y$$

$$\Rightarrow \quad y^2 + 2(\sqrt{2}+1)y - 1 = 0$$

$$\Rightarrow \quad y = \frac{-2(\sqrt{2}+1) \pm \sqrt{4(\sqrt{2}+1)^2+4}}{2}$$

$$\Rightarrow \quad y = -(\sqrt{2}+1) + \sqrt{(\sqrt{2}+1)^2+1}$$

$$\Rightarrow \quad y = -(\sqrt{2}+1) + \sqrt{4+2\sqrt{2}}$$

$$\Rightarrow \quad y = \sqrt{4+2\sqrt{2}} - (\sqrt{2}+1)$$

$$\Rightarrow \quad \tan\left(11\frac{1}{4}^\circ\right) = \sqrt{4+2\sqrt{2}} - (\sqrt{2}+1)$$

Thus, $a = 4$, $b = 2$, $c = 2$, $d = 1$

Hence, the value of $(a + b + c + d + 1)$

$= 4 + 2 + 2 + 1 + 1$

$= 10$

37. Given equation is $\frac{\cos\theta}{a} + \frac{\sin\theta}{b} = \frac{1}{c}$

$$\Rightarrow \quad bc\cos\theta + ac\sin\theta = ab$$

$$\Rightarrow \quad bc\left(\frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}\right) + ac\left(\frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}\right) = ab$$

$$\Rightarrow \quad bc(1-\tan^2(\theta/2)) + 2ac\tan(\theta/2) = ab(1+\tan^2(\theta/2))$$

$$\Rightarrow \quad (ab+bc)\tan^2(\theta/2) - 2ac\tan(\theta/2) + (ab-bc) = 0$$

Let its roots be $\tan(\alpha/2)$ and $\tan(\beta/2)$

$$\tan(\alpha/2) + \tan(\beta/2) = \frac{2ac}{b(a+c)}$$

and $\tan(\alpha/2)\cdot\tan(\beta/2) = \frac{(a-c)}{(a+c)}$

Now, $\tan\left(\frac{\alpha+\beta}{2}\right)$

$$= \frac{\tan(\alpha/2)+\tan(\beta/2)}{1-\tan(\alpha/2)\cdot\tan(\beta/2)}$$

$$= \frac{2ac/b(a+c)}{1-(a-c)/(a+c)}$$

$$= \frac{2ac/b}{a+c-a+c}$$

$$= \frac{2ac}{2bc} = \frac{a}{b}$$

Thus, $\cot\left(\frac{\alpha+\beta}{2}\right)=\frac{b}{a}$.

38. Let $\frac{\pi}{14}=\theta$

$\Rightarrow \quad 7\theta=\frac{\pi}{2}$

$\Rightarrow \quad 4\theta=\frac{\pi}{2}-3\theta$

$\Rightarrow \quad \sin(4\theta)=\sin\left(\frac{\pi}{2}-3\theta\right)=\cos(3\theta)$

$\Rightarrow \quad \sin(4\theta)=\cos(3\theta)$

$\Rightarrow \quad 2\sin(2\theta)\cos(2\theta)=4\cos^3\theta-3\cos\theta$

$\Rightarrow \quad 4\sin\theta\cos\theta(1-2\sin^2\theta)=\cos\theta(4\cos^3\theta-3)$

$\Rightarrow \quad 4\sin\theta(1-2\sin^2\theta)=(1-4\sin^2\theta)$

$\Rightarrow \quad 4\sin\theta-8\sin^3\theta=(1-4\sin^2\theta)$

$\Rightarrow \quad 8\sin^3\theta-4\sin^2\theta-4\sin\theta+1=0$

put $\sin\theta=x$

Hence, the required equation is

$8x^3-4x^2-4x+1=0$

39. Let $x=\tan A$, $y=\tan B$, $z=\tan C$

Given,

$\tan A+\tan B+\tan C=\tan A\cdot\tan B\cdot\tan C$

$\Rightarrow \quad \tan A+\tan B=-\tan C+\tan A\cdot\tan B\cdot\tan C$

$\Rightarrow \quad \tan A+\tan B=-\tan C(1-\tan A\cdot\tan B)$

$\Rightarrow \quad \frac{\tan A+\tan B}{(1-\tan A\cdot\tan B)}=-\tan C$

$\Rightarrow \quad \tan(A+B)=-\tan C$

$\Rightarrow \quad \tan(A+B)=\tan(\pi-C)$

$\Rightarrow \quad (A+B)=(\pi-C)$

$\Rightarrow \quad A+B+C=\pi$

$\Rightarrow \quad 2A+2B+2C=2\pi$

$\Rightarrow \quad 2A+2B=2\pi-2C$

$\Rightarrow \quad \tan(2A+2B)\tan(2\pi-2C)$

$\Rightarrow \quad \tan(2A+2B)=-\tan 2C$

$\Rightarrow \quad \frac{\tan 2A+\tan 2B}{1-\tan 2A\cdot\tan 2B}=-\tan 2C$

$\Rightarrow \quad \tan 2A+\tan 2B+\tan 2C=\tan 2A\cdot\tan 2B\cdot\tan 2C$

$\Rightarrow \quad \frac{2\tan A}{1-\tan^2 A}+\frac{2\tan B}{1-\tan^2 B}+\frac{2\tan C}{1-\tan^2 C}$

$=\frac{2\tan A}{1-\tan^2 A}\cdot\frac{2\tan B}{1-\tan^2 B}\cdot\frac{2\tan C}{1-\tan^2 C}$

$\Rightarrow \quad \frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}$

$=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}$

***Note:*** *No questions asked in 1984.*

40. Let $a=\cos\alpha+i\sin\alpha$, $b=\cos\beta+i\sin\beta$ and $c=\cos\gamma+i\sin\gamma$

Then $a+b+c$

$=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)$

$=0+i\cdot 0=0$

Therefore, $a^3+b^3+c^3=3abc$

$\Rightarrow \quad (\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\gamma+i\sin\gamma)^3$

$=3(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)$

$=3(\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma))$

$\Rightarrow \quad (\cos 3\alpha+i\sin 3\alpha)+(\cos 3\beta+i\sin 3\beta)+(\cos 3\gamma+i\sin 3\gamma)$

$=3(\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma))$

$\Rightarrow \quad (\cos 3\alpha+\cos 3\beta+\cos 3\gamma)+i(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)$

$=3(\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma))$

Comparing the real and imaginary parts, we get,

$\cos(3\alpha)+\cos(3\beta)+\cos(3\gamma)=3\cos(\alpha+\beta+\gamma)$

and $\sin(3\alpha)+\sin(3\beta)+\sin(3\gamma)=3\sin(\alpha+\beta+\gamma)$

Hence, the result.

***Note:*** No questions asked in 1986, 1987, 1988.

41. We have $\tan 9°-\tan 27°-\tan 63°+\tan 81°$

$=(\tan 9°+\tan 81°)-(\tan 27°+\tan 63°)$

$=(\tan 9°+\cot 9°)-(\tan 27°+\cot 27°)$

$=\left(\frac{\sin 9°}{\cos 9°}+\frac{\cos 9°}{\sin 9°}\right)-\left(\frac{\sin 27°}{\cos 27°}+\frac{\cos 27°}{\sin 27°}\right)$

$=\frac{1}{\sin 9°\cos 9°}-\frac{1}{\sin 27°\cos 27°}$

$=\frac{1\times 2}{2\sin 9°\cos 9°}-\frac{1\times 2}{2\sin 27°\cos 27°}$

$=\frac{2}{\sin 18°}-\frac{2}{\sin 54°}$

$=\frac{2}{\left(\frac{\sqrt{5}-1}{4}\right)}-\frac{2}{\left(\frac{\sqrt{5}+1}{4}\right)}$

$=\frac{8}{(\sqrt{5}-1)}-\frac{8}{(\sqrt{5}+1)}$

$=8\left(\frac{\sqrt{5}+1-\sqrt{5}+1}{5-1}\right)$

$=\frac{16}{4}$

$=4$

42. We have $\cos x+5\sin\left(x-\frac{\pi}{6}\right)$

$$= \cos x + 5\left(\sin x \cos\left(\frac{\pi}{6}\right) - \cos x \sin\left(\frac{\pi}{6}\right)\right)$$

$$= \cos x + 5\left(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right)$$

$$= \left(1 - \frac{5}{2}\right)\cos x + \frac{5\sqrt{3}}{2}\sin x$$

$$= -\frac{3}{2}\cos x + \frac{5\sqrt{3}}{2}\sin x$$

Maximum value $= \sqrt{\frac{9}{4} + \frac{75}{4}} = \sqrt{\frac{84}{4}} = \sqrt{21}$

Minimum value $= -\sqrt{\frac{9}{4} + \frac{75}{4}} = -\sqrt{\frac{84}{4}} = -\sqrt{21}$

Thus, $a = -\sqrt{21}, b = \sqrt{21}$

***Note:*** No questions asked in 1990, 1991.

43. $A = \cos^2\theta + \sin^4\theta$

$$= \frac{1}{2}(2\cos^2\theta) + \frac{1}{4}(2\sin^2\theta)^2$$

$$= \frac{1}{2}(1 + \cos 2\theta) + \frac{1}{4}(1 - \cos 2\theta)^2$$

$$= \frac{1}{2}(1 + \cos 2\theta) + \frac{1}{4}(1 - 2\cos 2\theta + \cos^2 2\theta)$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4}\cos^2 2\theta$$

$$= \frac{3}{4} + \frac{1}{4}\cos^2 2\theta$$

Maximum Value $= \frac{3}{4} + \frac{1}{4} \cdot 1 = 1$

Minimum Value $= \frac{3}{4} + \frac{1}{4} \cdot 0 = \frac{3}{4}$

Hence, the range of $A$ is $\left[\frac{3}{4}, 1\right]$

44. Given $\sin A \sin B \sin C = p$

$\cos A \cos B \cos C = q$

Thus, $\tan A \tan B \tan C = \frac{p}{q}$

Also, $A + B + C = \pi$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\tan A + \tan B + \tan C = \frac{p}{q}$$

Also, $\tan A \tan B + \tan B \tan C + \tan C \tan A$

$$= \frac{1+q}{q}$$

Hence, the required equation is

$$x^3 - \left(\frac{p}{q}\right)x^2 + \left(\frac{1+q}{q}\right)x - \left(\frac{p}{q}\right) = 0$$

$$qx^3 - px^2 + (1 + q)x - p = 0$$

***Note:*** No questions asked in 1993, 1994.

45. Given $x = \cos 10° \cos 20° \cos 40°$

$$= \frac{1}{2\sin 10°}(2\sin 10° \cos 10°)\cos 20° \cos 40°$$

$$= \frac{1}{4\sin 10°}(2\sin 20° \cos 20°)\cos 40°$$

$$= \frac{1}{8\sin 10°}(2\sin 40° \cos 40°)$$

$$= \frac{1}{8\sin 10°}(\sin 80°)$$

$$= \frac{1}{8\sin 10°}(\cos 10°)$$

$$= \frac{1}{8}\cot 10°$$

***Note:*** No questions asked in 1996, 1997, 1998, 1999.

46. Let $y = 27^{\cos 2x} \cdot 81^{\sin 2x}$

$= 3^{3\cos 2x + 4\sin 2x}$

Max value of $y = 3^5 = 243$

Min value of $y = 3^{-5} = \frac{1}{243}$

The expression $y = 27^{\cos 2x} \cdot 81^{\sin 2x}$ is minimum when

Let $f(x) = 3\cos 2x + 4\sin 2x$

$f'(x) = -6\sin 2x + 8\cos 2x$

Now, $f'(x) = 0$ gives, $\tan 2x = -\frac{8}{6} = -\frac{3}{4}$

$\tan 2x = -\frac{3}{4} = \tan\alpha$

$2x = n\pi + \alpha$, where $\alpha = \tan^{-1}\left(-\frac{3}{4}\right)$

47. Given, $e^{i\theta - \log\cos(x - iy)} = 1$

$\Rightarrow$ $e^{i\theta} = \cos(x - iy)$

$\Rightarrow$ $(\cos\theta + i\sin\theta)$

$= \cos x \cos(iy) + \sin x \sin(iy)$

$= \cos x \cos(hy) + i \sin x \sin(hy)$

Thus, $\cos\theta = \cos x \cos(hy)$

and $\sin\theta = \sin x \sin(hy)$

Now, $\frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$

$\Rightarrow$ $\frac{\cos^2\theta}{1 - p^2} - \frac{\sin^2\theta}{p^2} = 1, \sin x = p$

$\Rightarrow$ $p^2(1 - p^2) = p^2\cos^2\theta + (p^2 - 1)\sin^2\theta$

$= p^2 - \sin^2\theta$

$\Rightarrow$ $p^4 = \sin^2\theta$

$\Rightarrow$ $p = \pm(\sin\theta)^{1/2}$

$\Rightarrow$ $\sin x = \pm(\sin\theta)^{1/2}$

$\Rightarrow$ $x = 2n\pi \pm (\sin\theta)^{1/2}$

Also, $\sin(hy) = \frac{\sin\theta}{\pm(\sin\theta)^{1/2}} = \pm(\sin\theta)^{1/2}$

$\Rightarrow$ $y = h^{-1}(\sin^{-1}(\pm(\sin\theta)^{1/2}))$

## Level IV

1. We have

$$\tan x = \tan(2x - x)$$
$$= \frac{\tan 2x - \tan x}{1 + \tan x \tan 2x}$$
$$\tan x(1 + \tan x \tan 2x) = \tan 2x - \tan x$$
$$(1 + \tan x \tan 2x) = \cot x(\tan 2x - \tan x)$$
$$\tan x \tan 2x = \cot x(\tan 2x - \tan x) - 1$$

Similarly,

$$\tan 2x \tan 3x = \cot x(\tan 3x - \tan 2x) - 1$$
$$\tan 3x \tan 4x = \cot x(\tan 4x - \tan 3x) - 1$$
$$\vdots$$
$$\tan nx \tan(n+1)x = \cot x(\tan(n+1)x \tan nx) - 1$$

Hence, the required sum is

$$= \cot x(\tan(n+1)x) - \tan x) - n$$
$$= \cot x(\tan(n+1)x) - (n+1)$$

2. We have

$$\operatorname{cosec} x + \operatorname{cosec} 2x + \operatorname{cosec} 4x + \ldots \text{ to } n \text{ terms}$$
$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{\sin(x/2)}{\sin x \cdot \sin(x/2)}$$
$$= \frac{\sin\left(x - \frac{x}{2}\right)}{\sin x \cdot \sin(x/2)}$$
$$= \frac{\sin x \cos(x/2) - \cos x \sin(x/2)}{\sin x \cdot \sin(x/2)}$$
$$= \cot(x/2) - \cot x$$

Similarly, $\operatorname{cosec}(2x) = \cot(x) - \cot(2x)$

$$\operatorname{cosec}(4x) = \cot(2x) - \cot(4x)$$
$$\vdots$$
$$\operatorname{cosec}(2^{n-1}x) = \cot(2^{n-2}x) - \cot(2^{n-1}x)$$

Adding all, we get,

$$\operatorname{cosec} x + \operatorname{cosec} 2x + \operatorname{cosec} 4x + \ldots \text{ to } n \text{ terms}$$
$$= \cot\left(\frac{x}{2}\right) - \cot(2^{n-1}x)$$

3. Now, $\cot(A)\cos(B) - 1 = \dfrac{\cos(A \pm B)}{\sin A \sin B}$

So, $\cot(16°)\cot(44°) - 1 = \dfrac{\cos(60°)}{\sin(16°)\sin(44°)}$

$$\cot(76°)\cot(44°) - 1 = \frac{\cos(120°)}{\sin(76°)\sin(44°)}$$
$$\cot(76°)\cot(16°) - 1 = \frac{\cos(60°)}{\sin(76°)\sin(16°)}$$

Now, LHS

$$= \frac{\cos(60°)}{\sin(16°)\sin(44°)} + \frac{\cos(120°)}{\sin(76°)\sin(44°)} - \frac{\cos(60°)}{\sin(76°)\sin(16°)}$$
$$= \frac{\cos(60°)\sin(76°) + \cos(120°)\sin(16°) - \cos(60°)\sin(44°)}{\sin(16°)\sin(55°)\sin(76°)}$$
$$= \frac{1}{2}\left[\frac{\sin(76°) - \sin(16°) - \sin(44°)}{\sin(16°)\sin(44°)\sin(76°)}\right]$$
$$= \frac{1}{2}\left[\frac{(\sin(76°) - \sin(16°)) - \sin(44°)}{\sin(16°)\sin(44°)\sin(76°)}\right]$$
$$= \frac{1}{2}\left[\frac{2\sin(30°)\cos(46°) - \sin(44°)}{\sin(16°)\sin(44°)\sin(76°)}\right]$$
$$= \frac{1}{2}\left[\frac{\cos(46°) - \sin(44°)}{\sin(16°)\sin(44°)\sin(76°)}\right]$$
$$= \frac{1}{2}\left[\frac{\cos(46°) - \cos(44°)}{\sin(16°)\sin(44°)\sin(76°)}\right]$$
$$= 0$$

Hence, the result.

5. As we know

$$\cos\theta \cdot \cos(2\theta) \cdot \cos(2^2\theta)\cos(2^3\theta) \ldots \cos(2^{n-1}\theta)$$
$$= \frac{\sin(2^n\theta)}{2^n \sin(\theta)}$$
$$= \frac{\sin(\pi + \theta)}{2^n \sin(\theta)}$$
$$= -\frac{\sin(\theta)}{2^n \sin(\theta)}$$
$$= -\frac{1}{2^n}$$

Hence, the value of

$$2^n \cos\theta \cdot \cos(2\theta) \cdot \cos(2^2\theta)\cos(2^3\theta) \ldots \cos(2^{n-1}\theta)$$
$$= -1$$

6. Let $y = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)$

$$y^2 = \left(\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)\right)^2$$
$$= \sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right) + \sin^2\left(\frac{8\pi}{7}\right)$$
$$+ 2\left(\sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right)\sin\left(\frac{8\pi}{7}\right) + \sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{8\pi}{7}\right)\right)$$
$$= y_1 + y_2 \text{ (say)}$$

Now,

$$y_1 = \sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right) + \sin^2\left(\frac{8\pi}{7}\right)$$

$$= \sin^2\left(\frac{\pi}{7}\right) + \sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right)$$

$$= \frac{1}{2}\left[2\sin^2\left(\frac{\pi}{7}\right) + 2\sin^2\left(\frac{2\pi}{7}\right) + 2\sin^2\left(\frac{4\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left[3 - \cos\left(\frac{2\pi}{7}\right) - \cos\left(\frac{4\pi}{7}\right) - \cos\left(\frac{8\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left[3 - \cos\left(\frac{2\pi}{7}\right) - \cos\left(\frac{4\pi}{7}\right) - \cos\left(\frac{6\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left[3 - \left(\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)\right)\right]$$

$$= \frac{1}{2}\left[3 - \left(-\frac{1}{2}\right)\right]$$

$$= \frac{7}{4}$$

Now,

$$y_2 = 2\left(\sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right)\sin\left(\frac{8\pi}{7}\right) + \sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{8\pi}{7}\right)\right)$$

$$= \left[\cos\left(\frac{2\pi}{7}\right) - \cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) - \cos\left(\frac{12\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) - \cos\left(\frac{10\pi}{7}\right)\right]$$

$$= \left[\cos\left(\frac{2\pi}{7}\right) - \cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) - \cos\left(\frac{4\pi}{7}\right)\right]$$

$$= 0$$

Thus, $y^2 = \left[\sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right) + \sin^2\left(\frac{8\pi}{7}\right)\right]$

$$\Rightarrow \quad y^2 = \frac{7}{4}$$

$$\Rightarrow \quad y = \frac{\sqrt{7}}{2}$$

Hence, the value of

$\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)$ is $\frac{\sqrt{7}}{2}$

7. We have

$$= \tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{2\pi}{16}\right) + \ldots. + \tan^2\left(\frac{7\pi}{16}\right)$$

$$= \left(\tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{7\pi}{16}\right)\right) + \left(\tan^2\left(\frac{2\pi}{16}\right) + \tan^2\left(\frac{6\pi}{16}\right)\right) + \left(\tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{7\pi}{16}\right)\right) + \left(\tan^2\left(\frac{4\pi}{16}\right)\right)$$

$$= \left(\tan^2\left(\frac{\pi}{16}\right) + \cot^2\left(\frac{\pi}{16}\right)\right) + \left(\tan^2\left(\frac{2\pi}{16}\right) + \cot^2\left(\frac{2\pi}{16}\right)\right) + \left(\tan^2\left(\frac{\pi}{16}\right) + \cot^2\left(\frac{\pi}{16}\right)\right) + \left(\tan^2\left(\frac{\pi}{4}\right)\right)$$

$$= \left(\tan\left(\frac{\pi}{16}\right) + \cot\left(\frac{\pi}{16}\right)\right)^2 + \left(\tan\left(\frac{2\pi}{16}\right) + \cot\left(\frac{2\pi}{16}\right)\right)^2 + \left(\tan\left(\frac{\pi}{16}\right) + \cot\left(\frac{\pi}{16}\right)\right)^2 - 2 - 2 - 2 + 1$$

$$= \left(\frac{1}{\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)}\right)^2 + \left(\frac{1}{\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)}\right)^2 + \left(\frac{1}{\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)}\right)^2 - 5$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\sin^2\left(\frac{2\pi}{8}\right)} + \frac{4}{\sin^2\left(\frac{3\pi}{8}\right)} - 5$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\sin^2\left(\frac{\pi}{4}\right)} + \frac{4}{\sin^2\left(\frac{3\pi}{8}\right)} - 5$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\cos^2\left(\frac{\pi}{8}\right)} + 8 - 5$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)} + 3$$

$$= \frac{16}{4\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)} + 3$$

$$= \frac{16}{\left(2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right)^2} + 3$$

$$= \frac{16}{\sin^2\left(\frac{\pi}{4}\right)} + 3$$

$$= 32 + 3 = 35$$

8. Let $\theta = \frac{\pi}{7}$

$4\theta = \pi - 3\theta$

$\tan(4\theta) = \tan(\pi - 3\theta)$

$\tan(4\theta) = -\tan(3\theta)$

$$\frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = -\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\frac{4x - 4x^3}{1 - 6x^2 + x^4} = -\frac{3x - x^3}{1 - 3x^2}$$

On simplification, we get,

$x^6 - 21x^4 + 35x^2 - 7 = 0$

$y^3 - 21y^2 + 35y - 7 = 0,\ y = x^2 = \tan^2\theta$ …(i)

Let its roots be $\tan^2\theta$, $\tan^2(2\theta)$, $\tan^2(3\theta)$

Thus, $\tan^2\theta + \tan^2(2\theta) + \tan^2(3\theta) = 21$

$$\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right) = 21$$

Replacing $y$ by $1/y$ in (i), we get,

$$\frac{1}{y^3} - \frac{21}{y^2} + \frac{35}{y} - 7 = 0$$

$-7y^3 + 35y^2 - 21y + 1 = 0$

$7y^3 - 35y^2 + 21y - 1 = 0$

Let its roots be $\cot^2(\theta)$, $\cot^2(2\theta)$, $\cot^2(3\theta)$

Thus, $\cot^2(\theta) + \cot^2(2\theta) + \cot^2(3\theta) = \frac{35}{7} = 5$

$$\cot^2\left(\frac{\pi}{7}\right) + \cot^2\left(\frac{2\pi}{7}\right) + \cot^2\left(\frac{3\pi}{7}\right) = 5$$

Hence, the value of

$$\left(\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right)\right)$$
$$\times\left(\cot^2\left(\frac{\pi}{7}\right) + \cot^2\left(\frac{2\pi}{7}\right) + \cot^2\left(\frac{3\pi}{7}\right)\right)$$

$= 35 \times 3 = 105$

9. We have

$$\frac{3 + \cot(76°)\cot(16°)}{\cot(76°) + \cot(16°)}$$

$$= \frac{2 + 1 + \cot(76°)\cot(16°)}{\cot(76°) + \cot(16°)}$$

$$= \frac{2 + 1 + \frac{\cos(76°)\cos(16°)}{\sin(76°)\sin(16°)}}{\frac{\cos(76°)}{\sin(76°)} + \frac{\cos(16°)}{\sin(16°)}}$$

$$= \frac{2(\sin(76°)\sin(16°)) + \cos(76° - 16°)}{\cos(76°)\sin(16°) + \sin(76°)\cos(16°)}$$

$$= \frac{2(\sin(76°)\sin(16°)) + \cos(76° - 16°)}{\sin(76° + 16°)}$$

$$= \frac{2(\sin(76°)\sin(16°)) + \cos(60°)}{\sin(92°)}$$

$$= \frac{\cos(60°) - \cos(92°) + \cos(60°)}{\sin(92°)}$$

$$= \frac{\frac{1}{2} - \cos(92°) + \frac{1}{2}}{\sin(92°)}$$

$$= \frac{1 - \cos(92°)}{\sin(92°)}$$

$$= \frac{2\sin^2(46°)}{2\sin(46°)\cos(46°)}$$

$= \tan(46°)$

$= \cot(44°)$

10. Let $a = \cos x$, $b = \cos y$, $c = \cos z$

So, $a + b + c = 0$

then, $a^3 + b^3 + c^3 = 3abc$

Now, $\cos(3x) + \cos(3y) + \cos(3z)$

$= 4(\cos^3 x + \cos^3 y + \cos^3 z) - 3(\cos x + \cos y + \cos z)$

$= 4(\cos^3 x + \cos^3 y + \cos^3 z)$

$= 4 \cdot 3\cos x\cos y\cos z$

$= 12\cos x\cos y\cos z$

11. Let $\theta = \frac{\pi}{9}$

$\Rightarrow\ 3\theta = \frac{\pi}{3}$

$\Rightarrow\ \tan(3\theta) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\Rightarrow\ \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

$\Rightarrow\ (3\tan\theta - \tan^3\theta)^2 = 3(1 - 3\tan^2\theta)^2$

$\Rightarrow\ 9\tan^2\theta - 6\tan^4\theta + \tan^6\theta = 3(1 - 6\tan^2\theta + \tan^4\theta)$

$\Rightarrow\ \tan^6\theta - 9\tan^4\theta + 27\tan^2\theta = 3$

Hence, the result.

12. Let $a = \cos A + i\sin A$, $b = \cos B - i\sin B$, $c = \cos C + i\sin C$

Clearly, $a + b + c = 0$

Also, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

$= (\cos A - \sin A) + (\cos B - \sin B) + (\cos C - \sin C)$

$= 0$

Now, $(a + b + c)^2 = 0$

$\Rightarrow\ a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$

$\Rightarrow\ a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$

$\Rightarrow\ a^2 + b^2 + c^2 = 0$

$\Rightarrow\ (\cos A + i\sin A)^2 + (\cos B + i\sin B)^2 + (\cos B + i\sin B)^2 = 0$

$\Rightarrow\ (\cos 2A + i\sin 2A) + (\cos 2B + i\sin 2B) + (\cos 2C + i\sin 2C) = 0$

$\Rightarrow$ (cos 2$A$ + cos 2$B$ + cos 2$C$) + $i$(sin 2$A$ + sin 2$B$ + sin 2$C$) = 0

$\Rightarrow$ (cos 2$A$ + cos 2$B$ + cos 2$C$) = 0 and (sin 2$A$ + sin 2$B$ + sin 2$C$) = 0

$\Rightarrow$ (cos 2$A$ + cos 2$B$ + cos 2$C$) = 0

$\Rightarrow$ $2\cos^2 A - 1 + 2\cos^2 B - 1 + 2\cos^2 C - 1 = 0$

$\Rightarrow$ $\cos^2 A + \cos^2 B + \cos^2 C = \frac{3}{2}$

$\Rightarrow$ $1 - \sin^2 A + 1 - \sin^2 B + 1 - \sin^2 C = \frac{3}{2}$

$\Rightarrow$ $\sin^2 A + \sin^2 B + \sin^2 C = 3 - \frac{3}{2} = \frac{3}{2}$

Hence, the result.

13. Given $A = \frac{\pi}{4}$

$\Rightarrow$ $B + C = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Now, $p = \tan B \tan C$

$= \tan B \tan\left(\frac{3\pi}{4} - B\right)$

$= \tan B\left(\frac{-1 - \tan B}{1 - \tan B}\right)$

$= \left(\frac{-\tan B - \tan^2 B}{1 - \tan B}\right)$

$\Rightarrow$ $\tan^2 B + (1 - p)\tan B + p = 0$

since angles are real, so $D \geq 0$

$\Rightarrow$ $(1 - p)^2 - 4p \geq 0$

$\Rightarrow$ $p^2 - 6p + 1 \geq 0$

$\Rightarrow$ $(p - 3)^2 - 8\ \ 0$

$\Rightarrow$ $(p - 3 + 2\sqrt{2})(p - 3 - 2\sqrt{2}) \geq 0$

$\Rightarrow$ $p \leq 3 - 2\sqrt{2},\ p \geq 3 + 2\sqrt{2}$

14. Given, $k = \frac{\tan 3A}{\tan A}$

$\Rightarrow$ $k - 1 = \frac{\tan 3A}{\tan A} - 1 = \frac{\tan 3A - \tan A}{\tan A}$

$\Rightarrow$ $k - 1 = \frac{\sin(2A)}{\cos 3A \sin A}$

$\Rightarrow$ $k - 1 = \frac{2\cos A}{\cos 3A}$

Now, $\frac{2k}{k-1} = \frac{2\tan 3A}{\tan A} \cdot \frac{\cos 3A}{2\cos A}$

$\Rightarrow$ $\frac{2k}{k-1} = \frac{\sin 3A}{\sin A}$

$\Rightarrow$ $\frac{\sin 3A}{\sin A} = \frac{2k}{k-1}$

Also, $\frac{3\sin A - 4\sin^3 A}{\sin A} = \frac{2k}{k-1}$

$\Rightarrow$ $3 - 4\sin^2 A = \frac{2k}{k-1}$

$\Rightarrow$ $4\sin^2 A = 3 - \frac{2k}{k-1} = \frac{k-3}{k-1}$

$\Rightarrow$ $\sin^2 A = \frac{k-3}{4(k-1)}$

Clearly, $0 \leq \frac{k-3}{4(k-1)} \leq 1$

On simplification, we get,

$k \leq \frac{1}{3}$ and $k \geq 3$

15. Now, $\cos(A + B + C) = \cos(\pi) = -1$

$\Rightarrow$ $\cos A \cos B \cos C[1 - \tan A \tan B - \tan B \tan C - \tan C \tan A] = -1$

$\Rightarrow$ $\cos A \cos B \cos C + 1 = \cos A \sin B \sin C + \cos B \sin C \sin A + \cos C \sin A \sin B$

Dividing both sides by $\sin A \sin B \sin C$, we get,

cosec $A$ cosec $B$ cosec $C$ + cot $A$ cot $B$ cot $C$ = cot $A$ + cot $B$ + cot $C$

$\Rightarrow$ cot $A$ + cot $B$ + cot $C$ – cosec $A$ cosec $B$ cosec $C$ = cot $A$ cot $B$ cot $C$

Hence, the result.

16. Now, $\tan(\alpha + \beta)$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\frac{1}{\sqrt{x(x^2 + x + 1)}} + \frac{\sqrt{x}}{\sqrt{(x^2 + x + 1)}}}{1 - \frac{1}{x^2 + x + 1}}$$

$$= \frac{(1 + x)(x^2 + x + 1)}{(x^2 + x)\sqrt{x(x^2 + x + 1)}}$$

$$= \frac{\sqrt{(x^2 + x + 1)}}{x\sqrt{x}} = \tan\gamma$$

Thus, $\alpha + \beta = \gamma$

17. We have

$$\cos 2\alpha = \frac{3\cos 2\beta - 1}{3 - \cos 2\beta}$$

$\Rightarrow$ $\frac{1 - \tan^2\alpha}{1 + \tan^2\alpha} = \frac{3\left(\frac{1 - \tan^2\beta}{1 + \tan^2\beta}\right) - 1}{3 - \left(\frac{1 - \tan^2\beta}{1 + \tan^2\beta}\right)}$

$\Rightarrow$ $\frac{1 - \tan^2\alpha}{1 + \tan^2\alpha} = \frac{3 - 3\tan^2\beta - 1 - \tan^2\beta}{3 + 3\tan^2\beta - 1 + \tan^2\beta}$

$\Rightarrow$ $\frac{1 - \tan^2\alpha}{1 + \tan^2\alpha} = \frac{2 - 4\tan^2\beta}{2 + 4\tan^2\beta}$

$\Rightarrow \quad \dfrac{1-\tan^2\alpha}{1+\tan^2\alpha} = \dfrac{1-2\tan^2\beta}{1+2\tan^2\beta}$

$\Rightarrow \quad \dfrac{2}{-2\tan^2\alpha} = \dfrac{2}{-4\tan^2 G}$

$\Rightarrow \quad \dfrac{1}{-\tan^2\alpha} = \dfrac{1}{-2\tan^2\beta}$

$\Rightarrow \quad \tan^2\alpha = 2\tan^2\beta$

$\Rightarrow \quad \dfrac{\tan^2\alpha}{\tan^2\beta} = 2$

$\Rightarrow \quad \dfrac{\tan\alpha}{\tan\beta} = \sqrt{2} = \dfrac{\sqrt{2}}{1}$

Hence, the result.

18. Now, $\tan\left(\dfrac{\pi}{4}+\alpha\right) = \dfrac{1+\tan\alpha}{1-\tan\alpha} = \dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}$

$\tan^2\left(\dfrac{\pi}{4}+\alpha\right) = \dfrac{1+\sin(2\alpha)}{1-\sin(2\alpha)}$

So, $\tan^3\left(\dfrac{\alpha}{2}+\dfrac{\pi}{4}\right) = \tan\left(\dfrac{\beta}{2}+\dfrac{\pi}{4}\right)$

$\Rightarrow \quad \dfrac{(1+\sin\alpha)^3}{(1-\sin\alpha)^3} = \dfrac{1+\sin\beta}{1-\sin\beta}$

Applying, componendo and dividendo, we get,

$\dfrac{2\sin\beta}{2} = \dfrac{2(3\sin\alpha+\sin^3\alpha)}{2(1+3\sin^2\alpha)}$

$\sin\beta = \dfrac{(3\sin\alpha+\sin^3\alpha)}{(1+3\sin^2\alpha)}$

Hence, the result.

19. Given, $\sin\beta = \dfrac{1}{5}\sin(2\alpha+\beta)$

$\Rightarrow \quad \dfrac{\sin\beta}{\sin(2\alpha+\beta)} = \dfrac{1}{5}$

$\Rightarrow \quad \dfrac{\sin\beta+\sin(2\alpha+\beta)}{\sin\beta-\sin(2\alpha+\beta)} = \dfrac{1+5}{1-5}$

$\Rightarrow \quad \dfrac{2\sin(\alpha+\beta)\cos(\alpha)}{2\cos(\alpha+\beta)\sin(-\alpha)} = -\dfrac{3}{2}$

$\Rightarrow \quad \tan(\alpha+\beta)\cot\alpha = \dfrac{3}{2}$

$\Rightarrow \quad \tan(\alpha+\beta) = \dfrac{3}{2}\tan\alpha$

20. Given $\sin x + \sin y = 3(\cos x - \cos y)$

$\Rightarrow \quad 3\cos x + \sin x = 3\cos y - \sin y$

Put $3 = r\cos\alpha$, $1 = r\sin\alpha$

$\Rightarrow \quad r = \sqrt{10}$ and $\tan\alpha = \dfrac{1}{3}$

Now $3\cos x + \sin x = 3\cos y - \sin y$

$\Rightarrow \quad r\cos(x-\alpha) = r\cos(y+\alpha)$

$\Rightarrow \quad \cos(x-\alpha) = \cos(y+\alpha)$

$\Rightarrow \quad (x-\alpha) = (y+\alpha)$

$\Rightarrow \quad x = -y,\ x = y + 2\alpha$

$\Rightarrow \quad x = -y$ satisfies the given equation

$\Rightarrow \quad 3x = -3y$

$\Rightarrow \quad \sin(3x) = \sin(-3y)$

$\Rightarrow \quad \sin(3x) = -\sin(3y)$

$\Rightarrow \quad \sin(3x) + \sin(3y) = 0$

Hence, the result.

21. Given, $\sec(\varphi-\alpha)$, $\sec\varphi$, $\sec(\varphi+\alpha)$ are in AP

$\Rightarrow \quad 2\sec\varphi = \sec(\varphi-\alpha) + \sec(\varphi+\alpha)$

$\Rightarrow \quad \dfrac{2}{\cos\varphi} = \dfrac{1}{\cos(\varphi-\alpha)} + \dfrac{1}{\cos(\varphi+\alpha)}$

$\Rightarrow \quad \dfrac{2}{\cos\varphi} = \dfrac{\cos(\varphi+\alpha)+\cos(\varphi-\alpha)}{\cos(\varphi-\alpha)\cos(\varphi+\alpha)}$

$\Rightarrow \quad \dfrac{1}{\cos\varphi} = \dfrac{\cos\varphi\cos\alpha}{\cos^2(\varphi)-\sin^2(\alpha)}$

$\Rightarrow \quad \cos^2(\varphi) - \sin^2(\alpha) = \cos^2\varphi\cos\alpha$

$\Rightarrow \quad \cos^2(\varphi)(1-\cos\alpha) = \sin^2(\alpha)$

$\Rightarrow \quad \cos^2(\varphi) = \dfrac{\sin^2(\alpha)}{(1-\cos\alpha)}$

$\Rightarrow \quad \cos^2(\varphi) = \dfrac{4\sin^2(\alpha/2)\cos^2(\alpha/2)}{2\sin^2(\alpha/2)}$

$\Rightarrow \quad \cos^2(\alpha) = 2\cos^2(\alpha/2)$

$\Rightarrow \quad \cos(\varphi) = \sqrt{2}\cos(\alpha/2)$

Hence, the result.

22. Given, $\tan\left(\dfrac{x+y}{2}\right)$, $\tan z$, $\tan\left(\dfrac{x-y}{2}\right)$ are in GP

$\Rightarrow \quad \tan^2 z = \tan\left(\dfrac{x+y}{2}\right)\tan\left(\dfrac{x-y}{2}\right)$

$\Rightarrow \quad \tan^2 z = \dfrac{\sin^2\left(\dfrac{x}{2}\right)-\sin^2\left(\dfrac{y}{2}\right)}{\cos^2\left(\dfrac{x}{2}\right)-\sin^2\left(\dfrac{y}{2}\right)}$

$\Rightarrow \quad \tan^2 z = \dfrac{\cos y - \cos x}{\cos y + \cos x}$

$\Rightarrow \quad \dfrac{\tan^2 z}{1} = \dfrac{\cos y - \cos x}{\cos y + \cos x}$

$\Rightarrow \quad \dfrac{1}{\tan^2 z} = \dfrac{\cos y + \cos x}{\cos y - \cos x}$

Applying componendo and dividendo, we get,

$\Rightarrow \quad \dfrac{1-\tan^2 z}{1+\tan^2 z} = \dfrac{2\cos x}{2\cos y}$

$\cos(2z) = \dfrac{\cos x}{\cos y}$

$\Rightarrow \quad \cos x = \cos y\cos(2z)$

23. Let $y = \dfrac{\sec^2\theta - \tan\theta}{\sec^2\theta + \tan\theta}$

$\Rightarrow \quad y = \dfrac{\tan^2\theta - \tan\theta + 1}{\tan^2\theta + \tan\theta + 1}$

$\Rightarrow \quad y = \dfrac{x^2 - x + 1}{x^2 + x + 1}, x = \tan\theta$

$\Rightarrow \quad y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$

$\Rightarrow \quad (x^2 + x + 1)y = (x^2 - x + 1)$

$\Rightarrow \quad (y - 1)x^2 + (y + 1)x + (y - 1) = 0$

For all real $\theta$, $D \geq 0$

$\Rightarrow \quad (y + 1)^2 - 4(y - 1)^2 \geq 0$

$\Rightarrow \quad (y + 1)^2 - (2y - 2)^2 \geq 0$

$\Rightarrow \quad (y + 1 + 2y - 2)(y + 1 - 2y + 2) \geq 0$

$\Rightarrow \quad (3y - 1)(-y + 3) \geq 0$

$\Rightarrow \quad (3y - 1)(y - 3) \leq 0$

$\Rightarrow \quad \dfrac{1}{3} \leq y \leq 3$

$\Rightarrow \quad \dfrac{1}{3} \leq \left(\dfrac{\sec^2\theta - \tan\theta}{\sec^2\theta + \tan\theta}\right) \leq 3$

24. As we know

$\cos\theta \cdot \cos(2\theta) \cdot \cos(2^2\theta)\cos(2^3\theta) \ldots \cos(2^{n-1}\theta)$

$= \dfrac{\sin(2^n\theta)}{2^n \sin(\theta)}$

$= \dfrac{\sin(\pi - \theta)}{2^n \sin(\theta)}$

$= \dfrac{\sin(\theta)}{2^n \sin(\theta)}$

$= \dfrac{1}{2^n}$

Hence, the value of

$2^n \cos\theta \cdot \cos(2\theta) \cdot \cos(2^2\theta)\cos(2^3\theta) - \cos(2^{n-1}\theta)$

$= 1$

25. We have $\tan(6°)\tan(42°)\tan(66°)\tan(78°)$

$= \{\tan(6°)\tan(42°)\}\{\tan(66°)\tan(78°)\}$

$=$

$\dfrac{1}{\tan(54°)\tan(18°)}\{\tan(54°)\tan(6°)\tan(66°)\}$

$\times \{\tan(42°)\tan(18°)\tan(78°)\}$

$= \dfrac{\tan(54°)\tan(18°)}{\tan(54°)\tan(18°)}$

$= 1$

26. As we know $\dfrac{\tan(\alpha + \beta - \gamma)}{\tan(\alpha - \beta + \gamma)} = \dfrac{\tan\gamma}{\tan\beta}$

So, $\tan A \pm \tan B = \dfrac{\sin(A \pm B)}{\cos A \cos B}$

Applying, componendo and dividendo, we get

$\dfrac{\tan(\alpha + \beta - \gamma) + \tan(\alpha - \beta + \gamma)}{\tan(\alpha + \beta - \gamma) - \tan(\alpha - \beta + \gamma)} = \dfrac{\tan\gamma + \tan\beta}{\tan\gamma - \tan\beta}$

$\dfrac{\sin(\alpha + \beta - \gamma + \alpha - \beta + \gamma)}{\sin(\alpha + \beta - \gamma - \alpha + \beta - \gamma)} = -\dfrac{\sin(\beta + \gamma)}{\sin(\beta - \gamma)}$

$\dfrac{\sin(2\alpha)}{\sin 2(\beta - \gamma)} = -\dfrac{\sin(\beta + \gamma)}{\sin(\beta - \gamma)}$

$\dfrac{\sin(2\alpha)}{2\sin(\beta - \gamma)\cos(\beta - \gamma)} = -\dfrac{\sin(\beta + \gamma)}{\sin(\beta - \gamma)}$

$\sin(\beta - \gamma)[\sin(2\alpha) + 2\cos(\beta - \gamma)\sin(\beta + \gamma)] = 0$

$\sin(\beta - \gamma)[\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma)] = 0$

$\sin(\beta - \gamma) = 0, [\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma)] = 0$

Hence, the result.

27. Now, $\cot A + \dfrac{\sin A}{\sin B \sin C}$

$= \cot A + \dfrac{\sin(\pi - (B + C))}{\sin B \sin C}$

$= \cot A + \dfrac{\sin(B + C)}{\sin B \sin C}$

$= \cot A + \dfrac{\sin B \cos C + \cos B \sin C}{\sin B \sin C}$

$= \cot A + \dfrac{\sin B \cos C}{\sin B \sin C} + \dfrac{\cos B \sin C}{\sin B \sin C}$

$= \cot A + \dfrac{\cos B \sin C}{\sin B \sin C} + \dfrac{\sin B \cos C}{\sin B \sin C}$

$= \cot A + \cot B + \cot C$

Hence, the result.

28. Given, $\dfrac{\sin(\theta + A)}{\sin(\theta + B)} = \sqrt{\dfrac{\sin(2A)}{\sin(2B)}}$

$\Rightarrow \quad \dfrac{\sin\theta\cos A + \cos\theta\sin A}{\sin\theta\cos B + \cos\theta\sin B} = \sqrt{\dfrac{\sin A\cos A}{\sin B\cos B}}$

$\Rightarrow \quad \dfrac{\tan\theta\cos A + \sin A}{\tan\theta\cos B + \sin B} = \sqrt{\dfrac{\sin A\cos A}{\sin B\cos B}}$

$\Rightarrow \quad \tan\theta(\cos A\sqrt{\sin B\cos B} - \cos B\sqrt{\sin A\cos A})$

$= (\sin B\sqrt{\sin A\cos A} - \sin A\sqrt{\sin B\cos B})$

$\Rightarrow \quad \tan\theta\sqrt{\cos A\cos B}\,(\sqrt{\cos A\sin B} - \sqrt{\cos B\sin A})$

$= \sqrt{\sin A\sin B}\,(\sqrt{\cos A\sin B} - \sqrt{\cos B\sin A})$

$\Rightarrow \quad \tan\theta = \dfrac{\sqrt{\sin A\sin B}}{\sqrt{\cos A\cos B}} = \sqrt{\tan A\tan B}$

$\Rightarrow \quad \tan\theta = \sqrt{\tan A\tan B}$

$\Rightarrow \quad \tan^2\theta = \tan A\tan B$

Hence, the result.

29. Given, $\cos(x-y) = -1$

$\Rightarrow \quad \cos x \cos y + \sin x \sin y = -1$

$\Rightarrow \quad 2\cos x \cos y + 2\sin x \sin y = -2$

$\Rightarrow \quad (1 + 2\cos x \cos y) + (1 + 2\sin x \sin y) = 0$

$\Rightarrow \quad (\cos^2 x + \cos^2 y + 2\cos x \cos y) + (\sin^2 x + \sin^2 y + 2\sin x \sin y) = 0$

$\Rightarrow \quad (\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 0$

$\Rightarrow \quad (\cos x + \cos y)^2 = 0, (\sin x + \sin y)^2 = 0$

$\Rightarrow \quad (\cos x + \cos y) = 0, (\sin x + \sin y) = 0$

Hence, the result.

30. Given $\sqrt{2}\cos A = \cos B + \cos^3 B$

and $\sqrt{2}\sin A = \sin B - \sin^3 B$

squaring and adding, we get,

$(\cos B + \cos^3 B)^2 + (\sin B - \sin^3 B)^2 = 2$

$(\sin^6 B + \cos^6 B)^2 - 2(\sin^4 B - \cos^4 B) + 1 = 2$

$(1 - 3\sin^2 B\cos^2 B) - 2(\sin^2 B - \cos^2 B) + 1 = 2$

$(3\sin^2 B\cos^2 B) + 2(\sin^2 B - \cos^2 B) = 0$

$(3\sin^2 B\cos^2 B) = 2\cos(2B)$

$\frac{3}{4}(\sin^2 2B) = 2\cos(2B)$

$3 - 3\cos^2 2B = 8\cos(2B)$

$3\cos^2 2B + 8\cos(2B) - 3 = 0$

$3\cos^2(2B) + 9\cos(2B) - \cos(2B) - 3 = 0$

$3\cos(2B)(\cos(2B) + 3) - (\cos(2B) + 3) = 0$

$(3\cos(2B) - 1)(\cos(2B) + 3) = 0$

$(3\cos(2B) - 1) = 0, (\cos(2B) + 3) = 0$

$(3\cos(2B) - 1) = 0$

$\cos(2B) = \frac{1}{3}$

$\sin(2B) = \pm\sqrt{1 - \frac{1}{9}} = \pm\frac{2\sqrt{2}}{3}$ ...(i)

On simplification, we get,

$\sin(A-B) = -\frac{1}{2\sqrt{2}}\sin(2B)$ ...(ii)

From (i) and (ii), we get,

$\sin(A-B) = \pm\frac{1}{3}$

Hence, the result.

31. We have,

$16\sin^2(9°) = 8(2\sin^2(9°))$

$= 8(1 - \cos(18°))$

$= 8\left(1 - \frac{\sqrt{10+2\sqrt{5}}}{4}\right)$

$= (8 - 2\sqrt{10+2\sqrt{5}})$

$= (8 - 2\sqrt{(3+\sqrt{5})(5-\sqrt{5})})$

$= ((\sqrt{(3+\sqrt{5})})^2 + (\sqrt{(5-\sqrt{5})})^2 - 2\sqrt{(3+\sqrt{5})(5-\sqrt{5})})$

$= ((\sqrt{(3+\sqrt{5})}) - (\sqrt{(5-\sqrt{5})}))^2$

Thus, $4\sin(9^{\circ}) = ((\sqrt{(3+\sqrt{5})}) - (\sqrt{(5-\sqrt{5})}))$

$\Rightarrow \quad \sin(9°) = \frac{1}{4}((\sqrt{(3+\sqrt{5})}) - (\sqrt{(5-\sqrt{5})}))$

Hence, the result.

32. The function $f(x)$ will provide us the maximum value if $x = 0$ and minimum value if $x = \frac{\pi}{6}$

Hence, range is

$= \left[f\left(\frac{\pi}{6}\right), f(0)\right]$

$= \left[0, \sin\left(\frac{\pi}{6}\right)\right]$

$= \left[0, \frac{1}{2}\right]$

33. We have

$\sum_{k=1}^{6}\left(\sin\left(\frac{2k\pi}{7}\right) - i\cos\left(\frac{2k\pi}{7}\right)\right)$

$= -i\sum_{k=1}^{6}\left(\cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)\right)$

$= -i\sum_{k=1}^{6} e^{i\left(\frac{2k\pi}{7}\right)}$

$= -i\left(e^{i\frac{2\pi}{7}} + e^{i\frac{4\pi}{7}} + \ldots + e^{i\frac{12\pi}{7}}\right)$

$= -ie^{i\frac{2\pi}{7}}\left(1 + e^{i\frac{2\pi}{7}} + e^{i\frac{4\pi}{7}} + \ldots + e^{i\frac{10\pi}{7}}\right)$

$= -ie^{i\frac{2\pi}{7}}\left(\frac{1 - e^{i\frac{12\pi}{7}}}{1 - e^{i\frac{2\pi}{7}}}\right)$

$= -i\left(\frac{e^{i\frac{2\pi}{7}} - e^{i\frac{12\pi}{7} + i\frac{2\pi}{7}}}{1 - e^{i\frac{2\pi}{7}}}\right)$

$= -i\left(\frac{e^{i\frac{2\pi}{7}} - e^{i(2\pi)}}{1 - e^{i\frac{2\pi}{7}}}\right)$

$= -i\left(\frac{e^{i\frac{2\pi}{7}} - 1}{1 - e^{i\frac{2\pi}{7}}}\right)$

$= i\left(\frac{1 - e^{i\frac{2\pi}{7}}}{1 - e^{i\frac{2\pi}{7}}}\right) = i$

34. Given, $\cos\theta + \cos\varphi = a$
and $\sin\theta + \sin\varphi = b$

Now, $\dfrac{\sin\theta + \sin\varphi}{\cos\theta + \cos\varphi} = \dfrac{b}{a}$

$$\Rightarrow \frac{2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)}{2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)} = \frac{b}{a}$$

$$\Rightarrow \frac{2\sin\left(\frac{\theta+\varphi}{2}\right)}{2\cos\left(\frac{\theta+\varphi}{2}\right)} = \frac{b}{a}$$

$$\Rightarrow \tan\left(\frac{\theta+\varphi}{2}\right) = \frac{b}{a}$$

On simplification, we get,

and $\cos\left(\dfrac{\theta-\varphi}{2}\right) = \dfrac{\sqrt{a^2+b^2}}{2}$

$$\cos\left(\frac{\theta+\varphi}{2}\right) = \frac{a^2+b^2}{\sqrt{a^2+b^2}+2}$$

Now,

$$\tan\left(\frac{\theta}{2}\right) + \tan\left(\frac{\varphi}{2}\right) = \frac{\sin\left(\frac{\theta+\varphi}{2}\right)}{\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\varphi}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\theta+\varphi}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\varphi}{2}\right)}$$

$$= \frac{2\sin\left(\frac{è+\varphi}{2}\right)}{\cos\left(\frac{\theta+\varphi}{2}\right) + \cos\left(\frac{\theta-\varphi}{2}\right)}$$

$$= \frac{2\tan\left(\frac{\theta+\varphi}{2}\right)}{1+\dfrac{\cos\left(\frac{\theta-f}{2}\right)}{\cos\left(\frac{\theta+\varphi}{2}\right)}}$$

$$= 1 + \frac{\dfrac{\sqrt{a^2+b^2}}{2}}{\dfrac{a^2+b^2}{\sqrt{a^2+b^2}+2}}$$

$$= 1 + \frac{\sqrt{a^2+b^2}+2}{\sqrt{a^2+b^2}}$$

$$= \left(2 + \frac{2}{\sqrt{a^2+b^2}}\right)$$

35. We have $\dfrac{\tan\theta}{\tan\theta - \tan 3\theta} = \dfrac{1}{3}$

$$\Rightarrow \frac{1}{1-\dfrac{\tan 3\theta}{\tan\theta}} = \frac{1}{3}$$

$$\Rightarrow \frac{1}{1-\dfrac{3-\tan^2\theta}{1-3\tan^2\theta}} = \frac{1}{3}$$

$$\Rightarrow \frac{1-3\tan^2\theta}{1-3\tan^2\theta - 3 + \tan^2\theta} = \frac{1}{3}$$

$$\Rightarrow \frac{1-3\tan^2\theta}{2+2\tan^2\theta} = -\frac{1}{3}$$

$\Rightarrow$ $2 + 2\tan^2\theta = -3 + 9\tan^2\theta$

$\Rightarrow$ $7\tan^2\theta = 5$

$\Rightarrow$ $\tan^2\theta = \dfrac{5}{7}$

Now, $\dfrac{\cot\theta}{\cot\theta - \cot(3\theta)}$

$$= \frac{1}{1-\left(\dfrac{\cot(3\theta)}{\cot\theta}\right)}$$

$$= \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

## Integer Type Questions

1. Given, $\dfrac{x}{\cos\theta} = \dfrac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \dfrac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)} = k$

Now, $x + y + z$

$$= k\left[\cos\theta + \cos\left(\theta - \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{2\pi}{3}\right)\right]$$

$$= k\left[\cos\theta + 2\cos\theta\cos\left(\frac{2\pi}{3}\right)\right]$$

$$= k\left[\cos\theta - 2\cos\theta \cdot \frac{1}{2}\right]$$

$= k[\cos\theta - \cos\theta]$

$= 0$

Hence, the value of $(x + y + z + 4) = 4$.

2. We have $\displaystyle\sum_{r=0}^{9} \sin^2\left(\frac{\pi r}{18}\right)$

$$= \sin^2\left(\frac{\pi}{18}\right) + \sin^2\left(\frac{2\pi}{18}\right) + \sin^2\left(\frac{3\pi}{18}\right) + \ldots + \sin^2\left(\frac{9\pi}{18}\right)$$

$= \sin^2(10°) + \sin^2(20°) + \sin^2(30°) + \ldots + \sin^2(90°)$
$= 4 \times 1 + 1 = 5$

3. We have

$$\frac{\tan x}{\tan y} = \frac{1}{3}$$

$$\frac{\tan x}{1} = \frac{\tan y}{3}$$

$$\text{Now, } \tan x = \frac{1}{3}\sqrt{\frac{1-\frac{9}{4}}{\frac{1}{4}-1}} = \frac{1}{3}\sqrt{\frac{5}{3}}$$

We have

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$= \frac{\tan x + 3\tan x}{1 - \tan x \cdot 3\tan x}$$
$$= \frac{4\tan x}{1 - 3\tan^2 x}$$
$$= \frac{\frac{4}{3}\sqrt{\frac{5}{3}}}{1 - \frac{3}{9} \times \frac{5}{3}}$$
$$= \frac{4\sqrt{5}}{3\sqrt{3}} \times \frac{9}{4}$$
$$= \frac{3\sqrt{5}}{\sqrt{3}} = \sqrt{15}$$

Thus, $\frac{\tan^2(x+y)}{5} = 3$

4. Given, $\cos(x-y)$, $\cos x$, $\cos(x+y)$ are in H.P

$\Rightarrow \quad \cos x = \frac{2\cos(x-y)\cos(x+y)}{\cos(x-y)+\cos(x+y)}$

$\Rightarrow \quad \cos x = \frac{\cos 2x + \cos 2y}{2\cos x \cos y}$

$\Rightarrow \quad 2\cos^2 x \cos y = 2\cos^2 x + 2\cos^2 y - 2$

$\Rightarrow \quad \cos^2 x \cos y = \cos^2 x + \cos^2 y - 1$

$\Rightarrow \quad \cos^2 x (\cos y - 1) = \cos^2 y - 1$

$\Rightarrow \quad \cos^2 x = \cos y + 1$

$\Rightarrow \quad \cos^2 x = 2\cos^2\left(\frac{y}{2}\right)$

$\Rightarrow \quad \cos^2 x \sec^2\left(\frac{y}{2}\right) = 2$

$\Rightarrow \quad \left|\cos x \sec\left(\frac{y}{2}\right)\right| = \sqrt{2}$

Thus, $m = \sqrt{2}$

Hence, the value of $(m^2 + 2) = 4$

5. We have

$$\tan x + \tan\left(\frac{\pi}{3}+x\right) + \tan\left(\frac{2\pi}{3}+x\right)$$
$$= \tan x + \tan\left(\frac{\pi}{3}+x\right) + \tan\left(\pi - \frac{\pi}{3}+x\right)$$
$$= \tan x + \tan\left(\frac{\pi}{3}+x\right) - \tan\left(\frac{\pi}{3}-x\right)$$
$$= \tan x + \frac{\sqrt{3}+\tan x}{1-\sqrt{3}\tan x} - \frac{\sqrt{3}-\tan x}{1+\sqrt{3}\tan x}$$
$$= \tan x + \frac{8\tan x}{1-3\tan^2 x}$$
$$= \frac{\tan x - 3\tan^3 x + 8\tan x}{1-3\tan^2 x}$$
$$= \frac{9\tan x - 3\tan^3 x}{1-3\tan^2 x}$$
$$= \frac{3(3\tan x - \tan^3 x)}{1-3\tan^2 x}$$
$$= 3\tan(3x)$$

Thus, $k = 3$

6. Given, $f(\theta) = \sin^2\theta + \sin^2\left(\frac{2\pi}{3}+\theta\right) + \sin^2\left(\frac{4\pi}{3}+\theta\right)$

$$= \sin^2\theta + \sin^2\left(\frac{2\pi}{3}+\theta\right) + \sin^2\left(\pi + \frac{\pi}{3}+\theta\right)$$
$$= \sin^2\theta + \sin^2\left(\frac{2\pi}{3}+\theta\right) + \sin^2\left(\frac{\pi}{3}+\theta\right)$$
$$= \sin^2\theta + \sin^2\left(\frac{\pi}{3}-\theta\right) + \sin^2\left(\frac{\pi}{3}+\theta\right)$$
$$= \frac{1}{2}\left[1 - \cos(2\theta) + 1 - \cos\left(\frac{2\pi}{3}-2\theta\right) + 1 - \cos\left(\frac{2\pi}{3}+2\theta\right)\right]$$
$$= \frac{1}{2}\left[3 - \cos(2\theta) - \cos\left(\frac{2\pi}{3}-2\theta\right) - \cos\left(\frac{2\pi}{3}+2\theta\right)\right]$$
$$= \frac{1}{2}\left[3 - \cos(2\theta) - 2\cos\left(\frac{2\pi}{3}\right)\cos(2\theta)\right]$$
$$= \frac{1}{2}\left[3 - \cos(2\theta) - 2 \times -\frac{1}{2} \times \cos(2\theta)\right]$$
$$= \frac{3}{2}$$

Hence, the value of $2f\left(\frac{\pi}{15}\right) = 2 \times \frac{3}{2} = 3$.

7. We have

$$m = \sqrt{3}\operatorname{cosec}(20°) - \sec(20°) = 4$$

and $\quad n = \sin(12°)\sin(48°)\sin(54°) = \frac{1}{8}$

Hence, the value of $(m + 8n + 2)$

$= 4 + 1 + 2$

$= 7$

8. Here, $\tan(15°) + \tan(30°) = -p$

and $\tan(15°) + \tan(30°) = -p$

We have $\tan(45°) = 1$

$\Rightarrow \tan(30° + 15°) = 1$

$\Rightarrow \dfrac{\tan(30°) + \tan(15°)}{1 - \tan(30°)\tan(15°)} = 1$

$\Rightarrow \dfrac{-p}{1-q} = 1$

$\Rightarrow -p = 1 - q$

$\Rightarrow q - p = 1$

$\Rightarrow 2 + q - p = 2 + 1 = 3$

Hence, the value of $(2 + q - p)$ is 3.

9. We have $x = \dfrac{\sum_{n=1}^{44}\cos(n°)}{\sum_{n=1}^{44}\sin(n°)}$

$$= \frac{\cos\left(1° + \frac{43°}{2}\right)}{\sin\left(1° + \frac{43°}{2}\right)}$$

$$= \frac{\cos\left(\frac{45°}{2}\right)}{\sin\left(\frac{45°}{2}\right)}$$

$$= \frac{2\cos^2\left(\frac{45°}{2}\right)}{\sin(45°)}$$

$$= \frac{(1 + \cos 45°)}{\sin(45°)} = (\sqrt{2} + 1)$$

Hence, the value of $[x + 3]$

$= [\sqrt{2} + 1 + 3]$

$= [\sqrt{2}] + 4 = 1 + 4 = 5$

10. We have $\sin(25°)\sin(35°)\sin(85°)$

$$= \frac{1}{4}[4\sin(35°)\sin(25°)\sin(85°)]$$

$$= \frac{1}{4} \times \sin(75°)$$

$$= \frac{1}{4} \times \cos(15°)$$

$$= \frac{1}{4} \times \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{16}$$

Hence, the value of $\left(\dfrac{c}{a + \mathrm{b}} + 2\right) = 4$

11. We have $m = \sum_{k=1}^{17}\cos\left(\dfrac{k\pi}{9}\right)$

$$= \cos\left(\frac{\pi}{9}\right) + \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{3\pi}{9}\right) + \ldots + \cos\left(\frac{17\pi}{9}\right)$$

$$= \frac{\sin\left(\frac{17\pi}{18}\right)}{\sin\left(\frac{\pi}{18}\right)} \times \cos\left(\frac{\pi}{9} + \frac{16\pi}{18}\right)$$

$$= \cos\left(\frac{18\pi}{18}\right) = -1$$

Hence, the value of $(m^2 + m + 2) = 4$.

12. We have $\tan(55°)\tan(65°)\tan(75°)$

$$= \frac{1}{\tan(5°)} \times [\tan(55°)\tan(5°)\tan(65°)] \times \tan(75°)$$

$$= \frac{1}{\tan(5°)} \times \tan(15°) \times \tan(75°)$$

$$= \frac{1}{\tan(5°)} \times \tan(15°) \times \cot(15°)$$

$$= \frac{1}{\tan(5°)}$$

$= \cot(5°)$

Thus, $x = 5$

Also,

$\tan(27°) + \tan(18°) + \tan(27°)\tan(18°)$

$= \tan(27° + 18°)$

$= \tan(45°)$

$= 1$

So, $m = 1$

Hence, the value of $(m + x + 1)$ is 7.

## Previous Years' JEE-Advanced Examinations

1. Now, $\sin x \sin y \sin(x - y)$

$$= \frac{1}{2}(2\sin x \sin y)\sin(x - y)$$

$$= \frac{1}{2}(\cos(x - y) - \cos(x + y))\sin(x - y)$$

$$= \frac{1}{4}(2\cos(x + y)\sin(x - y) - 2\sin(x - y)\cos(x + y))$$

$$= \frac{1}{4}(\sin(2x - 2y) - \sin 2x + \sin 2y)$$

Similarly, $\sin y \sin z \sin(y - z)$

$$= \frac{1}{4}(\sin(2y - 2z) - \sin 2y + \sin 2z)$$

and $\sin z \sin x \sin(z - x)$

$$= \frac{1}{4}(\sin(2z - 2x) - \sin 2z + \sin 2x)$$

Therefore, the given expression reduces to

$\frac{1}{4}(\sin 2A + \sin 2B + \sin 2C) + \sin A \cdot \sin B \cdot \sin C$

where $A = x - y$, $B = y - z$, $C = z - x$

$$= \frac{1}{4}(\sin 2A + \sin 2B + \sin 2C) + \sin A \cdot \sin B \cdot \sin C$$

$$= \frac{1}{4}(2\sin(A+B)\cos(A-B) + \sin 2C) + \sin A \cdot \sin B \cdot \sin C$$

$$= \frac{1}{4}(2\sin(-C)\cos(A-B) + \sin 2C) + \sin A \cdot \sin B \cdot \sin C$$

$$= \frac{1}{4}(-2\sin(C)\cos(A-B) + 2\sin C\cos C) + \sin A \cdot \sin B \cdot \sin C$$

$$= \frac{1}{4}(-2\sin(C)\cos(A-B) + 2\sin C\cos C) + \sin A \cdot \sin B \cdot \sin C$$

$$= -\frac{1}{4} \times 2\sin C(\cos(A-B) - \cos(A+B)) + \sin A \cdot \sin B \cdot \sin C$$

$$= -\frac{1}{4} \times 2\sin C \times 2\sin A\sin B + \sin A \cdot \sin B \cdot \sin C$$

$$= -\sin A\sin B\sin C + + \sin A \cdot \sin B \cdot \sin C$$

$$= 0$$

2. We have $\tan 2\alpha$

$$= \tan((\alpha+\beta) + (\alpha-\beta))$$

$$= \frac{\tan(\alpha+\beta) + \tan(\alpha-\beta)}{1 - \tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$= \frac{\left(\frac{3}{4} + \frac{5}{12}\right)}{1 - \left(\frac{3}{4} \cdot \frac{5}{12}\right)}$$

$$= \frac{36+20}{48-15}$$

$$= \frac{56}{33}$$

3. We have $A = \sin^2\theta + \cos^4\theta$

$$= \frac{1}{2}(2\sin^2\theta) + \frac{1}{4}(2\cos^2\theta)^2$$

$$= \frac{1}{2}(1 - \cos 2\theta) + \frac{1}{4}(1 + \cos 2\theta)^2$$

$$= \frac{1}{2}(1 - \cos 2\theta) + \frac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta)$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4}\cos^2 2\theta$$

$$= \frac{3}{4} + \frac{1}{4}\cos^2 2\theta$$

Maximum value $= \frac{3}{4} + \frac{1}{4} \cdot 1 = 1$

Minimum value $= \frac{3}{4} + \frac{1}{4} \cdot 0 = \frac{3}{4}$

Therefore, $\frac{3}{4} \le A \le 1$

4. Given $\tan A = \frac{1 - \cos B}{\sin B}$

$$\tan A = \frac{2\sin^2(B/2)}{2\sin(B/2)\cos(B/2)} = \tan\left(\frac{B}{2}\right)$$

Now, $\tan 2A$

$$= \frac{2\tan A}{1 - \tan^2 A}$$

$$= \frac{2\tan(B/2)}{1 - \tan^2(B/2)}$$

$$= \tan B$$

Hence, the result.

5. We have $\sin^3 x \sin 3x$

$$= (\sin^2 x)(\sin 3x \sin x)$$

$$= \frac{1}{4}(2\sin^2 x)(2\sin 3x \sin x)$$

$$= \frac{1}{4}(1 - \cos 2x)(\cos 2x - \cos 4x)$$

$$= \frac{1}{4}(\cos 2x - \cos^2 2x - \cos 4x + \cos 2x \cdot \cos 4x)$$

$$= \frac{1}{8} 2\cos 2x - 2\cos^2 2x - 2\cos 4x + 2\cos 2x \cdot \cos 4x]$$

$$= \frac{1}{8}[2\cos 2x - (1 + \cos 4x) - 2\cos 4x + \cos 6x + \cos 2x]$$

$$= \frac{1}{8}(-1 + 3\cos 2x - 3\cos 4x + \cos 6x)$$

Thus, $n = 6$

6. We have

$\sin 12° \sin 54° \sin 48°$

$$= (\sin 12° \sin 48°)\sin 54°$$

$$= \frac{1}{4\sin 72°}(4\sin 48° \sin 12° \sin 72°)\sin 54°$$

$$= \frac{1}{4\sin 72°}(\sin 36° \cos 36°)$$

$$= \frac{1}{4 \times 2\sin 72°}(2\sin 36° \cos 36°)$$

$$= \frac{1}{4 \times 2\sin 72°}(\sin 72°)$$

$$= \frac{1}{8}$$

7. We have

$\sin^2\alpha + \sin^2\beta - \sin^2\gamma$

$= \sin^2\alpha + \sin(\beta+\gamma)\sin(\beta-\gamma)$

$= \sin^2\alpha + \sin(\pi-\alpha)\sin(\beta-\gamma)$

$= \sin^2\alpha + \sin\alpha\sin(\beta-\gamma)$

$= \sin\alpha(\sin\alpha + \sin(\beta-\gamma))$

$= \sin\alpha(\sin(\pi-(\beta+\gamma)) + \sin(\beta-\gamma))$

$= \sin\alpha(\sin(\beta+\gamma) + \sin(\beta-\gamma))$

$= \sin\alpha \times 2\sin\beta\sin\gamma$

$= 2\sin\alpha\sin\beta\sin\gamma$

Hence, the result

8. We have

$$16\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)\cos\left(\frac{16\pi}{15}\right)$$

$$= -16\cos\left(\frac{\pi}{15}\right)\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)$$

$$= \frac{-16}{2\sin\left(\frac{\pi}{15}\right)}\sin\left(\frac{2\pi}{15}\right)\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)$$

$$= \frac{-8}{2\sin\left(\frac{\pi}{15}\right)}\sin\left(\frac{4\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)$$

$$= \frac{-4}{2\sin\left(\frac{\pi}{15}\right)}\sin\left(\frac{8\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)$$

$$= \frac{-1}{\sin\left(\frac{\pi}{15}\right)}\sin\left(\frac{16\pi}{15}\right)$$

$$= \frac{-1}{\sin\left(\frac{\pi}{15}\right)}\sin\left(\pi + \frac{\pi}{15}\right)$$

$$= \frac{-1}{\sin\left(\frac{\pi}{15}\right)} \times -\sin\left(\frac{\pi}{15}\right) = 1$$

9. We have

$$\left(1+\cos\left(\frac{\pi}{8}\right)\right)\left(1+\cos\left(\frac{3\pi}{8}\right)\right)$$
$$\left(1+\cos\left(\frac{5\pi}{8}\right)\right)\left(1+\cos\left(\frac{7\pi}{8}\right)\right)$$

$$= \left(1+\cos\left(\frac{\pi}{8}\right)\right)\left(1+\cos\left(\frac{3\pi}{8}\right)\right)$$
$$\left(1-\cos\left(\frac{3\pi}{8}\right)\right)\left(1-\cos\left(\frac{\pi}{8}\right)\right)$$

$$= \left(1-\cos^2\left(\frac{\pi}{8}\right)\right)\left(1-\cos^2\left(\frac{3\pi}{8}\right)\right)$$

$$= \sin^2\left(\frac{\pi}{8}\right)\sin^2\left(\frac{3\pi}{8}\right)$$

$$= \frac{1}{4}\left(2\sin^2\left(\frac{\pi}{8}\right)\right)\left(2\sin^2\left(\frac{3\pi}{8}\right)\right)$$

$$= \frac{1}{4}\left(1-\cos\left(\frac{\pi}{4}\right)\right)\left(1-\cos\left(\frac{3\pi}{4}\right)\right)$$

$$= \frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{4}\left(1-\frac{1}{2}\right)$$

$$= \frac{1}{4} \times \frac{1}{2}$$

$$= \frac{1}{8}$$

10. ***Note:*** No questions asked in 1985.

11. The given expression reduces to

$3(\cos^4\alpha + \sin^4\alpha) - 2(\cos^6\alpha + \sin^6\alpha)$

$= 3(1 - 2\sin^2\alpha\cos^2\alpha) - 2(1 - 3\sin^2\alpha\cos^2\alpha)$

$= 3 - 2$

$= 1$

12. ***Note:*** No questions asked in 1987.

13. We have $\sqrt{3}\,\text{cosec}\,(20°) - \sec(20°)$

$$= \frac{\sqrt{3}}{\sin(20°)} - \frac{1}{\cos(20°)}$$

$$= \frac{\sqrt{3}\cos(20°) - \sin(20°)}{\sin(20°)\cos(20°)}$$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos(20°) - \frac{1}{2}\sin(20°)\right)}{2\sin(20°)\cos(20°)}$$

$$= \frac{4(\sin(60°)\cos(20°) - \cos(60°)\sin(20°))}{\sin(40°)}$$

$$= \frac{4(\sin(60° - 20°))}{\sin(40°)}$$

$$= \frac{4(\sin(40°))}{\sin(40°)}$$

$$= 4$$

14. $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$

$= \cot\alpha - (\cot\alpha - \tan\alpha) + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$

$= \cot\alpha - (2\cot 2\alpha - 2\tan 2\alpha) + 4\tan 4\alpha + 8\cot 8\alpha$

$= \cot\alpha - 2(\cot 2\alpha - \tan 2\alpha) + 4\tan 4\alpha + 8\cot 8\alpha$

$= \cot\alpha - 2\cdot 2\cot 4\alpha + 4\tan 4\alpha + 8\cot 8\alpha$

$= \cot\alpha - 4(\cot 4\alpha - \tan 4\alpha) + 8\cot 8\alpha$

$= \cot\alpha - 4\cdot 2\cot 8\alpha + 8\cot 8\alpha$

$= \cot\alpha - 8\cot 8\alpha + 8\cot 8\alpha$

$= \cot\alpha$

15. ***Note:*** No questions asked between 1989-1990.

16. $\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\sin\left(\frac{7\pi}{14}\right)$

$\sin\left(\frac{9\pi}{14}\right)\sin\left(\frac{11\pi}{14}\right)\sin\left(\frac{13\pi}{14}\right)$

$= \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\sin\left(\frac{\pi}{2}\right)$

$\sin\left(\frac{9\pi}{14}\right)\sin\left(\frac{11\pi}{14}\right)\sin\left(\frac{13\pi}{14}\right)$

$= \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\sin\left(\frac{\pi}{2}\right)$

$\sin\left(\pi - \frac{5\pi}{14}\right)\sin\left(\pi - \frac{3\pi}{14}\right)\sin\left(\pi - \frac{\pi}{14}\right)$

$= \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)$

$\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)$

$= \left(\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\right)^2$

$= \left(\cos\left(\frac{\pi}{2} - \frac{\pi}{14}\right)\cos\left(\frac{\pi}{2} - \frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2} - \frac{5\pi}{14}\right)\right)^2$

$= \left(\cos\left(\frac{6\pi}{14}\right)\cos\left(\frac{4\pi}{14}\right)\cos\left(\frac{2\pi}{14}\right)\right)^2$

$= \left(\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)\right)^2$

$= \left(\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\right)^2$

$= \left(\frac{1}{8}\right)^2$

$= \frac{1}{64}$

17. We have $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

$= \cos 9x + \cos 10x$

Now, $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{9\pi}{2}\right) + \cos(5\pi)$

$= 0 - 1 = -1$

$f(\pi) = \cos(9\pi) + \cos(10\pi) = -1 + 1 = 0$

$f(-\pi) = \cos(-9\pi) + \cos(-10\pi)$

$= \cos(9\pi) + \cos(10\pi)$

$= -1 + 1 = 0$

$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{9\pi}{4}\right) + \cos\left(\frac{10\pi}{4}\right)$

$= \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$

18. Ans. (i) $\rightarrow$ C; (ii) $\rightarrow$ A

19. We have

$k = \sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$

$= \sin(10°)\sin(50°)\sin(70°)$

$= \sin(50°)\sin(10°)\sin(70°)$

$= \frac{1}{4} \times (4\sin(60° - 10°)\sin(10°)\sin(60° + 10°))$

$= \frac{1}{4} \times (\sin(3 \times 10°))$

$= \frac{1}{4} \times \frac{1}{2}$

$= \frac{1}{8}$

20. Let $y = \tan A \tan B$

$= \tan A \tan\left(\frac{\pi}{3} - A\right)$

$= \tan A\left(\frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}\right)$

$= \left(\frac{\sqrt{3}\tan A - \tan^2 A}{1 + \sqrt{3}\tan A}\right)$

$= \left(\frac{\sqrt{3}t - t^2}{1 + \sqrt{3}t}\right)$

$\Rightarrow \frac{dy}{dt} = \frac{(1 + \sqrt{3}t)(\sqrt{3} - 2t) - (\sqrt{3}t - t^2)\sqrt{3}}{(1 + \sqrt{3}t)^2}$

$= \frac{\sqrt{3} + 3t - 2t - 2\sqrt{3}t^2 - 3t + \sqrt{3}t^2}{(1 + \sqrt{3}t)^2}$

$= \frac{\sqrt{3} - 2t - \sqrt{3}t^2}{(1 + \sqrt{3}t)^2}$

$= -\frac{\sqrt{3}t^2 + 2t - \sqrt{3}}{(1 + \sqrt{3}t)^2}$

$= -\frac{\sqrt{3}t^2 + 3t - t - \sqrt{3}}{(1 + \sqrt{3}t)^2}$

$= -\frac{\sqrt{3}t(t + \sqrt{3}) - (t + \sqrt{3})}{(1 + \sqrt{3}t)^2}$

$= -\frac{(\sqrt{3}t - 1)(1 + \sqrt{3}t)}{(1 + \sqrt{3}t)^2}$

$\frac{dy}{dt} = 0$ gives $t = \frac{1}{\sqrt{3}}, -\sqrt{3}$

Thus, $t = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \tan A = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad A = \dfrac{\pi}{6}$

When $A = \dfrac{\pi}{6}, B = \dfrac{\pi}{3} - A = \dfrac{\pi}{3} - \dfrac{\pi}{6} = \dfrac{\pi}{6}$

Thus, the minimum value of $\tan A \cdot \tan B$

$= \dfrac{1}{\sqrt{3}} \cdot \dfrac{1}{\sqrt{3}}$

$= \dfrac{1}{3}$

21. We have $(\sec 2x - \tan 2x)$

$$= \left(\frac{1 - \sin 2x}{\cos 2x}\right)$$

$$= \left(\frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x}\right)$$

$$= \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

$$= \left(\frac{1 - \tan x}{1 + \tan x}\right)$$

$$= \tan\left(\frac{\pi}{4} - x\right)$$

22. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$

$= 3[\sin^4 x - 4\sin^3 x \cos x + 6\sin^2 x \cos^2 x$
$\quad - 4\sin x \cos^3 x + \cos^4 x] + 6(1 + 2\sin x \cos x)$
$\quad + 4(1 - 3\sin^2 x \cos^2 x)$

$= 3[(\sin^4 x + \cos^4 x) - 4\sin x \cos x(\sin^2 x + \cos^2 x)$
$\quad + 6\sin^2 x \cos^2 x] + 6(1 + 2\sin x \cos x)$
$\quad + 4(1 - 3\sin^2 x \cos^3 x)$

$= 3[1 - 2\sin^2 x \cos^2 x + 4\sin x \cos x$
$\quad + 6\sin^2 x \cos^2 x] + 6(1 + 2\sin x \cos x)$
$\quad + 4(1 - 3\sin^2 x \cos^2 x)$

$= 3 + 6 + 4$

$= 13$

23. As we know

$(\sin\alpha + \sin\beta + \sin\gamma) < \sin(\alpha + \beta + \gamma)$

$= \sin(\pi) = 0$

Thus, $\sin\alpha + \sin\beta + \sin\gamma < 0$

24. we have $\sec^2\theta = \left(\dfrac{4xy}{(x + y)^2}\right)$

$$\Rightarrow \quad 1 + \tan^2\theta = \left(\frac{4xy}{(x + y)^2}\right)$$

$$\Rightarrow \quad \tan^2\theta = \left(\frac{4xy}{(x + y)^2} - 1\right)$$

$$\Rightarrow \quad \tan^2\theta = \left(\frac{4xy - (x + y)^2}{(x + y)^2}\right) = -\left(\frac{x - y}{x + y}\right)^2$$

$$\Rightarrow \quad -\left(\frac{x - y}{x + y}\right)^2 = \tan^2\theta \geq 0$$

$\Rightarrow \quad -(x - y)^2 \geq 0$

$\Rightarrow \quad (x - y)^2 \leq 0$

$\Rightarrow \quad (x - y)^2 = 0$

$\Rightarrow \quad x - y = 0$

$\Rightarrow \quad x = y$

Therefore, $x = y \neq 0$

25. We have $y = \cos x \cos(x + 2) - \cos^2(x + 1)$

$$= \frac{1}{2}(2\cos x \cos(x + 2) - 2\cos^2(x + 1))$$

$$= \frac{1}{2}(2\cos(2x + 2) + \cos(1) - (1 + \cos(2x + 2)))$$

$$= -\frac{1}{2}(1 - \cos(1))$$

$$= -\frac{1}{2} \times 2\sin^2(1)$$

$$= -\sin^2(1)$$

which is a straight line passing through $\left(\dfrac{\pi}{2}, -\sin^2 1\right)$ and parallel to $x$-axis.

26. We have $\sin 15° \cos 15°$

$$= \frac{1}{2}(2\sin 15° \cos 15°)$$

$$= \frac{1}{2}(\sin 30°)$$

$$= \frac{1}{4}$$

27. We have

$$f_n(\theta) = \tan\left(\frac{\theta}{2}\right)(1 + \sec\theta)(1 + \sec 2\theta) \ldots\ldots$$
$$(1 + \sec 4\theta) \ldots (1 + \sec 2^\circ\theta)$$

$$= \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \times \left(\frac{1 + \cos\theta}{\cos\theta}\right)(1 + \sec 2\theta)$$
$$(1 + \sec 4\theta) \ldots (1 + \sec 2^\circ\theta)$$

$$= \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\cos\theta}(1 + \sec 2\theta)(1 + \sec 4\theta)$$
$$(1 + \sec 8\theta) \ldots (1 + \sec 2^\circ\theta)$$

$$= \frac{\sin\theta}{\cos\theta} \times \left(\frac{1 + \cos 2\theta}{\cos 2\theta}\right)(1 + \sec 4\theta)$$
$$(1 + \sec 8\theta) \ldots (1 + \sec 2^\circ\theta)$$

$$= \frac{2\sin\theta\cos\theta}{\cos 2\theta} \times \left(\frac{1 + \cos 4\theta}{\cos 4\theta}\right)$$
$$(1 + \sec 8\theta) \ldots (1 + \sec 2^\circ\theta)$$

$$= \frac{\sin 2\theta}{\cos 2\theta} \times \frac{2\cos^2 2\theta}{\cos 4\theta}(1+\sec 8\theta)\ldots(1+\sec 2^n\theta)$$

$$= \frac{\sin 4\theta}{\cos 4\theta}\left(\frac{1+\cos 8\theta}{\cos 8\theta}\right)\ldots\ldots(1+\sec 2^n\theta)$$

$$= \frac{\sin 4\theta}{\cos 4\theta}\left(\frac{2\cos^2 4\theta}{\cos 8\theta}\right)\ldots\ldots(1+\sec 2^n\theta)$$

$$= \frac{\sin 8\theta}{\cos 8\theta}\ldots\ldots\ldots\ldots(1+\sec 2^n\theta)$$

$$= \frac{\sin(2^n\theta)}{\cos(2^n\theta)}$$

$$= \tan(2^n\theta)$$

Now, $f_2\left(\frac{\pi}{16}\right) = \tan\left(2^2 \times \frac{\pi}{16}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$f_3\left(\frac{\pi}{32}\right) = \tan\left(2^3 \times \frac{\pi}{32}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$f_4\left(\frac{\pi}{64}\right) = \tan\left(2^4 \times \frac{\pi}{64}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$f_5\left(\frac{\pi}{128}\right) = \tan\left(2^5 \times \frac{\pi}{128}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

Hence, the result.

28. We have $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$

$= \sin\theta(\sin 3\theta + \sin\theta)$

$= \sin\theta \times 2\sin 2\theta\cos\theta$

$= 2\sin\theta\cos\theta \times \sin 2\theta$

$= \sin 2\theta \times \sin 2\theta$

$= \sin^2 2\theta$

$\geq 0$ for all real $\theta$

29. Given, $\cot\alpha_1 \times \cot\alpha_2 \ldots \cot\alpha_n = 1$

$\Rightarrow \cos\alpha_1 \cdot \cos\alpha_2 \ldots \cos\alpha_n = \sin\alpha_1 \times \sin\alpha_2 \ldots \sin\alpha_n$

$\Rightarrow (\cos\alpha_1 \times \cos_2 \ldots \cos\alpha_n)^2$

$= (\cos\alpha_1 \sin\alpha_1)(\cos\alpha_2 \sin\alpha_2) \ldots (\cos\alpha_n \sin\alpha_n)$

$= \frac{1}{2^n}(2\cos\alpha_1\sin\alpha_1)(2\cos\alpha_2\sin\alpha_2)\ldots$

$(2\cos\alpha_n\sin\alpha_n) \leq \frac{1}{2^n}$

$\Rightarrow (\cos\alpha_1 \cdot \cos\alpha_2 \ldots \cos\alpha_n) \leq \frac{1}{2^{n/2}}$

30. ***Note:*** No questions asked in 2002.

31. Ans. (c)

Given $\alpha = \beta + \gamma$

$\Rightarrow \tan\alpha = \tan(\beta + \gamma)$

$\Rightarrow \tan\alpha = \frac{\tan\beta + \tan\gamma}{1 - \tan\beta\tan\gamma}$

$\Rightarrow \tan\alpha - \tan\alpha \times \tan\beta \times \tan\gamma = \tan\beta + \tan\gamma$

$\Rightarrow \tan\alpha - \tan\gamma = \tan\beta + \tan\gamma$

$\Rightarrow \tan\alpha = \tan\beta + 2\tan\gamma$

32. As we know that A.M. $\geq$ G.M.

$$\Rightarrow \frac{\left(\sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}}\right)}{2} \geq \sqrt{\sqrt{x^2+x} \cdot \frac{\tan^2\alpha}{\sqrt{x^2+x}}}$$

$$\Rightarrow \frac{\left(\sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}}\right)}{2} \geq \tan\alpha$$

$$\Rightarrow \left(\sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}}\right) \geq 2\tan\alpha$$

Hence, the result.

33. Given $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

Also, $\cos\varphi = \frac{1}{3} < \frac{1}{2}$

$\Rightarrow \frac{\pi}{2} < \varphi < \frac{\pi}{3}$

Thus, $\frac{\pi}{6} + \frac{\pi}{3} < \theta + \varphi < \frac{\pi}{6} + \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} < \theta + \varphi < \frac{2\pi}{3}$

34. Let $y = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1}$

$\Rightarrow 3x^2y - 2xy - y = 1 - 2x + 5x^2$

$\Rightarrow (3y - 5)x^2 + 2(1 - y)x - (1 + y) = 0$

As $x$ is real, so $(y - 1)^2 + (3y - 5)(y + 1) \geq 0$

$\Rightarrow y^2 - 2y + 1 + (3y^2 - 5y + 3y - 5) \geq 0$

$\Rightarrow 4y^2 - 4y - 4 \geq 0$

$\Rightarrow y^2 - y - 1 \geq 0$

$\Rightarrow (2y - 1)^2 \geq (\sqrt{5})^2$

$\Rightarrow (2y - 1) \geq \sqrt{5}$ or $(2y - 1) \leq -\sqrt{5}$

$\Rightarrow y \geq \left(\frac{\sqrt{5}+1}{2}\right)$ or $y \leq \left(\frac{1-\sqrt{5}}{2}\right)$

$\Rightarrow y \geq \left(\frac{\sqrt{5}+1}{2}\right)$ or $y \leq -\left(\frac{\sqrt{5}-1}{2}\right)$

$\Rightarrow 2\sin t \geq \left(\frac{\sqrt{5}+1}{2}\right)$ or $2\sin t \leq -\left(\frac{\sqrt{5}-1}{2}\right)$

$\Rightarrow \sin t \geq \left(\frac{\sqrt{5}+1}{4}\right)$ or $\sin t \leq -\left(\frac{\sqrt{5}-1}{4}\right)$

$\Rightarrow \sin \geq \sin(54°)$ or $\sin t \leq \sin(-18°)$

$\Rightarrow \sin t \geq \sin\left(\frac{3\pi}{10}\right)$ or $\sin t \leq \sin\left(-\frac{\pi}{10}\right)$

$\Rightarrow \frac{3\pi}{10} \leq t \leq \frac{\pi}{2}$ or $-\frac{\pi}{2} \leq t \leq -\frac{\pi}{10}$

Therefore, $t \in \left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$

35. Given, $\cos(\alpha-\beta)=1$ and $\cos(\alpha+\beta)=\dfrac{1}{e}$

$\Rightarrow \quad \cos(\alpha-\beta)=\cos(0)$

$\Rightarrow \quad \alpha=\beta$

Also, $\cos(\alpha+\beta)=\dfrac{1}{e}$

$\Rightarrow \quad \cos(2\alpha)=\dfrac{1}{e}$

Given $-\pi<\alpha<\pi$

$-2\pi<2\alpha<2\pi$

Thus, there are 4 values of the ordered pass of $(\alpha, \beta)$ satisfies the relation $\cos(2\alpha)=\dfrac{1}{e}$

36. As $0<\theta<\dfrac{\pi}{4}$

$\Rightarrow \quad 0<\tan\theta<1$ and $\cot\theta>1$

We also know that , if $0<x<1$ and $0<a<b$

$\Rightarrow$ then $x^b<x^a<\left(\dfrac{1}{x}\right)^a<\left(\dfrac{1}{x}\right)^b$

$\Rightarrow \quad (\tan\theta)^{\cot\theta}<(\tan\theta)^{\tan\theta}<(\cot\theta)^{\tan\theta}<(\cot\theta)^{\cot\theta}$

$\Rightarrow \quad t_2<t_1<t_3<t_4$

Hence, the result.

37. Given, $\dfrac{\sin^4 x}{2}+\dfrac{\cos^4 x}{3}=\dfrac{1}{5}$

$\Rightarrow \quad \dfrac{5}{2}\sin^4 x+\dfrac{5}{3}\cos^4 x=1$

$\Rightarrow \quad \left(1+\dfrac{3}{2}\right)\sin^4 x+\left(1+\dfrac{2}{3}\right)\cos^4 x=1$

$\Rightarrow \quad (\sin^4 x+\cos^4 x)+\left(\dfrac{3}{2}\sin^4 x+\dfrac{2}{3}\cos^4 x\right)=1$

$\Rightarrow \quad 1-2\sin^2 x\cos^2 x+\left(\dfrac{3}{2}\sin^4 x+\dfrac{2}{3}\cos^4 x\right)=1$

$\Rightarrow \quad \left(\dfrac{3}{2}\sin^4 x+\dfrac{2}{3}\cos^4 x-2\sin^2 x\cos^2 x\right)=0$

$\Rightarrow \quad \left(\sqrt{\dfrac{3}{2}}\sin^2 x-\sqrt{\dfrac{2}{3}}\cos^2 x\right)^2=0$

$\Rightarrow \quad \left(\sqrt{\dfrac{3}{2}}\sin^2 x-\sqrt{\dfrac{2}{3}}\cos^2 x\right)=0$

$\Rightarrow \quad \sqrt{\dfrac{3}{2}}\sin^2 x=\sqrt{\dfrac{2}{3}}\cos^2 x$

$\Rightarrow \quad \dfrac{\sin^2 x}{2}=\dfrac{\cos^2 x}{3}=\dfrac{1}{5}$

$\Rightarrow$ Thus, $\tan^2 x=\dfrac{2}{3}$

Now, $\dfrac{\sin^8 x}{8}+\dfrac{\cos^8 x}{27}$

$=\dfrac{(\sin^2 x)^4}{8}+\dfrac{(\cos^2 x)^4}{27}$

$=\dfrac{(2/5)^4}{8}+\dfrac{(3/5)^4}{27}$

$=\dfrac{2}{5^4}+\dfrac{3}{5^4}$

$=\dfrac{2+3}{5^4}=\dfrac{1}{5^3}=\dfrac{1}{125}$

38. Let $f(\theta)=\sin^2\theta+3\sin\theta\sin\theta+5\cos^2\theta$

$=1+3\sin\theta\cos\theta+4\cos^2\theta$

$=1+\dfrac{3}{2}\sin 2\theta+2(1+\cos 2\theta)$

$=3+\dfrac{3}{2}\sin 2\theta+2\cos 2\theta$

Maximum value $=\sqrt{\dfrac{9}{4}+4}+3=\dfrac{5}{2}+3=\dfrac{11}{2}$

Minimum value $=-\sqrt{\dfrac{9}{4}+4}+3=-\dfrac{5}{2}+3=\dfrac{1}{2}$

Therefore, the minimum value of

$\dfrac{1}{\sin^2\theta+3\sin\theta\cos\theta+5\cos^2\theta}$ is $\dfrac{2}{11}$

39. Now, $P:\sin\theta-\cos\theta=\sqrt{2}\cos\theta$

$\Rightarrow \quad \sin\theta=(\sqrt{2}+1)\cos\theta$

$\Rightarrow \quad \cos\theta=\dfrac{\sin\theta}{(\sqrt{2}+1)}$

$\Rightarrow \quad \cos\theta=\dfrac{\sin\theta}{(\sqrt{2}+1)}\times\dfrac{(\sqrt{2}-1)}{(\sqrt{2}-1)}$

$\Rightarrow \quad \cos\theta=(\sqrt{2}-1)\sin\theta$

$\Rightarrow \quad \cos\theta+\sin\theta=\sqrt{2}\sin\theta$

$Q:\cos\theta+\sin\theta=\sqrt{2}\sin\theta$

Thus, $P=Q$

40. Given, $\dfrac{1}{\sin\left(\dfrac{\pi}{n}\right)}=\dfrac{1}{\sin\left(\dfrac{2\pi}{n}\right)}+\dfrac{1}{\sin\left(\dfrac{3\pi}{n}\right)}$

Let $\dfrac{\pi}{n}=\theta$

Then $\dfrac{1}{\sin\theta}=\dfrac{1}{\sin 2\theta}+\dfrac{1}{\sin 3\theta}$

$\Rightarrow \quad \dfrac{1}{\sin\theta}-\dfrac{1}{\sin 3\theta}=\dfrac{1}{\sin 2\theta}$

$\Rightarrow \quad \dfrac{\sin 3\theta-\sin\theta}{\sin\theta\sin 3\theta}=\dfrac{1}{\sin 2\theta}$

$\Rightarrow \quad \dfrac{2\cos 2\theta\sin\theta}{\sin\theta\sin 3\theta}=\dfrac{1}{\sin 2\theta}$

$\Rightarrow \quad \dfrac{2\cos 2\theta}{\sin 3\theta}=\dfrac{1}{\sin 2\theta}$

$\Rightarrow \quad \sin 4\theta = \sin 3\theta$

$\Rightarrow \quad \sin 4\theta = \sin(\pi - 3\theta)$

$\Rightarrow \quad 4\theta = \pi - 3\theta$

$\Rightarrow \quad 7\theta = \pi$

$\Rightarrow \quad 7 \cdot \dfrac{\pi}{n} = \pi$

$\Rightarrow \quad n = 7$

42. Let $\cos 4\theta = \dfrac{1}{3}$

$\Rightarrow \quad 2\cos^2(2\theta) - 1 = \dfrac{1}{3}$

$\Rightarrow \quad 2\cos^2(2\theta) = 1 + \dfrac{1}{3} = \dfrac{4}{3}$

$\Rightarrow \quad \cos^2(2\theta) = \dfrac{2}{3}$

$\Rightarrow \quad \cos(2\theta) = \pm\sqrt{\dfrac{2}{3}}$

$\Rightarrow \quad 2\cos^2\theta - 1 = \pm\sqrt{\dfrac{2}{3}}$

$\Rightarrow \quad 2\cos^2\theta = 1 \pm \sqrt{\dfrac{2}{3}}$

Now, $f(\cos 4\theta) = \dfrac{2}{2 - \sec^2\theta} = \dfrac{2\cos^2\theta}{2\cos^2\theta - 1}$

$$\Rightarrow \quad f\left(\frac{1}{3}\right) = \frac{2\left(1 \pm \sqrt{\frac{2}{3}}\right)}{2\left(1 \pm \sqrt{\frac{2}{3}}\right) - 1}$$

$$= 1 \pm \sqrt{\frac{3}{2}}$$

CHAPTER

2

# Trigonometric Equations

## CONCEPT BOOSTER

### 2.1 Definition

An equation involving one or more trigonometrical ratios of unknown angle is called trigonometrical equation.

For example, $\sin^2 x - \cos x - 2 = 0$, $\tan x = 1$, $\cos^2 x + \cos x - 2 = 0$, etc. are trigonomteric equations.

### 2.2 Solution of a Trigonometric Equation

A value of the unknown angle which satisfies the given trigonometrical equation is called a solution or root of the equation

For example, $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

Types of solutions:

(i) Principal solution
(ii) General solution

Principal solution:
The smallest nutmerical value of the angle which satisfies the given equation is called the principal solutions.

General solution:
Since trigonometric functions are periodic function, therefore the solution of trigonometric equations can be genralised with the help of periodicity of a trigonometrical function. The solution consisting of all possible solution of a trigonometrical equation is called the general solution.

### 2.3 General solution of Trigonometric Equations

The general solution of the equations are

1. $\sin\theta = 0 \Rightarrow \theta = n\pi, n \in I$
2. $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}, n \in I$.
3. $\tan\theta = 0 \Rightarrow \theta = n\pi, n \in I$
4. $\sin\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{2}, n \in I$
5. $\cos\theta = 1 \Rightarrow \theta = 2n\pi, n \in I$
6. $\tan\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{4}, n \in I$
7. $\sin\theta = -1 \Rightarrow \theta = (4n-1)\frac{\pi}{2}, n \in I$
8. $\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi, n \in I$
9. $\tan\theta = -1 \Rightarrow \theta = (4n-1)\frac{\pi}{4}, n \in I$
10. $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha, n \in I$
11. $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha, n \in I$
12. $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha, n \in I$
13. $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$
14. $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$
15. $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$

### 2.4 A Trigonometric Equation is of the Form

$a\cos\theta \pm b\sin\theta = c$

**Rule:**

1. Divide by $\sqrt{a^2+b^2}$ on both the sides
2. Reduce the given equation into either $\sin(\theta \pm \alpha)$ or $\cos(\theta \pm \alpha)$
3. Simplify the given equation.

### 2.5 Principal Value

The numerically least angle is called the principal value.

For example, $\sin\theta = \frac{1}{2}$

Then,

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \cdot -\frac{11\pi}{6} \cdot -\frac{7\pi}{6}$$

Among all these values of $\theta$, $\dfrac{\pi}{6}$ is the numerically smallest.

So principal value of $\sin\theta = \dfrac{1}{2}$ is $\theta = \dfrac{\pi}{6}$.

## 2.6 Method to Find Out the Principal Value

(i) First draw a trigonometric circle and mark the quadrant in which the angle may lie.
(ii) Select anti-clockwise direction for 1st and 2nd quadrant and select clockwise direction for 3rd and 4th quadrants.
(iii) Find the angle in the first rotation.
(iv) Select the numerically least angle among these two values. The angle thus formed will be the principal value.
(v) In case, two angles, one with +ve sign and the other with –ve sign, qualify for the numerically least angle, then it is the conventional of mathematics, to consider the angle with +ve signs as principal value.

## 2.7 Solutions in Case of Two Equations are Given

If two equations are given and we have to find the value of $\theta$ which may satisfy both the given equations, like

$$\cos\theta = \cos\alpha,\ \sin\theta = \sin\alpha \text{ and } \tan\theta = \tan\alpha,$$

then the common solution is $\theta = 2n\pi + \alpha$, where $n \in Z$.

Similarly, $\sin\theta = \sin\alpha$, $\tan\theta = \tan\alpha$.

The common solution is $\theta = 2n\pi + \alpha$, $n \in Z$

**Rule:**

(i) Find the common value of $\theta$ between 0 and $2\pi$.
(ii) Add $2n\pi$ to this common value.
(iii) Then we shall get the general value of the given two equations.

## 2.8 Some Important Remarks to Keep in Mind while Solving a Trigonometric Equation

(i) Squaring the equation at any stage should be avoided as far as possible. If squaring is necessary, check the solutions for extraneous roots.
(ii) Never cancel terms containing unknown terms on both the sides, which are in product form. It may cause roots loose.
(iii) Domain should not changed. If it is changed, necessary correction must be made.
(iv) Check the denominator is non zero at any stage, while solving equations.
(v) The answer should not contain such values of angles, which may be any of the terms undefined.
(vi) Some times, we may find our answers differ from those in the package in their notations. This leads to the different methods of solving the same problem.
Whenever we come across such situation, we must check authenticity. This will ensure that our answer is correct.
(vii) Some times the two soltuion sets consist partly of common values. In all such cases the common part must be presented only once.

## 2.9 Types of Trigonometric Equations:

**Type 1**
A trigonometric equation reduces to quadratic/higher degree equations.

***Rules:***

1. Transform the terms to be a only one trigonometric ratio involving angles in same form.
2. Factorize the equation and express it in $f(x) \times g(x) = 0$
$\Rightarrow \quad f(x) = 0$ or $g(x) = 0$
3. Solve both the equations one by one to get general value of the variables.

**Type 2**
A trigonometric equation is solved by factorization method.

***Rules:*** Simply reduces to a single trigonometric ratio of the unknown angles and factorize by basic algebraic method.

**Type 3**
A trigonometric equation is solved by transformation as a sum or difference into a product.

**Rules:**

1. The given equation is reducible to $f(x) \times g(x) = 0$
$\Rightarrow \quad f(x) = 0, g(x) = 0$
2. Solve both the equations one by one to get the genral value of the variable $x$

**Type 4**
A trigonometric equation is solved by transformation as a product into a sum or difference:

***Rules:***

1. The given equation can be reduces to $f(x) = 0$, $g(x) = 0$
2. Solve both the equation one-by-one to get the general value of the variable $x$.

**Type 5**
A Trigonometric equation is of the form

$$b_0 \sin^n x + b_1 \sin^{n-1} x \cdot \cos x + b^2 \sin^{n-2} x \cdot \cos^2 x + \ldots + b_n \cos^n x = 0$$

where, $b_0, b_1, b_2, \ldots, b_n \in R$ is a homogenous equation of $\sin x$ and $\cos x$, where $\cos x$ is non zero.

***Rules:***

1. Divide both the sides by highest power of $\cos x$.
2. The given equation can be reduces to $b_0 \tan^n x + b_1 \tan^{n-1} x + \ldots + b_n = 0$
3. and then use factorization method.

**Type 6**
A trigonometric equation is of the form

$$R(\sin mx, \cos nx, \tan px, \cot qx) = 0$$

where $R$ is a rational function and $m, n, p, q \in N$, can be reduced to a rational function with respect to

$\sin x$, $\cos x$, $\tan x$ and $\cot x$.

***Rules:***

1. Use half angle formulae of tangents:

$$\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)},$$

$$\cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

$$\tan x = \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}$$

$$\cot x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{2\tan\left(\frac{x}{2}\right)}$$

2. Substitute $\tan\left(\frac{x}{2}\right) = t$, and then solve it.

**Type 7**

A trigonometric equation is of the form

$R(\sin x + \cos x, \sin x.\cos x) = 0$,

where $R$ is a rational function of the argument of $\sin x$ and $\cos x$.

***Rule:***

1. Put $\sin x + \cos x = t$
2. Use the identity $(\sin x + \cos x)^2 = 1 + 2\sin x\cos x$

$$\Rightarrow \quad \sin x \cos x = \frac{t^2-1}{2}$$

3. So, the given equation reduces to $R\left(t, \frac{t^2-1}{2}\right)$ and then solve it.

**Type 8**

A trigonometrical equation is based on extreme values of $\sin x$ and $\cos x$.

***Rule:***

1. Whenever terms are sin, cos in power 1 and all terms connected with plus sign and number of terms in LHS (with +ve or –ve sign) then each term must have in extreme value.
2. In such problems, each term will be +1 when the value of RHS is positive and each term will be (–1) when the value of RHS is negative.

**Type 9**

A trigonometrical equation involving with exponential, logarithmic and modulii terms:

***Rule:*** Whenever equation contains power term, then we should use the following method.

1. Equate the base if possible
2. If, it is not possible to equate the base, take log of both the sides and make its RHS is zero, then we proceed further.

**Type-10**

A Trigonometrical equation involving the terms of two sides are of different nature:

***Rules:***

1. Let $y$ = each side of the equation and break the equation in two parts.
2. Find the inequality for $y$ taking LHS of the equation and also for the RHS of the equation.
3. Use the AM $\geq$ GM on the right hand side of the equation.
4. If there is any value of $y$ satisfying both the inequalities, then the equation will have real solution, otherwise no solution.

## Exercises

### Level I

**(Problems Based on Fundamentals)**

1. Solve for $\theta$ : $\sin 3\theta = 0$
2. Solve for $\theta$: $\cos^2(5\theta) = 0$
3. Solve for $\theta$: $\tan\theta = \sqrt{3}$
4. Solve for $\theta$: $\sin 2\theta = \sin\theta$
5. Solve for $\theta$: $\sin(9\theta) = \sin\theta$
6. Solve for $\theta$: $5\sin^2\theta + 3\cos^2\theta = 4$
7. Solve for $\theta$: $\tan(\theta - 15°) = 3\tan(\theta + 15°)$
8. Solve for $\theta$: $\tan^2(\theta) + \cot^2(\theta) = 2$
9. Solve for $\theta$:
   $\cos(\theta) + \cos(2\theta) + \cos(3\theta) = 0$
10. Solved for $\theta$:
    $\sin(2\theta) + \sin(4\theta) + \sin(6\theta) = 0$
11. Solve for $\theta$:
    $\tan(\theta) + \tan(2\theta) + \tan(\theta) + \tan(2\theta) = 1$
12. Solve for $\theta$: $\tan(\theta) + \tan(2\theta) + \tan(3\theta)$
    $= \tan(\theta)\cdot\tan(2\theta)\cdot\tan(\theta)$
13. Solve for $\theta$: $\cot^2\theta + \dfrac{3}{\sin\theta} + 3 = 0$
14. Solve for $\theta$: $2\tan\theta - \cot\theta = -1$

15. Solve for $\theta$: $\tan^2\theta + (1-\sqrt{3})\tan\theta - \sqrt{3} = 0$
16. Solve for $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$
17. Solve for $\theta$: $3\tan(\theta - 60°) = \tan(\theta + 60°)$
18. Solve for $\theta$: $\tan\theta + \tan 2\theta + \tan 3\theta = 0$
19. Solve for $\theta$: $\cos 2\theta \cos 4\theta = \frac{1}{2}$
20. Solve for $\theta$: $\cot\theta - \tan\theta = \cos\theta - \sin\theta$
21. Solve for $\theta$: $(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$
22. Solve for $\theta$: $2\sin^2\theta + \sin^2 2\theta = 2$
23. Solve for $\theta$: $\sin 3\alpha = 4\sin\alpha \sin(\theta + \alpha)\sin(\theta - \alpha)$, $\alpha \neq n\pi, n \in Z$
24. Solve for $\theta$: $4\sin\theta \sin 2\theta \sin 4\theta = \sin 3\theta$

A trigonometric equation is of the form
**$a\cos\theta \pm b\sin\theta = c$**

25. Solve for $\theta$ : $\sin(\theta) + \cos(\theta) = 1$
26. Solve for $\theta$: $\sqrt{3}\sin(\theta) + \cos(\theta) = 2$
27. Solve for $\theta$:
$\sin(2\theta) + \cos(2\theta) = \sin(\theta) + \cos(\theta) + 1 = 0$
28. Solve for $\theta$: $\sin^3\theta + \sin\theta\cos\theta + \cos^3\theta = 1$
29. Solve for $\theta$: $\sin\theta + \sqrt{3}\cos\theta = \sqrt{2}$
30. Solve for $\theta$: $\sqrt{2}\sec\theta + \tan\theta = 1$
31. Solve for $\theta$: $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$
32. Solve for $\theta$: $\sin\theta + \cos\theta = \sqrt{2}$
33. Solve for $\theta$: $\sqrt{3}\cos\theta + \sin\theta = 1$
34. Solve for $\theta$: $\sin\theta + \cos\theta = 1$
35. Solve for $\theta$: $\operatorname{cosec}\theta = 1 + \cot\theta$
36. Solve for $\theta$: $\tan\theta + \sec\theta = \sqrt{3}$
37. Solve for $\theta$: $\cos\theta + \sqrt{3}\sin\theta = 2\cos 2\theta$
38. Solve for $\theta$:
$\sqrt{3}(\cos\theta - \sqrt{3}\sin\theta) = 4\sin 2\theta \cdot \cos 3\theta \cdot$

## PRINCIPAL VALUE

39. Find the principal value of $\sin(\theta) = -\frac{1}{2}$.
40. Find the principal value of $\sin(\theta) = \frac{1}{\sqrt{2}}$
41. Find the principal value of $\tan(\theta) = -\sqrt{3}$
42. Find the principal values of $\tan\theta = -1$
43. Find the principal values of $\cos\theta = \frac{1}{2}$
44. Find the principal values of $\cos\theta = -\frac{1}{2}$
45. Find the principal values of $\tan\theta = -\sqrt{3}$
46. Find the principal values of $\sec\theta = \sqrt{2}$.

**Solutions in case if two equations are given:**

47. If $\sin(\theta) = \frac{1}{\sqrt{2}}$ and $\cos(\theta) = -\frac{1}{\sqrt{2}}$, then find the general values of $\theta$
48. If $\sin(\theta) = \frac{1}{\sqrt{2}}$ and $\tan(\theta) = -1$, then find the general values of $\theta$
49. If $\cos\theta = \frac{1}{\sqrt{2}}$ and $\tan\theta = -1$, then find the general value of $\theta$
50. Find the most general value of $\theta$ which satisfy the equations $\sin\theta = \frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$
51. If $(1 + \tan A)(1 + \tan B) = 2$, then find all the values of $A + B$
52. If $\tan(A - B) = 1$ and $\sec(A + B) = \frac{2}{\sqrt{3}}$, then find the smallest +ve values of $A$ and $B$ and their most general values.
53. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, then prove that, $\cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$.
54. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then prove that $\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$
55. If $\sin A = \sin B$ and $\cos A = \cos B$, then find the values of $A$ in terms of $B$.
56. If $A$ and $B$ are acute +ve angles satisfying the equations $3\sin^2 A + 2\sin^2 B = 1$ and $3\sin^2 A - 2\sin^2 B = 0$, then find $A + 2B$.
57. Solve $x + y = \frac{\pi}{4}$ and $\tan x + \tan y = 1$.
58. Solve $\sin x + \sin y = 1$, $\cos 2x - \cos 2y = 1$.
59. Solve $x + y = \frac{2\pi}{3}$ and $\sin x = 2\sin y$.
60. Solve $x + y = \frac{2\pi}{3}$ and $\cos x + \cos y = \frac{3}{2}$
61. If $r\sin\theta = 3$ and $r = 4(1 + \sin\theta)$, where $0 \leq \theta \leq 2\pi$, then find the value of $\theta$
62. Solve $\sin x + \sin y = 1$
$\cos 2x - \cos 2y = 1$
63. Find the co-ordinates of the point of inter section of the curves $y = \cos x$ and $y = \sin 2x$
64. Find all points of $x$, $y$ that satisfying the equations $\cos x + \cos y + \cos(x + y) = -\frac{3}{2}$
65. If $0 < \theta, \varphi < \pi$ and $8\cos\theta\cos\varphi\cos(\theta + \varphi) + 1 = 0$, then find $\theta$ and $\varphi$.

66. Find the set of values of $x$ for which $\dfrac{\tan 3x - \tan 2x}{1 + \tan 3x \cdot \tan 2x} = 1$
67. Find the number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$
68. Find the number of values of $x$ in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$
69. Find the smallest positive value of $x$ such that $\tan(x + 20°) = \tan(x - 10°) \cdot \tan x \cdot \tan(x + 10°)$
70. If $\sin^2 x + \cos^2 y = 2\sec^2 z$, then find $x$, $y$ and $z$.

## DIFFERENT TYPES OF TRIGONOMETRIC EQUATIONS

### Type 1

71. Solve $5\cos 2x + 2\cos^2\left(\dfrac{x}{2}\right) + 1 = 0$
72. Solve $4\sin^4 x + \cos^4 x = 1$
73. Solve $4\cos^2 x \sin x - 2\sin^2 x = 2\sin x$
74. Solve $\sin 3x + \cos 2x = 1$
75. Solve $2\cos 2x + \sqrt{2\sin x} = 2$
76. Solve $1 + \sin^2 x + \cos^2 x = \dfrac{3}{2}\sin 2x$
77. Solve $\sin^6 x + \cos^6 x = \dfrac{7}{16}$
78. Solve $\sin^8 x + \cos^8 x = \dfrac{17}{16}\cos^2 2x$
79. Solve $2\sin^3 x + 2 = \cos^2 3x$
80. Solve $\cos 4x = \cos^2 3x$
81. Solve $\cos 2x = 6\tan^2 x - 2\cos^2 x$

### Type 2

82. Solve $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$
83. Solve $2\sin^2 x + \sin x - 1 = 0$ where $0 \le x \le 2\pi$
84. Solve $5\sin^2 x + 7\sin x - 6 = 0$, where $0 \le x \le 2\pi$
85. Solve $\sin^2 x - \cos x = \dfrac{1}{4}$, where $0 \le x \le 2\pi$
86. Solve $\tan^2 x - 2\tan x - 3 = 0$
87. Solve $2\cos^2 x - \sqrt{3}\sin x + 1 = 0$

### Type 3

88. Solve $\sin x + \sin 3x + \sin 5x = 0$, $0 \le x \le \dfrac{\pi}{2}$
89. Solve $\cos x - \cos 2x = \sin 3x$
90. Solve $\sin 7x + \sin 4x + \sin x = 0$, $0 \le x \le \dfrac{\pi}{2}$
91. Solve $\cos 3x + \cos 2x = \sin\left(\dfrac{3x}{2}\right) + \sin\left(\dfrac{x}{2}\right)$, $0 \le x \le 2\pi$
92. Solve $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$, $-\pi \le x \le 2\pi$
93. Solve $\cos 2x + \cos 4x = 2\cos x$
94. Solve $\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$
95. Solve $\tan x + \tan 2x + \tan 3x = 0$
96. Solve $\tan 3x + \tan x = 2\tan 2x$
97. Solve $(1 - \tan x)(1 + \sin 2x) = (1 + \tan x)$
98. Solve $\sin x - 3\sin 2x + \sin 3x = \cos x\ 3\cos 2x + \cos 3x$

### Type 4

99. Solve $4\sin x \cdot \sin 2x \cdot \sin 4x = \sin 3x$
100. Solve $\cos x \cdot \cos 2x \cdot \cos 3x = 1/4$, $0 \le x \le 2\pi$
101. Solve $\sin 3\alpha = 4\sin\alpha \cdot \sin(x + \alpha) \cdot \sin(x - \alpha)$
102. Solve $\sin 2x \cdot \sin 4x + \cos 2x = \cos 6x$
103. Solve $\sec x \cdot \cos 5x + 1 = 0$, $0 \le x \le 2\pi$
105. Solve $\cos x \cdot \cos 6x = -1$

### Type 5

106. Solve $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -2$
107. Solve $5\sin^2 x - 7\sin x\cos x + 10\cos^2 x = 4$
108. Solve $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -3$
109. Solve $\sin^3 x\cos x + \sin^2 x\cos^2 x + \sin x\cos^3 x = 1$

### Type 6

110. Solve $1 + 2\operatorname{cosec} x = -\dfrac{\sec^2\left(\dfrac{x}{2}\right)}{2}$.
111. Solve $(\cos x - \sin x)(2\tan x + \sec x) + 2 = 0$
112. Solve $\dfrac{\sin^3\dfrac{x}{2} - \cos^3\dfrac{x}{2}}{2 + \sin x} = \dfrac{\cos x}{3}$
113. Solve $\cot\left(\dfrac{x}{2}\right) - \operatorname{cosec}\left(\dfrac{x}{2}\right) = \cot x$
114. If $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ be the four roots of the equation $\sin(\theta + \alpha) = k\sin 2\theta$, no two of which differ by a multiple of $2\pi$, then prove that
$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n + 1)\pi,\ n\ Z$$

### Type 7

115. Solve $\sin x + \cos x - 2\sqrt{2}\sin x\cos x = 0$
116. Solve $\sin^3 x + \sin x\cos x + \cos^3 x = 1$
117. Solve $\sin x + \cos x = 1 - \sin x\cos x$
118. Solve $1 + \sin^3 x + \cos^3 x = \dfrac{3}{2}\sin 2x$
119. Solve $\sin 2x - 12(\sin x - \cos x) + 12 = 0$, where $0 \le x \le 2\pi$
120. Solve $\sin 6x + \cos 4x + 2 = 0$

### Type 8

121. Solve $\sin^6 x = 1 + \cos^4 3x$
122. Solve $\sin^4 x = 1 + \tan^8 x$
123. Solve $\sin^2 x + \cos^2 y = 2\sec^2 z$

124. Solve $\sin 3x + \cos 2x + 2 = 0$
125. Solve $\cos 4x + \sin 5x = 2$

**Type 9**

126. Find the values of $x$ in $(-\pi, \pi)$ which satisfy the equation
$$8^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cos^4 x+|\cos x|^5+\cdots \text{to}\,\infty} = 64$$
127. Solve $2^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cos^4 x+|\cos x|^5+\cdots \text{to}\,\infty} = 4$
128. Solve $1 + \sin\theta + \sin^2\theta + \sin^3\theta + \ldots$ to $\infty$
$= 4 + 2\sqrt{3}$
129. Solve $|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$
130. Solve $e^{\sin x} - e^{-\sin x} - 4 = 0$
131. If $e^{[\sin^2 x+\sin^4 x+\sin^6 x+\cdots \text{to}\,\infty]\log_e 2}$ satisfies the equations $x^2 - 9x + 8 = 0$, then find the value of $\dfrac{\cos x}{\cos x + \sin x}, 0 < x < \dfrac{\pi}{2}$
132. Solve $\log_{\cos x} \tan x + \log_{\sin x} \cot x = 0$
133. Solve $3^{\sin 2x + 2\cos^2 x} + 3^{1-\sin 2x + 2\sin^2 x} = 28$
134. Solve $\log_{\cos x} \sin x + \log_{\sin x} \cos x = 2$, where $x > 0$

**Type 10**

135. Solve $2\cos^2\left(\dfrac{x}{2}\right)\sin^2 x = x^2 + \dfrac{1}{x^2}, 0 < x < \dfrac{\pi}{2}$
136. Solve $2\cos^2\left(\dfrac{x^2+x}{6}\right) = 2^x + 2^{-x}$

## Level II

### (Mixed Problems)

Solve the following equations and tick the correct one.

1. $\sin^2\theta - \cos\theta = \dfrac{1}{2}, 0 \le \theta \le 2\pi$

(a) $\dfrac{2\pi}{3}, \dfrac{\pi}{3}$ (b) $\dfrac{\pi}{3}, \dfrac{5\pi}{3}$

(c) $-\dfrac{\pi}{3}, \dfrac{2\pi}{3}$ (d) $\dfrac{2\pi}{3}, \dfrac{5\pi}{3}$

2. If $3\tan^2\theta - 2\sin\theta = 0$, then $\theta$ is

(a) $n\pi$ (b) $n\pi + (-1)^n \dfrac{\pi}{6}$

(c) $n\pi - (-1)^n \dfrac{\pi}{6}$ (d) $n\pi + \dfrac{\pi}{3}$

3. If $\tan^2 x + (1-\sqrt{3})\tan x - \sqrt{3} = 0$, then $x$ is

(a) $n\pi + \dfrac{\pi}{3}$ (b) $n\pi - \dfrac{\pi}{3}$

(c) $n\pi + \dfrac{\pi}{4}$ (d) $n\pi - \dfrac{\pi}{4}$

4. If $\tan^2\theta + \cot^2\theta = 2$, then $\theta$ is

(a) $n\pi + \dfrac{\pi}{6}$ (b) $n\pi - \dfrac{\pi}{6}$

(c) $n\pi + \dfrac{\pi}{4}$ (d) $n\pi - \dfrac{\pi}{4}$

5. If $\tan\theta + \cot\theta = 2$, then $\theta$ is

(a) $n\pi + \dfrac{\pi}{4}$ (b) $n\pi - \dfrac{\pi}{4}$

(c) (d) $n\pi - \dfrac{\pi}{3}$

6. The set of values of $x$ for which $\dfrac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$ is

(a) $\phi$ (b) $\dfrac{\pi}{4}$

(c) $n\pi + \dfrac{\pi}{3}$ (d) $2n\pi + \dfrac{\pi}{4}$

7. If $\sin 5x + \sin 3x + \sin x = 0$, then the value of $x$ other than zero, lying between $0 \le x \le \dfrac{\pi}{2}$

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{12}$

(c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{4}$

8. If $\alpha$ and $\beta$ are acute positive angles satisfying the equation $3\sin^2\alpha + 2\sin^2\beta = 1$ and $3\sin 2\alpha - 2\sin^2\beta = 0$, then $\alpha + 2\beta$ is

(a) 0 (b) $\dfrac{\pi}{4}$

(c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{2}$

9. If $2\sin^2 x + \sin^2 2x = 2, -\pi < x < \pi$, then $x$ is

(a) $\pm\dfrac{\pi}{2}$ (b) $\pm\dfrac{\pi}{4}$

(c) $\pm\dfrac{3\pi}{4}$ (d) None

10. The real roots of the equation
$\cos^7 x + \sin^4 x = 1$ in $(-\pi, \pi)$ are

(a) $-\dfrac{\pi}{2}, 0$ (b) $-\dfrac{\pi}{2}, 0, \dfrac{\pi}{2}$

(c) $0, \dfrac{\pi}{2}$ (d) $0, \dfrac{\pi}{4}, \dfrac{\pi}{2}$

11. The general solution of $\cos^5 x - \sin^5 x - 1 = 0$ is

(a) $n\pi$ (b) $2n\pi$

(c) $n\pi + \dfrac{\pi}{2}$ (d) $2n\pi + \dfrac{\pi}{2}$

12. If $4 \sin^4 x + \cos^4 x = 1$, then $x$ is
(a) $n\pi$ (b) $n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}}$
(c) $\frac{2n\pi}{3}$ (d) $2n\pi \pm \frac{\pi}{4}$

13. The number of points of intersection of $2y = 1$ and $y = \cos x$ in $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, is
(a) 1 (b) 2 (c) 3 (d) 4

14. The number of values of $x$ in the interval $[0, 3\pi]$ satisfying the equation $2 \sin^2 x + 5 \sin x - 3 = 0$ is
(a) 6 (b) 1 (c) 2 (d) 4

15. The number of values of $x$ in $[0, 5\pi]$ staisfying the equation $3 \sin^2 x - 7 \sin x + 2 = 0$ is
(a) 0 (b) 5 (c) 6 (d) 10.

16. The number of solution of the equation $|\cot x| = \cot x + \frac{1}{\sin x}$, $0 < x < 2\pi$, is
(a) 1 (b) 2 (c) 3 (d) 4.

17. The number of solution of $|\cos x| = \sin x$ such that $0 < x < \pi$, is
(a) 2 (b) 4 (c) 8 (d) None

18. The number of solution of the equation $\tan x \cdot \tan 4x = 1$, $0 < x < \pi$, is
(a) 1 (b) 2 (c) 5 (d) 8

19. The number of solution of the equation $12 \cos^3 x - 7 \cos^2 x + 4 \cos x - 9 = 0$, is
(a) 0 (b) 2
(c) infinity (d) None

20. The sum of all solution of the equation $\cos\theta \cdot \cos\left(\frac{\pi}{3} + \theta\right) \cdot \cos\left(\frac{\pi}{3} - \theta\right) = \frac{1}{4}$ is
(a) $15\pi$ (b) $30\pi$
(c) $\frac{100\pi}{3}$ (d) None

21. The number of solution of $16^{\sin 2x} + 16^{\cos 2x} = 10$, $0 < x < 2\pi$, is
(a) 2 (b) 4 (c) 6 (d) 8

22. The smallest positive value of $x$ such that $\tan(x + 20°) = \tan(x + 10°) \cdot \tan x \cdot \tan(x - 10°)$, is
(a) 30° (b) 45° (c) 60° (d) 75°

23. The maximum value of
$$\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$$
in is attained at
(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

24. The minimum value of $2^{\sin x} + 2^{\cos x}$ is
(a) 1 (b) $2^{1-\frac{1}{\sqrt{2}}}$
(c) $2^{-\frac{1}{\sqrt{2}}}$ (d) $\left(2 - \frac{1}{\sqrt{2}}\right)$

25. If $\cos p\theta + \cos q\theta = 0$, then the different values of $\theta$ are in AP, whose common difference is
(a) $\frac{\pi}{p+q}$ (b) $\frac{\pi}{p-q}$
(c) $\frac{2\pi}{p \pm q}$ (d) $\frac{3\pi}{p \pm q}$

26. If $\tan 2x \cdot \tan x = 1$, then $x$ is
(a) $\frac{\pi}{3}$ (b) $(6n \pm 1)\frac{\pi}{6}$
(c) $(4n \pm 1)\frac{\pi}{6}$ (d) $(2n \pm 1)\frac{\pi}{6}$

27. The maximum value of $5 \sin\theta + 3 \sin(\theta - \alpha)$ is 7, then the set of all possible values of $\alpha$ is
(a) $\left(2n\pi \pm \frac{\pi}{3}\right)$ (b) $\left(2n\pi \pm \frac{2\pi}{3}\right)$
(c) $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ (d) None

28. If $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\sin\theta\right)$, then $\sin\theta + \cos\theta$ is
(a) $2n - 1$ (b) $2n + 1$
(c) $2n$ (d) $n$

29. If $\sin\left(\frac{\pi}{4}\cot\theta\right) = \cos\left(\frac{\pi}{4}\tan\theta\right)$, then $\theta$ is
(a) $\left(n\pi + \frac{\pi}{4}\right)$ (b) $\left(2n\pi \pm \frac{\pi}{4}\right)$
(c) $\left(n\pi - \frac{\pi}{4}\right)$ (d) $\left(2n\pi \pm \frac{\pi}{6}\right)$

30. If $\tan(\pi \cos\theta) = \cot(\pi \sin\theta)$, then the values of $\cos\left(\theta - \frac{\pi}{4}\right)$ is (are)
(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$
(c) $\pm\frac{1}{2\sqrt{2}}$ (d) None

31. If $3 \tan(\theta - 15°) = \tan(\theta + 15°)$, then $\theta$ is
(a) $\left(n\pi + \frac{\pi}{4}\right)$ (b) $\left(n\pi + \frac{\pi}{8}\right)$
(c) $\left(n\pi + \frac{\pi}{3}\right)$ (d) None

32. If $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$, then $\theta$ is
(a) $(2n+1)\frac{\pi}{12}$ (b) $\left(n\pi \pm \frac{\pi}{3}\right)$
(c) $(4n+1)\frac{\pi}{12}$ (d) None

33. The equation $a \sin 2x + \cos 2x = 2a - 7$ posses a solution if
(a) $a > 6$ (b) $2 \le a \le 6$
(c) $a > 2$ (d) None

34. The number of all possible triplets $(a_1, a_2, a_3)$ such that $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all $x$, the number of possible 5-tuplets is
(a) 0 (b) 1 (c) 2 (d) None

35. If $a_1 + a_2 \sin x + a_3 \cos x + a_4 \sin 2x + a_5 \cos 2x = 0$ holds for all $x$, then the number of possible 5-tuplets is
(a) 0 (b) 1 (c) 2 (d) infinity

36. The number of solution of the equation $1 + \sin x \cdot \sin^2 \frac{x}{2} = 0$ in $[-\pi, \pi]$ is
(a) 0 (b) 1 (c) 2 (d) 3

37. The solution of $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ is solvable for
(a) $-\frac{1}{2} \le \alpha \le \frac{1}{2}$ (b) $-3 \le \alpha \le 1$
(c) $-\frac{3}{2} \le \alpha \le \frac{1}{2}$ (d) $-1 \le \alpha \le 1$

38. The equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ is solvable for
(a) $-\sqrt{3} \le a \le \sqrt{3}$ (b) $-\sqrt{2} \le a \le \sqrt{2}$
(c) $-1 \le a \le 1$ (d) None

39. The number of pairs $(x, y)$ satisfying the equations $\sin x + \sin y = \sin(x + y)$ and $|x| + |y| = 1$, is
(a) 2 (b) 4
(c) 6 (d) infinity

40. The value of '$a$' for which the equation $4\,\text{cosec}^2[\pi(a + x)] + a^2 - 4a = 0$, has a real solution, if
(a) $a = 1$ (b) $a = 2$ (c) $a = 3$ (d) None

41. If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$, $x \in [0, \pi]$, then
(a) $x = \frac{\pi}{4}, y = 1$ (b) $y = 0$
(c) $y = 2$ (d) $x = \frac{3\pi}{4}$

42. $|\tan x + \sec x| = |\tan x| + |\sec x|$, $x \in [0, 2\pi]$, if $x$ belongs to that interval
(a) $[0, \pi]$ (b) $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
(c) $\left[0, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$ (d) $(\pi, 2\pi]$

43. The number of solutions of $\sum_{r=1}^{5} \cos(rx) = 5$ in the interval $[0, 2\pi]$ are
(a) 0 (b) 1 (c) 5 (d) 10

44. If $f(x) = \max\{\tan x, \cot x\}$, the number of roots of the equation $f(x) = \frac{1}{2 + \sqrt{3}}$ in $(0, 2\pi)$ are
(a) 0 (b) 2 (c) 4 (d) $\infty$

45. If $\sin x + \cos x + \tan x + \cot x + \sec x + \text{cosec}\, x = 7$ and $\sin 2x = a - b\sqrt{c}$, then $a - b + 2c$ is
(a) 0 (b) 14 (c) 2 (d) $\frac{3}{2}$

46. If $\sin^4 x + \cos^4 x + 2 = 4 \sin x \cos y$ and $0 < x, y < \frac{\pi}{2}$, then $\sin x + \cos y$ is
(a) $-2$ (b) 0 (c) 2 (d) $\frac{3}{2}$

47. The equation $\cos^4 x - (\lambda + 2)\cos^2 x - (\lambda + 3) = 0$ possesses a solution if
(a) $\lambda > -3$ (b) $\lambda < -2$
(c) $-3 < \lambda < -2$ (d) $\lambda \in z^+$

48. If $0 < \theta < 2\pi$ and $2\sin^2\theta - 5\sin\theta + 2 > 0$, then the range of $\theta$ is
(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$
(b) $\left(0, \frac{5\pi}{6}\right) \cup (\pi, 2\pi)$
(c) $\left(0, \frac{\pi}{6}\right) \cup (\pi, 2\pi)$
(d) None

49. The number of values of $x$ for which $\sin 2x + \cos 4x = 2$ are
(a) 0 (b) 1 (c) 2 (d) $\infty$

50. The number of solutions of the equation $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 < x < 2\pi$ are
(a) 0 (b) 1 (c) 2 (d) 4

51. The number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ are
(a) 0 (b) 1 (c) 2 (d) 3

52. The number of solutions of the equation $2(\sin^4 2x + \cos^4 2x) + 3\sin^2 x \cos^2 x = 0$ are
(a) 0 (b) 1 (c) 2 (d) 3

53. $\cos 2x + a \sin x = 2a - 7$ possesses a solution for
(a) $a \,||\, a$ (b) $a > 6$
(c) $a < 2$ (d) $a \in [2, 6]$

54. If $0 < x < 2\pi$ and $81^{\sin 2x} + 81^{\cos 2x} = 30$, then $x$ is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\pi$ (d) $\frac{\pi}{4}$

55. If $1 + \sin\theta + \sin^2\theta + \ldots\ldots$ to $\infty = 4 + 2\sqrt{3}$, $0 < \theta < \pi$, $\theta \ne \frac{\pi}{2}$, then

(a) $\theta = \frac{\pi}{6}$ (b) $\theta = \frac{\pi}{3}$

(c) $\theta = \frac{\pi}{3}$ or $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$

56. If tan ($\pi$ cos $\theta$) = cot ($\pi$ sin $\theta$), then the value of $\cos\left(\theta - \frac{\pi}{4}\right)$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2\sqrt{2}}$ (d) None

57. The most general values of $x$ for which $\sin x + \cos x = \min_{a \in R} \{1, a^2 - 4a + 6\}$ are given by

(a) $2n\pi, n \in N$

(b) $2n\pi + \frac{\pi}{2}, n \in N$

(c) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}, n \in N$

(d) None

58. If $x \in [0, 2\pi]$ and $\sin x + \sin y = 2$ then the value of $x + y$ is

(a) $\pi$ (b) $\frac{\pi}{2}$ (c) $3\pi$ (d) None

59. The number of roots of the equation $x + 2\tan x = \frac{\pi}{2}$ in the interval $[0, 2\pi]$ are

(a) 1 (b) 2 (c) 3 (d) $\infty$

60. The number of solutions of the equation $\cos(\pi\sqrt{x-4}) \cos(\pi\sqrt{x}) = 1$ are

(a) None (b) 1 (c) 2 (d) >2

61. The number of solutions of the equation $\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}\,x + 4$ are

(a) it forms an empty set
(b) is only one
(c) is only two
(d) is greater then two

62. Number of real roots of the equation $\sec\theta + \operatorname{cosec}\theta = \sqrt{15}$ lying between 0 and $2\pi$ are

(a) 8 (b) 4 (c) 2 (d) 0

63. The general solution of the equation $\sin^{100} x - \cos^{100} x = 1$, is

(a) $2n\pi + \frac{\pi}{3}, n \in z$

(b) $n\pi + \frac{\pi}{2}, n \in z$

(c) $n\pi + \frac{\pi}{4}\ \pi, n \in z$

(d) $2n\pi\ \frac{\pi}{3}, n \in z$

64. The number of solutions of the equation $2^{\cos x} = |\sin x|$ in $[-2\pi, 2\pi]$ are

(a) 1 (b) 2 (c) 3 (d) 4

65. The general solution of the equation $2^{\cos 2x} + 1 = 3.2^{-\sin 2x}$ is

(a) $n\pi.\ n \in z$ (b) $(n + 1)\pi, n \in z$
(c) $(n - 1)\pi, n \in z$ (d) None

66. If $x \in (0, 1)$, the greatest root of the equation $\sin 2\pi x = \sqrt{2}\cos \pi x$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) None

67. The number of solutions of tan ($5\pi\cos\alpha$) = cot ($5\pi\sin\alpha$) for $\alpha \in (0, 2\pi)$ are

(a) 7 (b) 14 (c) 21 (d) 3

68. The number of solution of the equation $1 + \sin x \cdot \sin^2\left(\frac{x}{2}\right) = 0$ in $[-\pi, \pi]$ is

(a) 0 (b) 1 (c) 2 (d) 3

69. The number of solution of the equation $|\cot x| = \cot x + \frac{1}{\sin x}, \forall x \in [0, 2\pi]$ are

(a) 0 (b) 1 (c) 2 (d) 3

70. The real roots of the equation $\cos^7 x + \sin^4 x = 1$ in $(-\pi, \pi)$ are

(a) $-\frac{\pi}{2}, 0$ (b) $-\frac{\pi}{2}, 0, \frac{\pi}{2}$

(c) $\frac{\pi}{2}, 0$ (d) $0, \frac{\pi}{4}, \frac{\pi}{2}$

## Level III

### (Probems for JEE Advanced)

1. Solve for $x$: $\sec x - \operatorname{cosec} x = \frac{4}{3}$.
2. Solve for $x$: $\sin 2x + 12 = 12(\sin x - \cos x)$
3. Solve for $x$: $|\sec x + \tan x| = |\sec x| + |\tan x|$ in $[0, 2\pi]$.
4. Let $n$ be a positive integer such that $\sin\left(\frac{\pi}{2n}\right) + \cos\left(\frac{\pi}{2n}\right) = \frac{\sqrt{n}}{2}$, find $n$.
5. If $\cos 2x + a\sin x = 2a - 7$ possesses a solution then find $a$.
6. Solve for $x$: $\sin^{100} x - \cos^{100} x = 1$.
7. Solve for $x$: $\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$
8. Solve for $x$: $|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$
9. Find the number of solutions of $\cos(\pi\sqrt{x-4}) \cdot \cos(\pi\sqrt{x}) = 1$.

10. Find the number of solution of $x^4 - 2x^2\sin^2\left(\frac{\pi}{2}\right)x + 1 = 0$

11. If $\cos^4 x + a\cos^2 x + 1 = 0$ has atleast one real solution, then find the value of $a$.
12. If the equation $\tan^4 x - 2\sec^2 x + b^2 = 0$ has at least one real solution then find the value of $b$.
13. If $a, b \in [0, 2\pi]$ and the equation $x^2 + 4 + 3\sin(ax + b) = 2x$ has at-least one solution, then find $(a + b)$.
14. Find the number of ordered pairs $(a, b)$ satisfying the equations $|x| + |y| = 4$ and $\sin\left(\frac{\pi x^2}{2}\right) = 1$
15. Find the number of values of $x$ in $(-2\pi, 2\pi)$ and satisfying $\log_{|\cos x|}|\sin x| + \log_{|\sin x|}|\cos x| = 2$.
16 The number of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi)$ is
(a) 2 (b) 3 (c) 0 (d) 1
**[JEE Main, 2002]**

> ***Note***
> No questions asked between 2003 to 2005.

17. The number of values of $x$ in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ are
(a) 4 (b) 6 (c) 1 (d) 2
**[JEE Main, 2006]**

> ***Note***
> No questions asked in between 2007 to 2015.

18. Find all the angles $\theta$ between $\pi$ and $-\pi$ that satisfy the equation

$$5\cos(2\theta) + 2\cos^2\left(\frac{\theta}{2}\right) + 1 = 0$$

**[Roorkee, 1984]**

> ***Note***
> No questions asked between 1985 to 1986.

19. Find the general solution to the following equation $2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0$
**[Roorkee, 1987]**
20. Solve for $x$ and $y$;
$x\cos^3 y + 3x\cos y\sin^2 y = 14$,
$x\sin^3 y + 3x\cos^2 y\sin y = 13$
**[Roorkee, 1988]**
21. Solve for: $x$; $4\sin^4 x + \cos^4 x = 1$
**[Roorkee, 1989]**
22. Find all the values of '$a$' for which the equation $\sin^4 x + \cos^4 x + \sin 2x + a = 0$ is valid.
Also find the general solution of the equation.
**[Roorkee, 1990]**

> ***Note***
> No questions asked in 1991.

23. Find the general solution of the equation $(\sqrt{3} - 1)\sin\theta + (\sqrt{3} + 1)\cos\theta = 2$.
**[Roorkee, 1992]**

> ***Note***
> No questions asked in 1993.

24. Solve for $\theta$ $\sec\theta - \operatorname{cosec}\theta = \frac{4}{3}$
**[Roorkee, 1994]**

> ***Note***
> No questions asked in 1995.

25. If $32\tan^8\theta = 2\cos^2\alpha - 3\cos\alpha$ and $3\cos 2\theta = 1$, then find the general values of $\alpha$.
**[Roorkee, 1996]**

> ***Note***
> No questions asked in 1997.

26. Find the general values of $x$ and $y$ and satisfying the equations $5\sin x\cos y = 1$, $4\tan x = \tan y$.
**[Roorkee, 1998]**

> ***Note***
> No questions asked in 1999.

27. Find the smallest positive value of $x$ and $y$ satisfying $(x - y) = \frac{\pi}{4}$, $\cot x + \cot y = 2$.
**[Roorkee, 2000]**
28. Solve the following equations for $x$ and $y$:
(i) $5^{(\operatorname{cosec}^2 x - \sec^2 y)} = 1$
(ii) $2^{(2\operatorname{cosec} x + \sqrt{3}|\sec y|)} = 64$
**[Roorkee, 2001]**

## Level IV
### (Tougher Problems for JEE Advanced)

Solve the following trigonometric equations:

1. $\cot\left(\frac{x}{2}\right) - \operatorname{cosec}\left(\frac{x}{2}\right) = \cot x$
2. $8\cos x \cdot \cos 2x \cdot \cos 4x = \frac{\sin 6x}{\sin x}$
3. $\frac{\tan x}{\tan 2x} + \frac{\tan 2x}{\tan x} + 2 = 0$
4. $\cos x\cos(6x) = -1$
5. $\cos(4x) + \sin(5x) = 2$
6. $\sin 2x + 5\cos x + 5\sin x + 1 = 0$
7. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$ in the interval $0 \le x \le 2\pi$
8. $\sin^2 x\tan x + \cos^2 x\cot x - \sin 2x = 1 + \tan x\cot x$.
9. $\sin^2 4x + \cos^2 x = 2\sin 4x\cos^4 x$

10. $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \cos x$
11. $\sin^4 x + \cos^4 x$
$$= 2\cos\left(2x + \frac{\pi}{6}\right)\cos\left(2x - \frac{\pi}{6}\right)$$
12. $\sin^4 x + \sin^4\left(x + \frac{\pi}{4}\right) = \frac{1}{4}$
13. If $\cos\left(x + \frac{\pi}{3}\right) + \cos x = a$, then find all values of $a$ so that the equation has a real solution.
14. Find the number of roots of $\cos x - x + \frac{1}{2} = 0$ lies in $\left(0, \frac{\pi}{2}\right)$
15. Find the number of integral ordered pairs satisfy the equations $\begin{cases}\cos(xy) = x \\ \tan(xy) = y\end{cases}$
16 Find the number of real solutions of $\sin^{2016} x - \cos^{2016} x = 1$ in $[0, 2\pi]$
17. Find the number of ordered pairs which satisfy the equation $x^2 + 2x\sin(xy) + 1 = 0$ for $y \in [0, 2\pi]$
18. Find the number of solutions of the equation $\sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x$ in $[0, \pi]$
19. Find the number of solution of the equation $\cos 3x \cdot \tan 5x = \sin 7x$ lying in $\left[0, \frac{\pi}{2}\right]$
20. The angles $B$ and $C$ ($B > C$) of a triangle satisfying the equation $2\tan x - \lambda(1 + \tan^2 x) = 0$, then find the angle $A$, if $0 < \lambda < 1$
21. Determine all values of '$a$' for which the equation $\cos^4 x - (a + 2)\cos^2 x - (a + 3) = 0$ has a solution and find those.
22. Find all the solution of the equation $\sin x + \sin\frac{\pi}{8}(\sqrt{(1 - \cos x)^2 + \sin^2 x}) = 0$ in $\left[\frac{5\pi}{2}, \frac{7\pi}{2}\right]$
23. If the equation $\sin^4 x - (k + 2)\sin^2 x - (k + 3) = 0$ has a solution, then find the value of $k$.
24. Find the number of principal solutions of the equation $4.16^{\sin 2x} = 2^{6\sin x}$
25. Find the general solution of $\sec x = 1 + \cos x + \cos^2 x + \cos^3 x + \ldots$

## Integer Type Questions

1. Find the number of values of $x$ in $(0, 5\pi)$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$
2. Find the number of integral values of $k$, for which the equation $2\cos x + 3\sin x = k + 1$ has a solution
3. Find the number of distinct real roots of $\begin{vmatrix}\sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x\end{vmatrix} = 0$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
4. Find the number of pairs $(x, y)$ satisfying the equations $\sin x + \sin y = \sin(x + y)$ and $|x| + |y| = 1$
5. Find the maximum value of
$$f(x) = \begin{vmatrix}1 + \sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1 + \cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1 + 4\sin 2x\end{vmatrix}$$
6. Find the number of solutions of
$$\sin x = \frac{|x|}{10}$$
7. Find the number of solutions of $\tan x + \cot x = 2\operatorname{cosec} x$ in $[-2\pi, 2\pi]$
8. Find the number of solutions of
$$\cos x \cdot \cos 2x \cdot \cos 3x = \frac{1}{4} \text{ in } [0, \pi]$$
9. If $x, y \in [0, 2\pi]$, then find the number of ordered pairs $(x, y)$ satisfying the equation $\sin x \cdot \cos y = 1$
10. If $x \in [0, 2\pi]$, then find the number of values of $x$ satisfying the equation
$$|\cot x| = \cot x + \frac{1}{\sin x}$$
11. Find the number of solutions of $\tan x \tan(4x) = 1$, for $0 < x < \pi$
12. Find the number of integral values of $n$ for which the equation $\sin x(\sin x + \cos x) = n$ has atleast one solution.
13. Find the number of real solutions of $\sin\{x\} = \cos\{x\}$ in $[0, 2\pi]$
14. Find the number of solutions of $(\sqrt{3} + 1)^{2x} + (\sqrt{3} - 1)^{2x} = 2^{3x}$
15. Find the number of values of $y$ in $[-2\pi, 2\pi]$ for which $|\sin(2x)| + |\cos(2x)| = |\sin(y)|$

## Comprehensive Link Passages

**Passage I**

An equation is of the form

$f(\sin x \pm \cos x, \pm \sin x \cos x) = 0$

can be solved by changing the variable.

Let $\sin x \pm \cos x = 1$

$\Rightarrow \quad \sin^2 x + \cos^2 x \pm 2\sin x\cos x = t^2$

$\Rightarrow \quad 1 \pm 2\sin x \cos x = t^2$

Thus, the given equation is reducible to $f\left(t, \frac{t^2 - 1}{2}\right) = 0$. On the basis of above information, answer the following questions.

1. If $1 - \sin 2x = \cos x - \sin x$, then $x$ is

(a) $2n\pi, \left(2n\pi - \frac{\pi}{2}\right), n \in I$

(b) $2n\pi, \left(n\pi + \frac{\pi}{4}\right), n \in I$

(c) $\left(2n\pi - \frac{\pi}{2}\right), \left(n\pi + \frac{\pi}{4}\right), n \in I$

(d) None.

2. If $\sin x + \cos x = 1 + \sin x \cos x$, then $x$ is

(a) $2n\pi, \left(2n\pi + \frac{\pi}{2}\right), n \in I$

(b) $2n\pi, \left(n\pi + \frac{\pi}{4}\right), n \in I$

(c) $\left(2n\pi - \frac{\pi}{2}\right), \left(n\pi + \frac{\pi}{4}\right), n \in I$

(d) None

3. If $\sin^4 x + \cos^4 x = \sin x \cos x$, then $x$ is

(a) $n\pi, n \in 1$

(b) $(6n+1)\frac{\pi}{6}, n \in I$

(c) $(4n+1)\frac{\pi}{4}, n \in I$

(d) None

4. If $1 + \sin^3 x + \cos^3 x = \frac{3}{2}\sin 2x$, then $x$ is

(a) $(2n+1)\pi, \left(2n\pi + \frac{\pi}{4}\right), n \in I$

(b) $(2n+1)\pi, \left(2n\pi - \frac{\pi}{2}\right), n \in I$

(c) $\left(2n\pi + \frac{\pi}{4}\right), \left(2n\pi + \frac{\pi}{2}\right), n \in I$

(d) None

5. If $(\sin x + \cos x) = 2\sqrt{2}\sin x \cos x$, then $x$ is

(a) $\left(2n\pi + \frac{\pi}{4}\right), n \in I$ (b) $\left(2n\pi - \frac{\pi}{4}\right), n \in I$

(c) $\left(n\pi + \frac{\pi}{4}\right), n \in I$ (d) $\left(n\pi - \frac{\pi}{4}\right), n \in I$

**Passage II**

$\alpha$ is a root of the equation

$$(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$$

$\beta$ is a root of the equation

$$3\cos^3 x - 10\cos x + 3 = 0$$

and $\gamma$ is a root of the equation

$$1 - \sin 2x = \cos x - \sin x,$$

$$0 \le \alpha, \beta, \gamma \le \frac{\pi}{2}$$

On the basis of above information, answer the following questions.

1. $\cos\alpha + \cos\beta + \cos\gamma$ is equal to

(a) $\frac{3\sqrt{6} + 2\sqrt{2} + 6}{6\sqrt{2}}$ (b) $\frac{3\sqrt{3} + 8}{6}$

(c) $\frac{3\sqrt{3} + 2}{6}$ (d) None

2. $\sin\alpha + \sin\beta + \sin\gamma$ is equal to

(a) $\frac{14 + 3\sqrt{2}}{6\sqrt{2}}$ (b) $\frac{5}{6}$

(c) $\frac{3 + 4\sqrt{2}}{6}$ (d) $\frac{1 + \sqrt{2}}{2}$

3. $\sin(\alpha - \beta)$ is equal to

(a) 1 (b) 0

(c) $\frac{1 - 2\sqrt{6}}{6}$ (d) $\frac{\sqrt{3} - 2\sqrt{2}}{6}$

**Passage III**

Solutions of equations $a \sin x \pm b \cos x = c$. General value satisfying two equations. $a \cos\theta \pm b \sin\theta = c$, where $\theta$ satisfying two equations.

(a) The equations be first converted to, where $a = r\cos\theta$, $b = r\sin\theta$.

(b) Satisfying two equations, find the common value of lying between 0 and $2\pi$ and then add $2n\pi$.

On the basis of above information, answer the following questions.

1. The number of integral values of $k$ for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution is

(a) 4 (b) 8 (c) 10 (d) 12

2. If $\cos 3x + \sin\left(2x - \frac{7\pi}{6}\right) = -2$, then $x$ is

(a) $(6n+1)\frac{\pi}{3}, n \in I$

(b) $(6n-1)\frac{\pi}{3}, n \in I$

(c) $(2n+1)\frac{\pi}{3}, n \in I$

(d) None.

3. The value of $x$ such that $-\pi < x < \pi$ and satisfying the equation $8^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cdots} = 4^3$, then $x$ is

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $-\frac{\pi}{3}$ (d) $-\frac{2\pi}{3}$

4. The number of solutions of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ is

(a) 1 (b) 2 (c) 4 (d) 0

**Passage IV**

Suppose equation is $f(x) - g(x)$ or $y = f(x) = g(x)$, say, then draw the graphs of $y = f(x)$ and $y = g(x)$.

If graphs of $y = f(x)$ and $y = g(x)$ cuts at one, two, three, …, no points, then number of solutions are one, two, three, …, zero respectively.

On the basis of above information, answer the following questions:

1. The number of solutions of $\sin x = \frac{|x|}{10}$ is

(a) 4 (b) 6 (c) 5 (d) None

2. Total number of solutions of the equation $3x + 2\tan x = \frac{5\pi}{2}$, $x \in [0, 2\pi]$, is
(a) 1 (b) 2 (c) 3 (d) 4

3. Total number of solutions of $\sin\{x\} = \cos\{x\}$, where $\{\}$ = FPF, in $[0, 2\pi]$ is
(a) 3 (b) 5 (c) 7 (d) None

4. If $1 - \sin x = \frac{\sqrt{3}}{2}\left|x - \frac{\pi}{2}\right| + a$
has no solution, when $a \in R^+$, then
(a) $a \in R^+$
(b) $a > \frac{3}{2} + \frac{\pi}{\sqrt{3}}$
(c) $a \in \left(0, \frac{3}{2} + \frac{\pi}{\sqrt{3}}\right)$
(d) $a \in \left(\frac{3}{2}, \frac{3}{2} + \frac{\pi}{\sqrt{3}}\right)$

5. Total number of solutions of $\cos 2x = |\sin x|$, where $-\frac{\pi}{2} < x < \pi$, is
(a) 3 (b) 4 (c) 5 (d) 6

**Passage V**

Whenever the terms of two sides of the equation are of different nature, then equations are known as non standard form, some of them are in the form of an ordinary equation but can not be solved by standard procedures.

Non standard problems require high degree of logic, they also require the use of graphs, inverse properties of functions, in-equalities

On the basis of above information, answer the following questions:

1. The number of solutions of the equation $2\cos\left(\frac{x}{2}\right) = (3^x + 3^{-x})$ is
(a) 1 (b) 2 (c) 3 (d) None.

2. The number of solution of the equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = (x^2 + x^{-2})$, $0 \le x \le \frac{\pi}{2}$ is
(a) 1 (b) $> 1$ (c) 0 (d) None.

3. The number of real solutions of the equation $\sin(e^x) = 5x + 5^{-x}$ is
(a) 0 (b) 1
(c) 2 (d) infinitely many

4. If $0 \le x \le 2\pi$ and $2^{\mathrm{cosec}^2 x} \times \sqrt{\frac{y^2}{2} - y + 1} \le \sqrt{2}$, then the number of ordered pairs of $(x, y)$ is
(a) 1 (b) 2
(c) 3 (d) infinitely many

5. The number of solutions of the equation $\sin x = x^2 + x + 1$ is
(a) 0 (b) 1 (c) 2 (d) None

## Matrix Match
## (For JEE-Advanced Examination Only)

1. Match the following columns:

| | Column I | | Column II |
|---|---|---|---|
| (A) | The equation $\sin x + \cos x = 2$ has | (P) | 1 solution |
| (B) | The equation $\sqrt{3}\sin x + \cos x = 4$ has | (Q) | 2 solution |
| (C) | The equation $3\sin x + 4\cos x$ has | (R) | 3 solution. |
| (D) | The equation $\sin x \cos x = 2$ has | (S) | No solution |

2. Match the following columns:

| | Column I | | Column II |
|---|---|---|---|
| (A) | If $\begin{vmatrix} \cos(A+B) & -\sin(A+B) & \cos(2B) \\ \sin A & \cos A & \sin B \\ -\cos A & \sin A & \cos B \end{vmatrix} = 0$ then $B$ is | (P) | $n\pi$ |
| (B) | If $\begin{vmatrix} \cos\theta & \sin\theta & \cos\theta \\ -\sin\theta & \cos\theta & \sin\theta \\ -\cos\theta & -\sin\theta & \cos\theta \end{vmatrix} = 0$, then $\theta$ is | (Q) | $(2n+1)\frac{\pi}{2}$ |
| (C) | If $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$, then $\theta$ is | (R) | $(2n-1)\frac{\pi}{2}$ |
| | | (S) | $\frac{7\pi}{24}$ |

3. Match the following columns:

| | Column I | | Column II |
|---|---|---|---|
| (A) | If $4\sin^4 x + \cos^4 x = 1$, then $x$ is | (P) | $\frac{\pi}{4}$ |
| (B) | If $\sec x \cdot \cos(5x) + 1 = 0$, where $0 < x < 2\pi$, then $x$ is | (Q) | $\frac{\pi}{6}$ |
| (C) | If $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, where $0 < x < 2\pi$, then $x$ is | (R) | $-\frac{\pi}{4}$ |
| (D) | If $2\sin^2 x + \sin^2 2x = 2$ where $0 < x < 2\pi$, then $x$ is | (S) | $n\pi, n \in I$ |

4. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | If $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$, then $\theta$ is | (P) | $\frac{n\pi}{2} + \frac{\pi}{6}, n \in I$ |
| (B) | If $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ then $x$ is | (Q) | $n\pi + \frac{\pi}{2}, n \in I$ |
| (C) | If $\sin 4\theta - \sec 2\theta = 2$, then $\theta$ is | (R) | $\frac{n\pi}{5} + \frac{\pi}{10}, n \in I$ |
| (D) | If $\tan(x + 100°) = \tan(x + 50°)\ \tan x \cdot \tan(x - 50°)$ then $x$ is | (S) | $\frac{n\pi}{5}, n \in I$ |
| | | (T) | $\pm\left(\frac{\pi}{3}\right)$ |

5. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | The number of real roots of | | |
| | $\cos^7 x + \sin^4 x = 1$ in $(-\pi, \pi)$, is | (P) | 8 |
| (B) | The number of real roots of cosec $x = 1 + \cot x$ in $(-2\pi, 2\pi)$ is | (Q) | 4 |
| (C) | The number of integral values of $k$ for which, the equation $7\cos x + 5\sin x = 2k + 1$ has a solution is | (R) | 3 |
| (D) | The number of solutions of the pair of equations | (S) | 2 |
| | $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in $[0, 2\pi]$ is | (T) | 7 |

6. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | If $\cos 3x \cdot \cos^3 x + \sin 3x \cdot \sin^3 x = 0$, then $x$ is | (P) | $\left(n\pi \pm \frac{\pi}{3}\right), n \in I$ |
| (B) | If $\sin 3\alpha = 4\sin\alpha \sin(x + \alpha)\sin(x - \alpha)$, then $\alpha$ is, where $\alpha \neq n\pi$ | (Q) | $\left(n\pi + \frac{\pi}{4}\right), n \in I$ |
| (C) | If $\lvert 2\tan x - 1\rvert + \lvert 2\cot x - 1\rvert = 2$, then $x$ is | (R) | $\left(\frac{n\pi}{4} + \frac{\pi}{8}\right), n \in I$ |
| (D) | If $\sin^{10}x + \cos^{10}x = \frac{29}{16}\cos^4(2x)$, then $x$ is | (S) | $\left(\frac{n\pi}{2} \pm \frac{\pi}{4}\right), n \in I$ |

7. Match the following columns:
If $\alpha$ and $\beta$ are the roots of $a\cos\theta + b\sin\theta = c$, then

| Column I | | Column II | |
|---|---|---|---|
| (A) | $\sin\alpha + \sin\beta$ is | (P) | $\frac{2b}{a+c}$ |
| (B) | $\sin\alpha \cdot \sin\beta$ is | (Q) | $\frac{c-a}{c+a}$ |
| (C) | $\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)$ is | (R) | $\frac{2bc}{a^3+b^3}$ |
| (D) | $\tan\left(\frac{\alpha}{2}\right) \cdot \tan\left(\frac{\beta}{2}\right)$ is | (S) | $\frac{c^2-a^2}{a^2+b^2}$ |

8. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | If $\sin 5x = 16\sin^5 x$, then $x$ is | (P) | $(2n+1)\frac{\pi}{4}, n \in I$ |
| (B) | If $4\cos^2 x \cdot \sin x - 2\sin^2 x = 3\sin x$, then $x$ is | (Q) | $n\pi, n \in I$ |
| (C) | If $\tan^2(2x) + \cot^2(2x) + 2\tan(2x) + 2\cot(2x) = 0$ | (R) | $n\pi + \frac{\pi}{6}, n \in I$ |
| (D) | If $\tan^2 x \cdot \tan^2 3x \cdot \tan 4x = \tan^2 x - \tan^2 3x + \tan 4x$ then $x$ is | (S) | $\left(n\pi + (-1)^n\frac{\pi}{8}\right)$ |

9. Observe the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | If $\cos(6\theta) + \cos(4\theta) + \cos(2\theta) + 1 = 0$, then $\theta$ is | (P) | $2n\pi, n \in I$ |
| (B) | If $3 - 2\cos\theta - 4\sin\theta - \cos 2\theta + \sin 2\theta = 0$, then $\theta$ is | (Q) | $\frac{n\pi}{3}, n \in I$ |
| (C) | If $\cos\theta \cdot \cos 2\theta \cdot \cos 3\theta = \frac{1}{4}$, then $\theta$ is | (R) | $(4n+1)\frac{\pi}{2}, n \in I$ |
| (D) | If $\sin(5\theta) + \sin(\theta) = \sin(3\theta)$, then $\theta$ is | (S) | $(2n+1)\frac{\pi}{12}$ |

## Assertion and Reason

**Codes:**

(A) Both A and R are individually true and R is the correct explanation of A

(B) Both A and R are individually true and R is not the correct explanation of A.

(C) A is true but R is false.
(D) A is false but R is true.

1. Assertion (A): The number of real solutions of $\sin x = x^2 + x + 1$ is 1
Reason (R): since $|\sin x| \le 1$
(a) A (b) B (c) C (d) D
2. Assertion (A): The number of real solutions of $\cos x = 3^x + 3^{-x}$
Reason (R): since $|\cos x| \le 1$
(a) A (b) B (c) C (d) D
3. Assertion (A): The maximum value of $3 \sin x + 4 \cos x + 10$ is 15
Reason (R): The least value of $2 \sin^2 x + 4$ is 4
(a) A (b) B (c) C (d) D
4. Assertion (A): The greatest value of $\sin^4 x + \cos^2 x$ is 1
Reason (R): The range of the function $f(x) = \sin^2 x + \cos^2 x$ is 1
(a) A (b) B (c) C (d) D
5. Assertion (A): $a \cos x + b \cos 3x \le 1$ for every $x$ in $R$
Reason (R): since $|b| \le 1$
(a) A (b) B (c) C (d) D
6. Assertion (A): The set of values of $x$ for which $\dfrac{\tan 3x - \tan 2x}{1 + \tan 3x \cdot \tan 2x} = 1$ is $\phi$
Reason (R): Since $\tan x$ is not defined at $x = (2n+1)\dfrac{\pi}{2}, n \in I$
(a) A (b) B (c) C (d) D
7. Assertion (A): The number of solutions of the equation $\tan x + \sec x = 2 \cos x$ lying in the interval $[0, 2\pi]$ is 2.
Reason (R): The number of solutions of the equation $3 \sin^2 x - 7 \sin x + 2 = 0$ in $[0, 5\pi]$ is 6
(a) A (b) B (c) C (d) D
8. Assertion (A): The number of solutions of $\tan x \cdot \tan 4x = 1$ in $(0, \pi)$ is 5
Reason (R): The number of solutions of $|\cos x| = \sin x$ in $[0, 4\pi]$ is 4
(a) A (b) B (c) C (d) D
9. Assertion (A): If $\tan\left(\dfrac{\pi}{2}\sin\theta\right) = \cot\left(\dfrac{\pi}{2}\cos\theta\right)$, then $\sin\theta + \cos\theta = \pm\sqrt{2}$
Reason (R): $-\sqrt{2} \le \sin\theta + \cos\theta \le \sqrt{2}$
(a) A (b) B (c) C (d) D
10. Assertion (A): $\sin A = \sin B = \sin C = 2 \sin (18°)$
Reason (R): If $\cos A = \tan B$, $\cos B = \tan C$, $\cos C = \tan A$
(a) A (b) B (c) C (d) D

## Questions Asked In Previous Years' JEE-Advanced Examinations

1. The general solution of the trigonometric equation $\sin x + \cos x = 1$ is given by
(a) $x = 2n\pi, n \in I$
(b) $x = \left(2n\pi + \dfrac{\pi}{2}\right), n \in I$
(c) $x = \left(n\pi + (-1)^n \dfrac{\pi}{4}\right), n \in I$
(d) None of these
**[IIT-JEE, 1981]**
2. Find the point of intersections of the curves $y = \cos x$ and $y = \sin 3x$ where $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
**[IIT-JEE, 1982]**
3. Find all solutions of
$4 \cos^2 x \sin x - 2 \sin^2 x = 3 \sin x$
**[IIT-JEE, 1983]**
4. There exist a value of $\theta$ between 0 and $2\pi$ which satisfies the equation $\sin^4 \theta - 2 \sin^2 \theta - 1 = 0$?
**[IIT-JEE, 1984]**
5. No questions asked in 1985.
6. Find the solution set of $x + y = \dfrac{2\pi}{3}$, $\cos x + \cos y = \dfrac{3}{2}$, where $x$ and $y$ are real. **[IIT-JEE, 1986]**
7. Find the set of all $x$ in the interval $[0, \pi]$ for which $2 \sin^2 x - 3 \sin x + 1 \ge 0$.
**[IIT-JEE, 1987]**
8. The smallest +ve root of the equation $\tan x - x = 0$ lies in
(a) $\left(0, \dfrac{\pi}{2}\right)$ (b) $\left(\dfrac{\pi}{2}, \pi\right)$
(c) $\left(\pi, \dfrac{3\pi}{2}\right)$ (d) $\left(\dfrac{3\pi}{2}, 2\pi\right)$
**[IIT-JEE, 1987]**
9. The general solutions of
$\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$ is
(a) $n\pi + \dfrac{\pi}{8}, n \in I$
(b) $\dfrac{n\pi}{2} + \dfrac{\pi}{8}, n \in I$
(c) $(-1)^n \dfrac{n\pi}{2} + \dfrac{\pi}{8}, n \in I$
(d) $2n\pi + \cos^{-1}\left(\dfrac{2}{3}\right), n \in I$ **[IIT-JEE, 1989]**
10. No questions asked between 1990 to 1992.
11. The number of solutions of the equation $\tan x + \sec x = 2 \cos x$ lying in the interval $[0, 2\pi]$ is
(a) 0 (b) 1 (c) 2 (d) 3
**[IIT-JEE, 1993]**
12. Determine the smallest +ve value of $x$ (in degrees) for which
$\tan (x + 100°) = \tan (x + 50°) \tan x \tan (x - 50°)$
**[IIT-JEE, 1993]**

13. Let $n$ be a +ve integer such that
$\sin\left(\frac{\pi}{2n}\right)+\cos\left(\frac{\pi}{2n}\right)=\frac{\sqrt{n}}{2}$. Then,
(a) $6 \le n \le 8$ (b) $4 < n \le 8$
(c) $4 \le n \le 8$ (d) $4 < n < 8$
**[IIT-JEE, 1994]**

14. Let $2\sin^2 x + 3\sin x - 2 \ge 0$ and $x^2 - x - 2 < 0$ ($x$ is measured in radians). Then $x$ lies in the interval
(a) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (b) $\left(-1, \frac{5\pi}{6}\right)$
(c) $(-1, 2)$ (d) $\left(\frac{\pi}{6}, 2\right)$
**[IIT-JEE, 1994]**

15. Find the smallest +ve value of $p$ for which the equation $\cos(p\sin x) = \sin(p\cos x)$ has a solution for $x \in [0, 2\pi]$. **[IIT-JEE, 1995]**

16. Find all values of $\theta$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ satisfying the equation
$$(1 - \tan\theta)(1 + \tan\theta)\sec^2\theta + 2^{\tan 2\theta} = 0$$
**[IIT-JEE, 1996]**

17. Find the general value of $\theta$ satisfying the equation $\tan^2\theta + \sec 2\theta = 1$
**[IIT-JEE, 1997]**

18. Find the real roots of the equation $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$
**[IIT-JEE, 1997]**

19. The number of values of $x$ in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x = 2 = 0$ is
(a) 0 (b) 5 (c) 6 (d) 10
**[IIT-JEE, 1998]**

20. Let $n$ be an odd integer. If $\sin(n\theta)=\sum_{r=0}^{n} b_r \sin^r\theta$ for each value of $\theta$, then
(a) $b_0 = 1, b_1 = 3$
(b) $b_0 = 0, b_1 = n$
(c) $b_0 = -1, b_1 = n$
(d) $b_0 = -1, b_1 = n^2 - 3n + 3$
**[IIT-JEE, 1998]**

21. No questions asked between 1999 to 2001.

22. The number of values of $k$ for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution is
(a) 4 (b) 8 (c) 10 (d) 12
**[IIT-JEE, 2002]**

23. No questions asked between 2003 to 2004.

24. Let $(a, b) \in [-\pi, \pi]$ be such that $\cos(a - b) = 1$ and $\cos(a+b)=\frac{1}{e}$. The number of pairs of $a$, $b$ satisfying the system of equations is
(a) 0 (b) 1 (c) 2 (d) 4
**[IIT-JEE, 2005]**

25. Find the values of $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that
$$2\sin t = \frac{5x^2 - 2x + 1}{3x^2 - 2x - 1}, \forall x \in R - \left\{1, -\frac{1}{3}\right\}$$
**[IIT-JEE, 2005]**

26. If $0 \le \theta \le 2\pi$, $2\sin^2\theta - 5\sin\theta + 2 > 0$ then the range of $\theta$ is
(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$
(b) $\left(0, \frac{5\pi}{6}\right) \cup (\pi, 2\pi)$
(c) $\left(0, \frac{\pi}{6}\right) \cup (\pi, 2\pi)$
(d) None of these
**[IIT-JEE, 2006]**

27. The number of solutions of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$is
(a) 0 (b) 1 (c) 2 (d) 4
**[IIT-JEE, 2007]**

28. If $\sin\theta = \cos\varphi$, then the possible values of $\frac{1}{\pi}\left(\theta \pm \varphi - \frac{\pi}{2}\right)$ are..... **[IIT-JEE, 2008]**

29. For $0 < \theta < \frac{\pi}{2}$, the solutions of
$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$
is (are)
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$
**[IIT-JEE, 2009]**

30. The number of values of $\theta$ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\varphi \ne \frac{n\pi}{2}$ for $n \in I$ and $\tan\theta = \cot 5\theta$ as well as $\sin(2\theta) = \cos(4\theta)$ is... **[IIT-JEE, 2010]**

31. The +ve integer value of $n > 3$ satisfying the equation
$$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$
is... **[IIT-JEE, 2011]**

32. No questions asked between 2012 to 2013.

33. For $x \in (0, \pi)$, the equation $\sin x + 2 \sin 2x - \sin 3x = 3$ has
    (a) infinitely many solutions
    (b) three solutions
    (c) one solution
    (d) no solution **[IIT-JEE, 2014]**

34. The number of distinct solutions of the equation
$$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$$
in the interval $[0, 2\pi]$ is ______ **[IIT-JEE, 2015]**

# Answers

## Level II

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | (b) | 2. | (a, b) | 3. | (a, d) | 4. | (c) | 5. | (a) |
| 6. | (a) | 7. | (c) | 8. | (d) | 9. | (a,b,c) | 10. | (b) |
| 11. | (a) | 12. | (a,b) | 13. | (b) | 14. | (d) | 15. | (c) |
| 16. | (b) | 17. | (b) | 18. | (c) | 19. | (c) | 20. | (b) |
| 21. | (b) | 22. | (c) | 23. | (a) | 24. | (b) | 25. | (c) |
| 26. | (b) | 27. | (a) | 28. | (b) | 29. | (a) | 30. | (c) |
| 31. | (a) | 32. | (c) | 33. | (b) | 34. | (d) | 35. | (b) |
| 36. | (a) | 37. | (c) | 38. | (b) | 39. | (c) | 40. | (b) |
| 41. | (a) | 42. | (b) | 43. | (b) | 44. | (a) | 45. | (c) |
| 46. | (c) | 47. | (c) | 48. | (a) | 49. | (a) | 50. | (b) |
| 51. | (c) | 52. | (a) | 53. | (d) | 54. | (a) | 55. | (d) |
| 56. | (c) | 57. | (c) | 58. | (a) | 59. | (c) | 60. | (b) |
| 61. | (b) | 62. | (b) | 63. | (b) | 64. | (d) | 65. | (a) |
| 66. | (c) | 67. | (b) | 68. | (a) | 69. | (c) | 70. | (b) |

## Level IV

1. $x = (4n+2)\frac{\pi}{3}, n \in I$
2. $x = (2n+1)\frac{\pi}{14}, x = n\pi, n \in I$
3. $x = \frac{n\pi}{3}, n \in I$
4. $x = 2n\pi + \pi = (2n+1)\pi, n \in I$
5. $x = \left(2n\pi + \frac{\pi}{2}\right) = (4n+1)\frac{\pi}{2}, n \in I$
6. $x = n\pi - \frac{\pi}{4}, n \in I$
7. $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{2\pi}{3}, \frac{4\pi}{3}$
8. $x = \frac{n\pi}{2} + (-1)^n \frac{\alpha}{2}$
9. $x = (2n+1)\frac{\pi}{2}$
10. $x = \frac{n\pi}{2} + (-1)^n\left(-\frac{\pi}{12}\right), n \in I$
11. $x = \frac{n\pi}{2} \pm \frac{\alpha}{2}, n \in I, \alpha = \sin^{-1}\left(\frac{1}{3}\right)$
12. $x = n$ð, $x = n\pi - \frac{\pi}{4}, n \in I$
13. $-\sqrt{3} \le a \le \sqrt{3}$
14. 1
15. 1
16. 1
17. $\left(1, \frac{\pi}{2}\right), \left(-1, \frac{3\pi}{2}\right)$
18. 5
19. 2
20. $90°$
21. $-3 \le a \le -2$
22. $x = \frac{13\pi}{4}$.
23. $[-3, -2]$
24. 3
25. $x = 2n\pi \pm \frac{\pi}{3}, n \in I$

### INTEGER TYPE QUESTIONS

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | 6 | 2. | 7 | 3. | 1 | 4. | 4 | 5. | 6 |
| 6. | 6 | 7. | 4 | 8. | 6 | 9. | 3 | 10. | 2 |
| 11. | 5 | 12. | 2 | 13. | 6 | 14. | 1 | 15. | 4 |

### COMPREHENSIVE LINK PASSAGES

Passage-I: 1. (d) 2. (a) 3. (c) 4. (b) 5. (a)
Passage-II: 1. (a, b) 2. (a, c) 3. (c)
Passage-III: 1. (b) 2. (b) 3. (a, b, c, d) 4. (d)
Passage-IV: 1. (b) 2. (c) 3. (b) 4. (b) 5. (b)
Passage-V: 1. (a) 2. (c) 3. (a) 4. (b) 5. (a)

### MATRIX MATCH

1. (A) $\to$ (S), (B) $\to$ (S), (C) $\to$ (S), (D) $\to$ (S)
2. (A) $\to$ (Q), (B) $\to$ (R), (C) $\to$ (S).

3. (A) → (S), (B) → (P), (C) → (Q),(D) → (P, Q)
4. (A) → (S), (B) → (P), (C) → (Q, R),(D)→ (T)
5. (A) → (R), (B) → (S), (C) → (P), (D) → (S)
6. (A) → (S), (B) → (P), (C) → (Q), (D) → (R)
7. (A) → (R), (B) → (S), (C) → (P), (D) → (Q)
8. (A) → (Q), (B) → (R), (C) → (S), (D) → (P)
9. (A) → (S), (B) → (R), (C) → (Q), (D) → (P).

**ASSERTION AND REASON**

| | | | | |
|---|---|---|---|---|
| 1. (a) | 2. (a) | 3. (b) | 4. (b) | 5. (a) |
| 6. (a) | 7. (b) | 8. (b) | 9. (d) | 10. (a) |

# Hints and Solutions

## Level 1

1. We have, $\sin 3\theta = 0$

$\Rightarrow \quad 3\theta = n\pi$

$\Rightarrow \quad \theta = \frac{n\pi}{3}$, where $n \in I$

2. We have, $\cos^2 (5\theta) = 0$

$\Rightarrow \quad \cos^2(5\theta) = \cos^2\left(\frac{\pi}{2}\right)$

$\Rightarrow \quad (5\theta) = n\pi \pm \left(\frac{\pi}{2}\right)$

$\Rightarrow \quad \theta = \frac{1}{5}\left(n\pi \pm \left(\frac{\pi}{2}\right)\right)$, where $n \in I$

3. We have, $\tan\theta = \sqrt{3}$

$\Rightarrow \quad \tan\theta = \tan\left(\frac{\pi}{3}\right)$

$\Rightarrow \quad \theta = n\pi + \left(\frac{\pi}{3}\right)$, where $n \in I$

4. We have, $\sin 2\theta = \sin\theta$

$\Rightarrow \quad 2\sin\theta\cos\theta = \sin\theta$

$\Rightarrow \quad \sin\theta(2\cos\theta - 1) = 0$

$\Rightarrow \quad \sin\theta = 0$ and $(2\cos\theta - 1) = 0$

$\Rightarrow \quad \sin\theta = 0$ and $\cos\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = n\pi$ and $\theta = 2n\pi \pm \frac{\pi}{3}$, where $n \in I$

5. We have, $\sin(9\theta) = \sin\theta$

$\Rightarrow \quad \sin(9\theta) - \sin\theta = 0$

$\Rightarrow \quad 2\cos\left(\frac{9\theta + \theta}{2}\right)\sin\left(\frac{9\theta - \theta}{2}\right) = 0$

$\Rightarrow \quad 2\cos(5\theta)\sin(4\theta) = 0$

$\Rightarrow \quad \cos(5\theta) = 0$ and $\sin(4\theta) = 0$

$\Rightarrow \quad (5\theta) = (2n+1)\frac{\pi}{2}$ and $(4\theta) = n\pi$

$\Rightarrow \quad \theta = (2n+1)\frac{\pi}{10}$ and $\theta = \left(\frac{n\pi}{4}\right)$

where $n \in I$

6. We have, $5\sin^2\theta + 3\cos^2\theta = 4$

$\Rightarrow \quad 2\sin^2\theta + 3(\sin^2\theta + \cos^2\theta) = 4$

$\Rightarrow \quad 2\sin^2\theta + 3 = 4$

$\Rightarrow \quad 2\sin^2\theta = 1$

$\Rightarrow \quad \sin^2\theta = \frac{1}{2}$

$\Rightarrow \quad \sin^2\theta = \left(\frac{1}{\sqrt{2}}\right)^2 = \sin^2\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad \theta = n\pi \pm \left(\frac{\pi}{4}\right)$, where $n \in I$

7. We have, $\tan(\theta - 15°) = \tan(\theta + 15°)$

$\Rightarrow \quad \frac{\tan(\theta - 15°)}{\tan(\theta + 15°)} = \frac{3}{1}$

$\Rightarrow \quad \frac{\tan(\theta - 15°) + \tan(\theta + 15°)}{\tan(\theta - 15°) - \tan(\theta + 15°)} = \frac{3+1}{3-1}$

$\Rightarrow \quad \frac{\sin(\theta + 15° + \theta - 15°)}{\sin(\theta + 15° - \theta + 15°)} = \frac{3+1}{3-1}$

$\Rightarrow \quad 2\sin(2\theta) = 2$

$\Rightarrow \quad \sin(2\theta) = 1$

$\Rightarrow \quad \theta = (4n+1)\frac{\pi}{4}, n \in I$

8. We have, $\tan^2(\theta) + \cot^2(\theta) = 2$

$\Rightarrow \quad \tan^2(\theta) + \frac{1}{\tan^2(\theta)} = 2$

$\Rightarrow \quad \tan^4(\theta) - 2\tan^2(\theta) + 1 = 0$

$\Rightarrow \quad (\tan^2(\theta) - 1)^2 = 0$

$\Rightarrow \quad (\tan^2(\theta) - 1) = 0$

$\Rightarrow \quad \theta = n\pi \pm \left(\frac{\pi}{4}\right), n \in I$

9. We have, $(\cos(3\theta) + \cos(\theta)) + \cos(2\theta) = 0$

$\Rightarrow \quad (\cos(3\theta) + \cos(\theta)) + \cos(2\theta) = 0$

$\Rightarrow \quad 2\cos(2\theta)\cos(\theta) + \cos(2\theta) = 0$

$\Rightarrow \quad \cos(2\theta)(2\cos(\theta) + 1) = 0$

$\Rightarrow \quad \cos(2\theta) = 0$ and $(2\cos(\theta) + 1) = 0$

$\Rightarrow \quad \cos(2\theta) = 0$ and $\cos(\theta) = -\dfrac{1}{2}$

$\Rightarrow \quad \theta = (2n+1)\left(\dfrac{\pi}{4}\right)$ and $\theta = n\pi \pm \left(\dfrac{2\pi}{3}\right)$

10. We have $\sin(2\theta) + \sin(4\theta) + \sin(6\theta) = 0$

$\sin(6\theta) + \sin(2\theta) + \sin(4\theta) = 0$

$2\sin(4\theta), \cos(2\theta) + \sin(4\theta) = 0$

$\sin(4\theta)(2\cos(2\theta) + 1) = 0$

$\sin(4\theta) = 0$ and $(2\cos(2\theta) + 1) = 0$

$\Rightarrow \quad (4\theta) = n\pi$ and $\cos(2\theta) = -\dfrac{1}{2}$

$\Rightarrow \quad \theta = \left(\dfrac{n\pi}{4}\right)$ and $(2\theta) = n\pi \pm \left(\dfrac{2\pi}{3}\right)$

$\Rightarrow \quad \theta = \left(\dfrac{n\pi}{4}\right)$ and $\theta = \left(\dfrac{n\pi}{2}\right) \pm \left(\dfrac{\pi}{3}\right), n \in I$

11. We have

$\tan(\theta) + \tan(2\theta) + \tan(\theta)\tan(2\theta) = 1$

$\Rightarrow \quad \tan(2\theta) + \tan(\theta) = 1 - \tan(\theta)\tan(2\theta)$

$\Rightarrow \quad \dfrac{\tan(2\theta) + \tan(\theta)}{1 - \tan(\theta)\tan(2\theta)} = 1$

$\Rightarrow \quad \tan(3\theta) = 1$

$\Rightarrow \quad \tan(3\theta) = \tan\left(\dfrac{\pi}{4}\right)$

$\Rightarrow \quad (3\theta) = n\pi + \left(\dfrac{\pi}{4}\right)$

$\Rightarrow \quad \theta = \left(\dfrac{n\pi}{3}\right) + \left(\dfrac{\pi}{12}\right), n \in I$

12. We have

$\tan(\theta) + \tan(2\theta) + \tan(3\theta)$

$= \tan(\theta), \tan(2\theta), \tan(3\theta)$

$\Rightarrow \quad \tan(\theta) + \tan(2\theta)$

$= -\tan(3\theta) + \tan(\theta)\tan(2\theta)\tan(3\theta)$

$\Rightarrow$ tna (q) + $\tan(2\theta) = -\tan(3\theta)(1 - \tan(\theta), \tan(2\theta))$

$\Rightarrow \quad \left(\dfrac{\tan(\theta) + \tan(2\theta)}{(1 - \tan(\theta)\cdot\tan(2\theta))}\right) = -\tan(3\theta)$

$\Rightarrow \quad \tan(3\theta) = -\tan(3\theta)$

$\Rightarrow \quad 2\tan(3\theta) = 0$

$\Rightarrow \quad (3\theta) = n\pi$

$\Rightarrow \quad \theta = \left(\dfrac{n\pi}{3}\right), n \in I$

13. Given equation is

$\cot^2\theta + \dfrac{3}{\sin\theta} + 3 = 0$

$\Rightarrow \quad \cot^2\theta + 3(1 + \operatorname{cosec}\theta) = 0$

$\Rightarrow \quad (\operatorname{cosec}^2\theta - 1) + 3(1 + \operatorname{cosec}\theta) = 0$

$\Rightarrow \quad (\operatorname{cosec}\theta - 1 + 3)(1 + \operatorname{cosec}\theta) = 0$

$\Rightarrow \quad (\operatorname{cosec}\theta + 2)(1 + \operatorname{cosec}\theta) = 0$

$\Rightarrow \quad \operatorname{cosec}\theta = -1, -2$

$\Rightarrow \quad \sin\theta = -1, \sin\theta = -\dfrac{1}{2}$

$\Rightarrow \quad \theta = (4n-1)\dfrac{\pi}{2}, \theta = n\pi + (-1)^n\left(-\dfrac{\pi}{6}\right), n \in I$

14. Given equation is

$2\tan\theta - \cot\theta = -1$

$\Rightarrow \quad 2\tan\theta = \cot\theta - 1$

$\Rightarrow \quad 2\tan\theta = \dfrac{1}{\tan\theta} - 1$

$\Rightarrow \quad 2\tan^2\theta + \tan\theta - 1 = 0$

$\Rightarrow \quad 2\tan^2\theta + 2\tan\theta - \tan\theta - 1 = 0$

$\Rightarrow \quad 2\tan\theta(\tan\theta + 1) - (\tan\theta + 1) = 0$

$\Rightarrow \quad (2\tan\theta - 1)(\tan\theta + 1) = 0$

$\Rightarrow \quad (2\tan\theta - 1) = 0, (\tan\theta + 1) = 0$

$\Rightarrow \quad \tan\theta = -1, \dfrac{1}{2}$

$\Rightarrow \quad \theta = \left(n\pi - \dfrac{\pi}{4}\right), \theta = n\pi + \alpha, \alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$

15. Given equation is

$\tan^2\theta + (1 - \sqrt{3})\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan^2\theta + \tan\theta - \sqrt{3}(\tan\theta + 1) = 0$

$\Rightarrow \quad \tan\theta(\tan\theta + 1) - \sqrt{3}(\tan\theta + 1) = 0$

$\Rightarrow \quad (\tan\theta - \sqrt{3})(\tan\theta + 1) = 0$

$\Rightarrow \quad \tan\theta = \sqrt{3}, \tan\theta = -1$

$\Rightarrow \quad \theta = n\pi + \dfrac{\pi}{3}, \theta = n\pi - \dfrac{\pi}{4}, n \in I$

16. Given equation is

$\tan\theta + \tan\left(\theta + \dfrac{\pi}{3}\right) + \tan\left(\theta + \dfrac{2\pi}{3}\right) = 3$

$\Rightarrow \quad \tan\theta + \tan\left(\dfrac{\pi}{3} + \theta\right) + \tan\left(\pi - \left(\dfrac{\pi}{3} - \theta\right)\right) = 3$

$\Rightarrow \quad \tan\theta + \tan\left(\dfrac{\pi}{3} + \theta\right) - \tan\left(\dfrac{\pi}{3} - \theta\right) = 3$

$\Rightarrow \quad \tan\theta + \dfrac{\sqrt{3} + \tan\theta}{1 - \sqrt{3}\tan\theta} - \dfrac{\sqrt{3} - \tan\theta}{1 + \sqrt{3}\tan\theta} = 3$

$\Rightarrow \quad \tan\theta + \dfrac{8\tan\theta}{1 - 3\tan^2\theta} = 3$

$\Rightarrow \quad \dfrac{9\tan\theta - 3\tan^3\theta}{1 - 3\tan^2\theta} = 3$

$\Rightarrow \quad \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 1$

$\Rightarrow \quad \tan(3\theta) = 1$

$\Rightarrow \quad 3\theta = n\pi + \dfrac{\pi}{4}$

$\Rightarrow \quad \theta = \dfrac{n\pi}{3} + \dfrac{\pi}{12}, n \in I$

17. Given equation is

$3\tan(\theta - 60°) = \tan(\theta + 60°)$

$\Rightarrow \quad 3 = \dfrac{\tan(\theta + 60°)}{\tan(\theta - 60°)}$

$\Rightarrow \quad \dfrac{\tan(\theta + 60°)}{\tan(\theta - 60°)} = 3$

$\Rightarrow \quad \dfrac{\tan(\theta + 60°)}{\tan(\theta - 60°)} = \dfrac{3}{1}$

$\Rightarrow \quad \dfrac{\tan(\theta + 60°) + \tan(\theta - 60°)}{\tan(\theta + 60°) - \tan(\theta - 60°)} = \dfrac{3+1}{3-1}$

$\Rightarrow \quad \dfrac{\sin(\theta + 60° + \theta - 60°)}{\sin(\theta + 60° - \theta + 60°)} = 2$

$\Rightarrow \quad \dfrac{\sin(2\theta)}{\sin(120°)} = 2$

$\Rightarrow \quad \sin(2\theta) = 2\sin(120°)$

$\Rightarrow \quad \sin(2\theta) = 2 \times \dfrac{\sqrt{3}}{2} = \sqrt{3}$

It is not possible.
Hence, the equation has no solution.

18. Given equation is

$\tan\theta + \tan 2\theta + \tan 3\theta = 0$

$\Rightarrow \quad \tan\theta + \tan 2\theta + \tan(2\theta + \theta) = 0$

$\Rightarrow \quad \tan\theta + \tan 2\theta + \dfrac{\tan(2\theta) + \tan(\theta)}{1 - \tan(2\theta)\tan(\theta)} = 0$

$\Rightarrow \quad (\tan\theta + \tan 2\theta)\left(1 + \dfrac{1}{1 - \tan(2\theta)\tan(\theta)}\right) = 0$

$\Rightarrow \quad (\tan\theta + \tan 2\theta) = 0, \left(1 + \dfrac{1}{1 - \tan(2\theta)\tan(\theta)}\right) = 0$

when $(\tan\theta + \tan 2\theta) = 0$

$\Rightarrow \quad \tan\theta + \dfrac{2\tan\theta}{1 - \tan^2\theta} = 0$

$\Rightarrow \quad \tan\theta\left(1 + \dfrac{2}{1 - \tan^2\theta}\right) = 0$

$\Rightarrow \quad \tan\theta = 0, \left(1 + \dfrac{2}{1 - \tan^2}\right) = 0$

$\Rightarrow \quad \tan\theta = 0, \dfrac{2}{1 - \tan^2\theta} = -1$

$\Rightarrow \quad \tan\theta = 0, 1 - \tan^2\theta = -2$

$\Rightarrow \quad \tan\theta = 0, \tan^2\theta = 3$

$\Rightarrow \quad \theta = n\pi, \theta = n\pi \pm \dfrac{\pi}{3}, n \in I$

when $\left(1 + \dfrac{1}{1 - \tan(2\theta)\tan(\theta)}\right) = 0$

$\Rightarrow \quad \dfrac{1}{1 - \tan\theta\tan 2\theta} = -1$

$\Rightarrow \quad 1 - \tan\theta\tan 2\theta = -1$

$\Rightarrow \quad \tan\theta\tan 2\theta = 2$

$\Rightarrow \quad \tan\theta\left(\dfrac{2\tan\theta}{1 - \tan^2\theta}\right) = 2$

$\Rightarrow \quad \tan^2\theta = 1 - \tan^2\theta$

$\Rightarrow \quad \tan^2\theta = \dfrac{1}{2} = \tan^2\alpha, \alpha = \tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad \theta = n\pi \pm \alpha, n \in I$

19. Given equation is

$\Rightarrow \quad \cos 2\theta\cos 4\theta = \dfrac{1}{2}$

$\Rightarrow \quad 2\cos(4\theta)\cos(2\theta) = 1$

$\Rightarrow \quad \cos(6\theta) + \cos(2\theta) = 1$

$\Rightarrow \quad \cos(6\theta) = 1 - \cos(2\theta)$

20. Given equation is

$\cot\theta - \tan\theta = \cos\theta - \sin\theta$

$\Rightarrow \quad (\cos\theta - \sin\theta)\left(\dfrac{(\cos\theta + \sin\theta)}{\sin\theta\cos\theta} - 1\right) = 0$

$\Rightarrow \quad (\cos\theta - \sin\theta) = 0, (\cos\theta + \sin\theta) = \sin\theta\cos\theta$

$\Rightarrow \quad \tan\theta = 1, (\cos\theta + \sin\theta) = \sin q\cos\theta$

when $\tan\theta = 1$

$\Rightarrow \quad \theta = n\pi + \dfrac{\pi}{4}, n \in I$

when $(\cos\theta + \sin\theta) = \sin\theta\cos\theta$

No real value of $\theta$ satisfies the given equation.

21. Given equation is

$(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$

$\Rightarrow \quad (\cos\theta - \sin\theta)(\cos\theta + \sin\theta)^2 = (\cos\theta + \sin\theta)$

$\Rightarrow \quad (\cos\theta + \sin\theta)(\cos 2\theta - 1) = 0$

$\Rightarrow \quad \tan(\theta) = -1, \sin^2\theta = 0$

$\Rightarrow \quad \tan(\theta) = -1, \sin(\theta) = 0$

$\Rightarrow \quad \theta = n\pi - \dfrac{\pi}{4}, \theta = n\pi, n \in I$

22. Given equation is

$2\sin^2\theta + \sin^2 2\theta = 2$

$\Rightarrow \quad 2\sin^2\theta + 4\sin^2\theta\cos^2\theta = 2$

$\Rightarrow \quad \sin^2\theta + 2\sin^2\theta\cos^2\theta = 1$

$\Rightarrow \quad 2\sin^2\theta\cos^2\theta = 1 - \sin^2\theta$

$\Rightarrow$ $2\sin^2\theta\cos^2\theta = \cos^2\theta$

$\Rightarrow$ $(2\sin^2\theta - 1)\cos^2\theta = 0$

$\Rightarrow$ $(2\sin^2\theta - 1) = 0, \cos^2\theta = 0$

$\Rightarrow$ $\sin^2\theta = \frac{1}{2}, \cos\theta = 0$

$\Rightarrow$ $\theta = (2n+1)\frac{\pi}{2}, \theta = n\pi \pm \frac{\pi}{4}, n \in I$

23. Given equation is

$\sin(3\alpha) = 4\sin\theta\sin(\theta+\alpha)\sin(\theta+\alpha)$

$\Rightarrow$ $\sin(3\alpha) = 4\sin\theta(\sin^2\theta - \sin^2\alpha)$

$\Rightarrow$ $3\sin\alpha - 4\sin^2\alpha = 4\sin\alpha(\sin^2\theta - \sin^2\alpha)$

It is possible only when

$\sin^2\theta = \frac{3}{4}$

$\Rightarrow$ $\sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow$ $\theta = n\pi \pm \frac{\pi}{3}, n \in I$

24. Given equation is

$4\sin\theta\sin 2\theta\sin 4\theta = \sin 3\theta$

$\Rightarrow$ $4\sin\theta\sin(3\theta-\theta)\sin(3\theta+\theta) = \sin 3\theta$

$\Rightarrow$ $4\sin\theta[\sin^2(3\theta) - \sin^2(\theta)] = \sin 3\theta$

$\Rightarrow$ $4\sin\theta[\sin^2(3\theta) - \sin^2(\theta)] = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow$ $\sin\theta[4\sin^2(3\theta) - 4\sin^2(\theta) + 4\sin^2\theta - 3] = 0$

$\Rightarrow$ $\sin\theta[4\sin^2(3\theta) - 3] = 0$

$\Rightarrow$ $\sin\theta = 0, [4\sin^2(3\theta) - 3] = 0$

$\Rightarrow$ $\sin\theta = 0, \sin^2(3\theta) = \frac{3}{4}$

$\Rightarrow$ $\theta = n\pi, \theta = n\pi \pm \frac{\pi}{3}, n \in I$

25. We have $\sin(\theta) + \cos(\theta) = 1$

$\Rightarrow$ $\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(\theta) + \frac{1}{\sqrt{2}}\cos(\theta)\right) = 1$

$\Rightarrow$ $\left(\frac{1}{\sqrt{2}}\sin(\theta) + \frac{1}{\sqrt{2}}\cos(\theta)\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\sin\left(\theta + \frac{\pi}{4}\right) = \left(\sin\left(\frac{\pi}{4}\right)\right)$

$\Rightarrow$ $\left(\theta + \frac{\pi}{4}\right) = \left(n\pi + (-1)^n\left(\frac{\pi}{4}\right)\right)$

$\Rightarrow$ $\theta = \left(n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right), n \in I$

26. We have $\sqrt{3}\sin(\theta) + \cos(\theta) = 2$

$\Rightarrow$ $\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = 1$

$\Rightarrow$ $\sin\left(\theta + \frac{\pi}{6}\right) = 1$

$\Rightarrow$ $\sin\left(\theta + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right)$

$\Rightarrow$ $\left(\theta + \frac{\pi}{6}\right) = n\pi + (-1)^n\left(\frac{\pi}{2}\right)$

$\Rightarrow$ $\theta = n\pi + (-1)^n\left(\frac{\pi}{2}\right) - \frac{\pi}{6}$

27. We have

$\sin(2\theta) + \cos(2\theta) + \sin(\theta) + \cos(\theta) + 1 = 0$

$\Rightarrow$ $(\sin(\theta) + \cos(\theta)) + (1 + \sin(2\theta)) + \cos(2\theta) = 0$

$\Rightarrow$ $(\sin(\theta) + \cos(\theta)) + (\sin(\theta) + \cos(\theta))^2 + (\cos^2\theta - \sin^2\theta) = 0.$

$\Rightarrow$ $(\sin(\theta) + \cos(\theta)) + (\sin(\theta) + \cos(\theta))^2 + (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 0$

$\Rightarrow$ $(\sin(\theta) + \cos(\theta))(1 + (\sin(\theta) + \cos(\theta)) + (\cos\theta - \sin\theta)) = 0$

$\Rightarrow$ $(\sin(\theta) + \cos(\theta))(1 + 2\cos\theta) = 0$

$\Rightarrow$ $(\sin(\theta) + \cos(\theta)) = 0$ and $(1 + 2\cos\theta) = 0$

$\Rightarrow$ $\left(\sin\left(\frac{\pi}{4} + \theta\right)\right) = 0$ and $\cos\theta = -\frac{1}{2}$

$\Rightarrow$ $\left(\frac{\pi}{4} + \theta\right) = n\pi$ and $\theta = 2n\pi \pm \left(\frac{2\pi}{3}\right)$

$\Rightarrow$ $\theta = n\pi - \frac{\pi}{4}$ and $\theta = 2n\pi \pm \left(\frac{2\pi}{3}\right), n \in I$

28. We have $\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta = 1$

$\Rightarrow$ $(\sin^3\theta + \cos^3\theta) + \sin\theta\cos\theta = 1$

$\Rightarrow$ $(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) + \sin\theta\cos\theta = 1$

$\Rightarrow$ $(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) = (1 - \sin\theta\cos\theta)$

$\Rightarrow$ $(\sin\theta + \cos\theta - 1)(1 - \sin\theta\cos\theta) = 0$

$\Rightarrow$ $(\sin\theta + \cos\theta - 1) = 0$ and $(1 - \sin\theta\cos\theta) = 0$

$\Rightarrow$ $(\sin\theta + \cos\theta) = 1$ and $\sin(2\theta) = \frac{1}{2}$

$\Rightarrow$ $\left(\sin\left(\theta + \frac{\pi}{4}\right)\right) = \sin\left(\frac{\pi}{2}\right)$

and $\sin(2\theta) = \sin\left(\frac{\pi}{6}\right)$

$\Rightarrow$ $\theta = n\pi + (-1)^n\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{4}\right)$

and $\theta = \frac{1}{2}\left(n\pi + (-1)^n\left(\frac{\pi}{6}\right)\right)$, where $n \in I$

29. Given equation is

$\sin\theta + \sqrt{3}\cos\theta = \sqrt{2}$

$\Rightarrow$ $\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin\left(\theta + \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \left(\theta + \frac{\pi}{3}\right) = n\pi + (-1)^n \left(\frac{\pi}{4}\right)$

$\Rightarrow \quad \theta = n\pi + (-1)^n \left(\frac{\pi}{4}\right) - \frac{\pi}{3}, n \in I$

30. Do yourself.
31. Do yourself.
32. Do yourself.
33. Do yourself.
34. Do yourself.
35. Do yourself.
36. Do yourself.
37. Given equation is

$\cos\theta + \sqrt{3}\sin\theta = 2\cos 2\theta$

$\Rightarrow \quad \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta = \cos 2\theta$

$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{3}\right) = \cos 2\theta$

$\Rightarrow \quad \left(\theta - \frac{\pi}{3}\right) = 2n\pi \pm 2\theta$

Taking positive one, we get

$\theta = -\left(2n\pi + \frac{\pi}{3}\right)$

Taking negative one, we get,

$\Rightarrow \quad \theta = \frac{2n\pi}{3} + \frac{\pi}{9}, n \in I$

38. Given equation is

$\sqrt{3}(\cos\theta - \sqrt{3}\sin\theta) = 4\sin 2\theta \cdot \cos 3\theta$

$\Rightarrow \quad \sqrt{3}\cos\theta - 3\sin\theta = 2(\sin 5\theta - \sin\theta)$

$\Rightarrow \quad \sqrt{3}\cos\theta - \sin\theta = 2(\sin 5\theta)$

$\Rightarrow \quad \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = (\sin 5\theta)$

$\Rightarrow \quad \sin\left(\frac{\pi}{3} - \theta\right) = \sin 5\theta$

$\Rightarrow \quad 5\theta = n\pi + (-1)^n \left(\frac{\pi}{3} - \theta\right)$

when $n$ is even

$5\theta = 2k\pi + \left(\frac{\pi}{3} - \theta\right)$

$\Rightarrow \quad 6\theta = 2k\pi + \frac{\pi}{3}$

$\Rightarrow \quad \theta = \frac{k\pi}{3} + \frac{\pi}{18}, k \in I$

when $n$ is odd

$5\theta = (2k+1)\pi - \left(\frac{\pi}{3} - \theta\right)$

$\Rightarrow \quad 4\theta = (2k+1)\pi - \frac{\pi}{3}$

$\Rightarrow \quad \theta = (2k+1)\frac{\pi}{4} - \frac{\pi}{12}, k \in I$

39. We have $\sin(\theta) = -\frac{1}{2}$

$\Rightarrow \quad \theta = -\frac{\pi}{6}$

Hence, the principal value of $\theta$ is $\left(-\frac{\pi}{6}\right)$

40. We have $\sin(\theta) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Hence the principal value of $\theta$ is $\frac{\pi}{4}$

41. We have $\tan(\theta) = -\sqrt{3}$

$\Rightarrow \quad (\theta) = -\frac{\pi}{3}$

Hence, the principal value of $\theta$ is $-\frac{\pi}{3}$.

42. Given, $\tan\theta = -1$

$\Rightarrow \quad \theta = \frac{3\pi}{4}, -\frac{\pi}{4}$

Hence, the principal value of $\theta$ is $-\frac{\pi}{4}$.

43. Given, $\cos\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{3}, -\frac{\pi}{3}$

Hence, the principal value of $\theta$ is $\frac{\pi}{3}$.

44. Given, $\cos\theta = -\frac{1}{2}$

$\Rightarrow \quad \theta = \frac{2\pi}{3}, -\frac{2\pi}{3}$

Hence, the principal value of $\theta$ is $\frac{2\pi}{3}$.

45. Given, $\tan\theta = -\sqrt{3}$

$\Rightarrow \quad \theta = \frac{2\pi}{3}, -\frac{\pi}{3}$

Hence, the principal value of $\theta$ is $-\frac{\pi}{3}$.

46. Given, $\sec\theta = \sqrt{2}$.

$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{\pi}{4}, -\frac{\pi}{4}$

Hence, the principal value of $\theta$ is $\frac{\pi}{4}$.

47. Now $\sin(\theta) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

and $\cos(\theta) = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{3\pi}{4}$

Thus, the common value of $\theta$ is $\frac{3\pi}{4}$.

Hence, the general values of $\theta$ is

$$\left(2n\pi + \frac{3\pi}{4}\right)$$

48. We have $\sin(\theta) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Also, $\tan\theta = -1$

$\Rightarrow \quad \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

Thus, the common value of $\theta$ is $\frac{3\pi}{4}$.

Hence, the general values of $\theta$ is $\left(2n\pi + \frac{3\pi}{4}\right)$, where $n \in I$.

49. Given, $\cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{7\pi}{4}$

and $\tan\theta = -1$

$\Rightarrow \quad \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

Hence, the general solution is

$$\theta = 2n\pi + \frac{7\pi}{4}, n \in I$$

50. Given, $\sin\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

and $\tan\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = \frac{\pi}{3}, \frac{4\pi}{3}$

Hence, the general solution is

$$\theta = 2n\pi + \frac{\pi}{3}, n \in I$$

51. We have, $(1 + \tan A)(1 + \text{tab } B) = 2$

$\Rightarrow \quad 1 + \tan A + \tan B + \tan A \cdot \tan B = 2$

$\Rightarrow \quad \tan A + \tan B = 1 - \tan A \cdot \tan B$

$\Rightarrow \quad \left(\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}\right) = 1$

$\Rightarrow \quad \tan(A + B) = 1$

$\Rightarrow \quad \tan(A + B) = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad (A + B) = n\pi + \left(\frac{\pi}{4}\right)$, where $n \in I$

52. Given, $\tan(A - B) = 1$

$\Rightarrow \quad (A - B) = \frac{\pi}{4}, \frac{5\pi}{4}$

Also, $\sec(A + B) = \frac{2}{\sqrt{3}}$

$\Rightarrow \quad \cos(A + B) = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad (A + B) = \frac{\pi}{6}, \frac{11\pi}{6}$

Here, we observe that $A - B$ is positive

So, $\quad A > B$

$\Rightarrow \quad A + B > A - B$

$$\begin{cases} A + B = \frac{11\pi}{6} \\ A - B = \frac{\pi}{4} \end{cases} \text{or} \begin{cases} A + B = \frac{11\pi}{6} \\ A - B = \frac{5}{4} \end{cases}$$

On solving, we get,

$$\begin{cases} A = \frac{25\pi}{24} \\ B = \frac{19\pi}{24} \end{cases} \text{or} \begin{cases} A = \frac{19\pi}{24} \\ B = \frac{7\pi}{24} \end{cases}$$

General values of tan $\tan(A - B) = 1$

is $\quad (A - B) = n\pi + \frac{\pi}{4}, n \in I \qquad \ldots\text{(i)}$

General values of $\sec(A+B) = \frac{2}{\sqrt{3}}$

is $(A+B) = 2n\pi + \frac{\pi}{6}, n \in I$ ...(ii)

On solving (i) and (ii), we get

$$\begin{cases} A = (2n+m)\frac{\pi}{2} + \frac{\pi}{24} \\ B = (2n-m)\frac{\pi}{2} - \frac{5\pi}{24} \end{cases}$$

53. We have $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$

$\Rightarrow \sin(\pi\cos\theta) = \sin\left(\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow (\pi\cos\theta) = \left(\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow \cos\theta = \left(\frac{1}{2} - \sin\theta\right)$

$\Rightarrow \cos\theta + \sin\theta = \frac{1}{2}$

$\Rightarrow \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2\sqrt{2}}$

$\Rightarrow \cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

Similarly, we can prove that,

$\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

54. We have $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$

$\Rightarrow \tan(\pi\cos\theta) = \tan\left(\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow (\pi\cos\theta) = \left(\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow \cos(\theta) + \sin(\theta) = \frac{1}{2}$

$\Rightarrow \frac{1}{\sqrt{2}}\cos(\theta) + \frac{1}{\sqrt{2}}\sin(\theta) = \frac{1}{2\sqrt{2}}$

$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

55. Given, $\sin A = \sin B$ ...(i)

and $\cos A = \cos B$ ...(ii)

Dividing (i) and (ii), we get,

$\frac{\sin A}{\cos A} = \frac{\sin B}{\cos B}$

$\Rightarrow \tan A = \tan B$

$\Rightarrow A = n\pi + B$, where $n \in I$

56. Given equations are

$3\sin^2 A + 2\sin^2 B = 1$ ...(i)

and $3\sin 2A - 2\sin 2B = 0$ ...(ii)

From (ii), we get,

$3\sin 2A = 2\sin 2B$

$\Rightarrow \frac{\sin 2A}{2} = \frac{\sin 2B}{3}$

$\Rightarrow \frac{\sin 2B}{\sin 2A} = \frac{3}{2}$

From (i), we get

$\frac{3}{2}(2\sin^2 A) + (2\sin^2 B) = 1$

$\Rightarrow \frac{3}{2}(1 - \cos 2A) + (1 - \cos 2B) = 1$

$\Rightarrow \frac{3}{2}\cos 2A + \cos 2B = \frac{3}{2}$

$\Rightarrow \frac{\sin 2B}{\sin 2A}\cos 2A + \cos 2B = \frac{\sin 2B}{\sin 2A}$

$\Rightarrow \sin 2B\cos 2A + \sin 2A\cos 2B = \sin 2B$

$\Rightarrow \sin(2A + 2B) = \sin 2B$

$\Rightarrow \sin(2A + 2B) = \sin(\pi - 2B)$

$\Rightarrow (2A + 2) = (\pi = 2B)$

$\Rightarrow (2A + 4B) = \pi$

$\Rightarrow (A + 2B) = \frac{\pi}{2}$

57. Given, $x + y = \frac{\pi}{4}$ and $\tan x + \tan y = 1$

$\Rightarrow \tan(x+y) = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y} = 1$

$\Rightarrow 1 - \tan x \cdot \tan y = 1$

$\Rightarrow \tan x \cdot \tan y = 0$

$\Rightarrow \tan x = 0$ & $\tan y = 0$

$\Rightarrow x = n\pi = y$

Thus, no values of $x$ and $y$ satisfy the given equations. Therefore, the given equations have no solutions.

58. Given, $\sin x + \sin y = 1$ ...(i)

and $\cos 2x - \cos 2y = 1$ ...(ii)

From (ii), we get, $\cos 2x - \cos 2y = 1$

$\Rightarrow 1 - 2\sin^2 x - 1 + \sin^2 y = 1$

$\Rightarrow \sin^2 x - \sin^2 y = -\frac{1}{2}$

$\Rightarrow y = n\pi + (-1)^n \sin^{-1}\left(\frac{3}{4}\right),$

where $n \in I$

$\Rightarrow \quad \sin x - \sin y = -\dfrac{1}{2}$ ...(iii)

Adding (i) and (iii), we get,

$$2 \sin x = \frac{1}{2}$$

$\Rightarrow \quad \sin x = \dfrac{1}{4}$

$\Rightarrow \quad x = n\pi + (-1)^n \sin^{-1}\left(\dfrac{1}{4}\right), n \in I,$

Subtracting (i) and (iii), we get

$$2 \sin y = \frac{3}{2}$$

$\Rightarrow \quad \sin y = \dfrac{3}{4}$

$\Rightarrow \quad y = n\pi + (-1)^n \sin^{-1}\left(\dfrac{3}{4}\right), n \in I$

59. Given, $\sin x = 2 \sin y$

$\Rightarrow \quad \sin x = 2 \sin\left(\dfrac{2\pi}{3} - x\right)$

$\Rightarrow \quad \sin x = 2\left(\dfrac{\sqrt{3}}{2}\cos x + \dfrac{1}{2}\sin x\right)$

$\Rightarrow \quad \sin x = \sqrt{3} \cos x + \sin x$

$\Rightarrow \quad \sqrt{3} \cos x = 0$

$\Rightarrow \quad \cos x = 0$

$\Rightarrow \quad x = (2n+1)\dfrac{\pi}{2}$

when $x = (2n+1)\dfrac{\pi}{2}$, then $y = n\pi - \dfrac{\pi}{6}$

Hence, the solutions are

$$\left\{\begin{array}{l} x = (2n+1)\dfrac{\pi}{2} \\ y = n\pi - \dfrac{\pi}{6} \end{array}\right., n \in I$$

60. Given, $x + y = \dfrac{2\pi}{3}$ and $\cos x + \cos y = \dfrac{3}{2}$

Now $\cos x + \cos y = \dfrac{3}{2}$

$\Rightarrow \quad \cos x + \cos\left(\dfrac{2\pi}{3} - x\right) = \dfrac{3}{2}$

$\Rightarrow \quad \cos x - \dfrac{1}{2}\cos x + \dfrac{\sqrt{3}}{2}\sin x = \dfrac{3}{2}$

$\Rightarrow \quad \dfrac{1}{2}\cos x + \dfrac{\sqrt{3}}{2}\sin x = \dfrac{3}{2}$

$\Rightarrow \quad \cos x + \sqrt{3} \sin x = 3$

It is not possible, since the maximum value of LHS is 2.

So, the given system of equations has no solutions.

61. Given equations are

$r \sin \theta = 3$ ...(i)

and $r = 4(1 + \sin \theta)$ ...(ii)

Eliminating (i) and (ii), we get

$4(1 + \sin \theta) \sin \theta = 3$

$\Rightarrow \quad 4 \sin^2 \theta + 4 \sin \theta - 3 = 0$

$\Rightarrow \quad 4 \sin^2 \theta + 6 \sin \theta - 2 \sin \theta - 3 = 0$

$\Rightarrow \quad 2 \sin \theta(2 \sin \theta + 3) - 1(2 \sin \theta + 3) = 0$

$\Rightarrow \quad (2 \sin \theta + 3)(2 \sin \theta - 1) = 0$

$\Rightarrow \quad \sin \theta = -\dfrac{3}{2}, \dfrac{1}{2}$

$\Rightarrow \quad \sin \theta = \dfrac{1}{2}$

$\Rightarrow \quad \theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

62. Given equations are

$\sin x + \sin y = 1$ ...(i)

and $\cos 2x - \cos 2y = 1$

Now, $\cos 2x - \cos 2y = 1$

$\Rightarrow \quad 1 - 2 \sin^2 x - 1 + 2 \sin^2 y = 1$

$\Rightarrow \quad -2 \sin^2 x - 1 + 2 \sin^2 y = 0$

$\Rightarrow \quad 2(\sin^2 x - \sin^2 y) = -1$

$\Rightarrow \quad (\sin x + \sin y)(\sin x - \sin y) = -\dfrac{1}{2}$

$\Rightarrow \quad (\sin x - \sin y) = -\dfrac{1}{2}$ ...(ii)

On solving, we get

$\sin x = 0, \sin y = 1$

$\Rightarrow \quad x = n\pi, y = (4n+1)\dfrac{\pi}{2}, n \in I$

Hence, the solutions are

$$\left\{\begin{array}{l} x = n\pi \\ y = (4n+1)\dfrac{\pi}{2} \end{array}\right., n \in I$$

63. Given curves are $y = \cos x$ and $y = \sin 2x$

Thus, $\sin 2x = \cos x$

$\Rightarrow \quad 2$ isn $x \cos x = \cos x$

$\Rightarrow \quad (2 \sin x - 1) \cos x = 0$

$\Rightarrow \quad (2 \sin x - 1) = 0, \cos x = 0$

$\Rightarrow \quad \sin x = \dfrac{1}{2}, \cos x = 0$

$\Rightarrow \quad x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

then $\quad y = \dfrac{\sqrt{3}}{2}, -\dfrac{\sqrt{3}}{2}, 0$

Hence, the solutions are

$$\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right), \left(\frac{5\pi}{6}, -\frac{\sqrt{3}}{2}\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$$

64. Given equation is

$\Rightarrow \quad \cos x + \cos y + \cos(x+y) = -\frac{3}{2}$

$\Rightarrow \quad 2(\cos x + \cos y) + 2\cos(x+y) + 2 = 1$

$\Rightarrow \quad 4\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + 4\cos^2\left(\frac{x+y}{2}\right) = 1$

$\Rightarrow \quad 4\cos^2\left(\frac{x+y}{2}\right) + 4\cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) + 1 = 0$

For real $x$ and $y$,

$16\cos^2\left(\frac{x-y}{2}\right) - 16 \geq 0$

$\Rightarrow \quad \cos^2\left(\frac{x-y}{2}\right) \geq 1$

$\Rightarrow \quad \cos^2\left(\frac{x-y}{2}\right) = 1$

$\Rightarrow \quad \left(\frac{x-y}{2}\right) = 0$

$\Rightarrow \quad x = y$

The given equation

$4\cos^2\left(\frac{x+y}{2}\right) + 4\cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) + 1 = 0$

reduces to $4\cos^2(x) + \cos(x) + 1 = 0$

$\Rightarrow \quad (2\cos(x) + 1)^2 = 0$

$\Rightarrow \quad \cos(x) = -\frac{1}{2}$

$\Rightarrow \quad x = \frac{2\pi}{3} = y$

65. Given equation is

$8\cos\theta\cos\varphi\cos(\theta+\varphi) + 1 = 0$

$\Rightarrow \quad 2\cos\theta\cos\varphi\cos(\theta+\varphi) = -\frac{1}{4}$

$\Rightarrow \quad 4[\cos(\theta+\varphi) + \cos(\theta+\varphi)]\cos(\theta+\varphi) + 1 = 0$

$\Rightarrow \quad 4\cos^2(\theta+\varphi) + \cos(\theta+\varphi)\cos(\theta+\varphi) + 1 = 0$

For all real $0 < \theta, \varphi < \pi$,

$16\cos^2(\theta-\varphi) - 16 > 0$

$\Rightarrow \quad \cos^2(\theta-\varphi) \geq 1$

$\Rightarrow \quad \cos^2(\theta-\varphi) = 1$

$\Rightarrow \quad \theta - \varphi = 0$

$\Rightarrow \quad \theta = \varphi$

when $\theta = \varphi$, then the equation

$4\cos^2(\theta+\varphi) + 4\cos(\theta-\varphi)\cos(\theta+\varphi) + 1 = 0$

reduces to

$\Rightarrow \quad 4\cos^2(2\theta) + 4\cos(2\theta) + 1 = 0$

$\Rightarrow \quad (2\cos(2\theta) + 1)^2 = 0$

$\Rightarrow \quad (2\cos(2\theta) + 1) = 0$

$\Rightarrow \quad \cos(2\theta) = -\frac{1}{2}$

$\Rightarrow \quad (2\theta) = \frac{2\pi}{3}$

$\Rightarrow \quad \theta = \frac{\pi}{3} = \varphi$

66. We have $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \cdot \tan 2x} = 1$

$\Rightarrow \quad \tan(3x - 2x) = 1$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad x = n\pi + \frac{\pi}{4}$, where $n \in I$

But the values of $x$ do not satisfy the given equation.

Hence, the set of values of $x$ is $\phi$.

67. Given equation is $\tan x + \sec x = 2\cos x$

$\Rightarrow \quad (1 + \sin x) = 2\cos^2 x$

$\Rightarrow \quad (1 + \sin x) = 2(1 - \sin^2 x)$

$\Rightarrow \quad (1 + \sin x) = 2(1 + \sin x)\cdot(1 - \sin x)$

$\Rightarrow \quad (1 + \sin x)(1 - 2 + 2\sin x) = 0$

$\Rightarrow \quad (1 + \sin x)(2\sin x - 1) = 0$

$\Rightarrow \quad (1 + \sin x) = 0$ and $(2\sin x - 1) = 0$

$\Rightarrow \quad \sin x = -1$ and $\sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

But $x = \frac{\pi}{2}$ does not satisfy the given equation.

Thus, the values of $x$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Hence, the number of solutions is 2.

68. Given equation is $2\sin^2 x + 6\sin x - \sin x - 3 = 0$

$\Rightarrow \quad 2\sin^2 x + 6\sin x - \sin x - 3 = 0$

$\Rightarrow \quad 2\sin x(\sin x + 3) - 1(\sin x + 3) = 0$

$\Rightarrow \quad (\sin x + 3)(2\sin x - 1) = 0$

$\Rightarrow \quad \sin x = -3, \frac{1}{2}$

$\Rightarrow \quad \sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

Hence, the number of values of $x$ is 4.

69. Given

$\tan(x + 20°) = \tan(x - 10°)\tan x \cdot \tan(x + 10°)$

$\Rightarrow \quad \frac{\tan(x + 20°)}{\tan x} = \tan(x - 10°)\cdot\tan(x + 10°)$

$$\Rightarrow \quad \frac{\sin(x+20°)\cos x}{\cos(x+20°)\sin x} = \frac{\sin(x-10°)\sin(x+10°)}{\cos(x-10°)\cdot\cos(x+10°)}$$

$$\Rightarrow \quad \frac{\sin(x+20°)\cos x+\cos(x+20°)\sin x}{\sin(x+20°)\cos x-\cos(x+20°)\sin x}$$

$$= \frac{\sin(x-10°)\sin(x+10°)+\cos(x-10°)\cdot\cos(x+10°)}{\sin(x-10°)\sin(x+10°)-\cos(x-10°)\cdot\cos(x+10°)}$$

$$\Rightarrow \quad \frac{\sin(x+20°+x)}{\sin(x+20°-x)} = -\frac{\cos(x+10°-x+10°)}{\cos(x+10°+x-10°)}$$

$$\Rightarrow \quad \frac{\sin(2x+20°)}{\sin(20°)} = -\frac{\cos(20°)}{\cos(2x)}$$

$\Rightarrow$ $\sin(2x + 20°)\cos(2x) = -\sin(20°)\cos(20°)$
$\Rightarrow$ $2\sin(2x + 20°)\cos(2x) = -2\sin(20°)\cos(20°)$
$\Rightarrow$ $\sin(4x + 20°) + \sin(20°) = \sin(40°)$
$\Rightarrow$ $\sin(4x + 20°) = -\sin(40°) - \sin(20°)$
$\Rightarrow$ $\sin(4x + 20°) = -2\sin(30°)\cos(10°)$
$\Rightarrow$ $\sin(4xz + 20°) = -\cos(10°)$
$\Rightarrow$ $\sin(4x + 20°) = -\sin(80°)$
$\Rightarrow$ $\sin(4x + 20°) = \sin(-80°)$
$\Rightarrow$ $\sin(4x + 20°) = \sin(\pi - (-80°))$
$\Rightarrow$ $(4x + 20°) = (\pi - (-80°))$
$\Rightarrow$ $(4x + 20°) = 260°$
$\Rightarrow$ $4x = 260° - 20° = 240°$
$\Rightarrow$ $x = 60°$

Hence, the smallest positive value of $x$ is 60°

70. Given, $\sin^2 x + \cos^2 y = 2\sec^2 z$
Here, LHS $\leq 2$ and RHS $\geq 2$
It is possible only when
$\sin^2 x = 1, \cos^2 y = 1, \sec^2 z = 1$
$\Rightarrow$ $\cos^2 x = 0, \sin^2 y = 0, \cos^2 z = 1$
$\Rightarrow$ $\cos^2 x = 0, \sin^2 y = 0, \sin^2 z = 1$
$\Rightarrow$ $\cos x = 0, \sin y = 0, \sin z = 0$

$$\Rightarrow \quad x = (2n+1)\frac{\pi}{2}, y = m\pi, z = k\pi$$

where, $n, m, k \in I$.

71. The given equation can be expressed as
$5(2\cos^2 x - 1) + (1\ \ \cos x) + 1 = 0$
$\Rightarrow$ $10\cos^2 x + \cos x - 3 = 0$
$\Rightarrow$ $(5\cos x + 3)(2\cos x - 1) = 0$
$\Rightarrow$ $(5\cos x + 3) = 0, (2\cos x - 1) = 0$

$$\Rightarrow \quad \cos x = -\frac{3}{5} = \cos\alpha,$$

$$\Rightarrow \quad \cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad x = 2n\pi \pm \alpha = 2n\pi \pm \cos^{-1}\left(\frac{\pi}{3}\right),$$

$$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{3}, n \in Z$$

72. Given equation is
$4\sin^4 x + \cos^4 x = 1$
$\Rightarrow$ $4\sin^4 x = 1 - \cos^4 x$
$\Rightarrow$ $4\sin^4 x = (1 + \cos^2 x)\sin^2 x$
$\Rightarrow$ $\sin^2 x(4\sin^2 x - \cos^2 x - 1) = 0$
$\Rightarrow$ $\sin^2 x = 0, (5\sin^2 x - 2) = 0$

$$\Rightarrow \quad \sin x = 0, \sin^2 x = \frac{2}{5}$$

$$\Rightarrow \quad x = n\pi, x = n\pi \pm \alpha, \alpha = \sin^{-1}\left(\sqrt{\frac{2}{5}}\right)$$

73. Given equation is
$4\cos^2 x \sin x - 2\sin^2 x = 2\sin x$
$\Rightarrow$ $4(1 - \sin^2 x)\sin x - 2\sin^2 x = 2\sin x$
$\Rightarrow$ $2(1 - \sin^2 x)\sin x - \sin^2 x = \sin x$
$\Rightarrow$ $2\sin x - 2\sin^3 x - \sin^2 x - \sin x = 0$
$\Rightarrow$ $\sin x - 2\sin^3 x - \sin^2 x = 0$
$\Rightarrow$ $2\sin^3 x + \sin^2 x - \sin x = 0$
$\Rightarrow$ $\sin x(2\sin^2 x + \sin x - 1) = 0$
$\Rightarrow$ $\sin x = 0, (2\sin^2 x + \sin x - 1) = 0$

$$\Rightarrow \quad \sin x = 0, \sin x = \frac{-1 \pm 3}{2}$$

$\Rightarrow$ $\sin x = 0, \sin x = 1, \sin x = -2$
$\Rightarrow$ $\sin x = 0, \sin x = 1$

$$\Rightarrow \quad x = n\pi, x = (4n+1)\frac{\pi}{2}, n \in I$$

74. Given equation is
$\sin 3x + \cos 2x = 1$
$\Rightarrow$ $\sin 3x = 1 - \cos 2x$
$\Rightarrow$ $\sin x(3 - 4\sin^2 x) = 2\sin^2 x$
$\Rightarrow$ $\sin x(3 - 4\sin^2 x - 2\sin x) = 0$
$\Rightarrow$ $\sin x = 0, (4\sin^2 x + 2\sin x - 3) = 0$

$$\Rightarrow \quad \sin x = 0, \sin x = \frac{-2 \pm \sqrt{4+48}}{8}$$

$$\Rightarrow \quad \sin x = 0, \sin x = \frac{-1 \pm \sqrt{13}}{4}$$

$$\Rightarrow \quad \sin x = 0, \sin x = \frac{\sqrt{13}-1}{4}$$

$$\Rightarrow \quad x = n\pi, x = n\pi + (-1)^n\alpha, \alpha = \sin^{-1}\left(\frac{\sqrt{13}-1}{4}\right)$$

75. Given equation is

$$2\cos 2x + \sqrt{2\sin x} = 2$$

$$\Rightarrow \quad \sqrt{2}\sqrt{\sin x} = 2(1 - \cos 2x)$$

$$\Rightarrow \quad \sqrt{2}\sqrt{\sin x} = 4\sin^2 x$$

$$\Rightarrow \quad \sqrt{\sin x} = 2\sqrt{2}\sin^2 x$$

$$\Rightarrow \quad \sqrt{\sin x}\,(1 - 2\sqrt{2}\sin^{3/2} x) = 0$$

$\Rightarrow \quad \sqrt{\sin x} = 0, \sin^{3/2} x = \frac{1}{2\sqrt{2}}$

$\Rightarrow \quad \sin x = 0, \sin x = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = n\pi, n = n\pi + (-1)^n \left(\frac{\pi}{4}\right), n \in I$

76. Given equation is

$$1 + \sin^3 x + \cos^3 x = \frac{3}{2}\sin 2x$$

$\Rightarrow \quad 1 + (\sin^3 x + \cos^3 x) = 3 \sin x \cos x$

$\Rightarrow \quad 1 + (\sin^3 x + \cos^3 x) - 3 \sin x \cos x \cdot 1 = 0$

$\Rightarrow \quad (\sin x + \cos x + 1)(2 - \sin x \cos x - \sin x - \cos x) = 0$

$\Rightarrow \quad (\sin x + \cos x + 1) = 0$

$\Rightarrow \quad \sin x + \cos x = -1$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \left(x + \frac{\pi}{4}\right) = n\pi + (-1)^n \left(-\frac{\pi}{4}\right)$

$\Rightarrow \quad x = n\pi + (-1)^n \left(-\frac{\pi}{4}\right) - \frac{\pi}{4}, n \in I$

77. Given equation is

$$\sin^6 x + \cos^6 x = \frac{7}{16}$$

$\Rightarrow \quad 1 - 3\sin^2 x \cos^2 x = \frac{7}{16}$

$\Rightarrow \quad 3\sin^2 x \cos^2 x = 1 - \frac{7}{16} = \frac{9}{16}$

$\Rightarrow \quad \sin^2 x \cos^2 x = \frac{3}{16}$

$\Rightarrow \quad 4\sin^2 x \cos^2 x = \frac{3}{4}$

$\Rightarrow \quad \sin^2(2x) = \frac{3}{4}$

$\Rightarrow \quad (2x) = n\pi \pm \frac{\pi}{3}$

$\Rightarrow \quad x = \frac{n\pi}{3} \pm \frac{\pi}{6}, n \in I$

78. Given equation is

$$\sin^8 x + \cos^8 x = \frac{17}{16}\cos^2 2x$$

$\Rightarrow \quad (\sin^4 x + \cos^4 x)^2 - 2\sin^4 x \cos^4 x = \frac{17}{16}\cos^2 2x$

$\Rightarrow \quad (1 - 2\sin^2 x \cos^2 x)^2 - 2\sin^4 x \cos^4 x = \frac{17}{16}\cos^2 2x$

$\Rightarrow \quad (1 - 4\sin^2 x \cos^2 x) + 2\sin^4 x \cos^4 x = \frac{17}{16}\cos^2 2x$

$\Rightarrow \quad 16(1 - 4\sin^2 x \cos^2 x = 2\sin^4 x \cos^4 x)$

$\Rightarrow \quad 17(\cos^4 x + \sin^4 x - 2\sin^2 x \cos^2 x)$

$\Rightarrow \quad 17(1 - 4\sin^2 x \cos^2 x)$

$\Rightarrow \quad 32\,\text{sn}^4 x \cos^4 x + 4\sin^2 x \cos^2 x - 1 = 0$

$\Rightarrow \quad 2\sin^4(2x) + \sin^2 2x - 1 = 0$

$\Rightarrow \quad \sin^2(2x) = \frac{-1 \pm \sqrt{5}}{4}$

$\Rightarrow \quad \sin^2(2x) = \frac{\sqrt{5} - 1}{4}$

$\Rightarrow \quad 2x = n\pi \pm \alpha, \alpha = \sin^{-1}\left(\sqrt{\frac{\sqrt{5}-1}{4}}\right)$

$\Rightarrow \quad x = \frac{n\pi}{2} \pm \frac{\alpha}{2}, \alpha = \sin^{-1}\left(\sqrt{\frac{\sqrt{5}-1}{4}}\right)$

79. Given equation is

$2\sin^2 x + 2 = \cos^2 3x$

$\Rightarrow \quad 2\sin^3 x + 2 = 1 - \sin^2 3x$

$\Rightarrow \quad 2\sin^3 x + \sin^2 3x + 1 = 0$

$\Rightarrow \quad 2\sin^3 x + (3\sin x - 4\sin^3 x)^2 + 1 = 0$

$\Rightarrow \quad 2\sin^3 x + 9\sin^2 x - 24\sin^4 x + 16\sin^6 x + 1 = 0$

$\Rightarrow \quad 16\sin^6 x - 24\sin^4 x + 2\sin^3 x + 9\sin^2 x + 1 = 0$

$\Rightarrow \quad \sin x = -1$

$\Rightarrow \quad x = (4n - 1)\frac{\pi}{2}, n \in I$

80. Given equation is

$\cos 4x = \cos^2 3x$

$\Rightarrow \quad 2\cos^2 2x - 1 = \cos^2 3x$

$\Rightarrow \quad 2\cos^2 2x = 1 + \cos^2 3x$

It is possible only when

$\cos^2 2x = 1, \cos^2 3x = 1$

It is true for $x = 0$

Hence, the solution is $x = n\pi, n \in I$

81. Given equation is

$\cos 2x = 6\,\text{tran}^2 x - 2\cos^2 x$

$\Rightarrow \quad 2\cos^2 x - 1 = 6\left(\frac{\sin^2 x}{\cos^2 x}\right) - 2\cos^2 x$

$\Rightarrow \quad 2\cos^4 x - \cos^2 x = 6 - 6\cos^2 x - 2\cos^4 x$

$\Rightarrow \quad 4\cos^4 x +\_5\cos^2 x - 6 = 0$

$\Rightarrow \quad 4\cos^4 x + 8\cos^2 x - 3\cos^2 x - 6 = 0$

$\Rightarrow \quad 4\cos^2 x(\cos^2 x + 2) - 3(\cos^2 x + 2) = 0$

$\Rightarrow \quad (4\cos^2 x - 3)(\cos^2 x + 2) = 0$

$\Rightarrow \quad (4\cos^2 x - 3) = 0$

$\Rightarrow \quad \cos^2 x = \frac{3}{4}$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{6}, n \in I$

82. The given equation can be written as

$(2 \sin x - \cos x)(1 + \cos x)$

$= (1 - \cos x)(1 + \cos x)$

$\Rightarrow (1 + \cos x)(2 \sin x - \cos x - 1 + \cos x) = 0$

$\Rightarrow (1 + \cos x)(2 \sin x - 1) = 0$

$\Rightarrow \cos x = -1, \sin x = 1/2$

$\Rightarrow \cos x = -1 = \cos \pi,\ \sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$

$\Rightarrow x = 2n\pi \pm \pi,\ x = n\pi + (-1)^n \frac{\pi}{6}, n \in Z$

83. Given equation is

$2 \sin^2 x + \sin x - 1 = 0$

$\Rightarrow \sin x = \frac{-1 \pm 3}{4} = \frac{1}{2}, -1$

$\Rightarrow \sin x = \frac{1}{2}, \sin x = -1$

$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

84. Given equation is

$5 \sin^2 x + 7 \sin x - 6 = 0$

$\Rightarrow 5 \sin^2 x + 10 \sin x - 3 \sin x - 6 = 0$

$\Rightarrow 5 \sin x (\sin x + 2) - 3(\sin x + 2) = 0$

$\Rightarrow (5 \sin x - 3)(\sin x + 2) = 0$

$\Rightarrow (5 \sin x - 3) = 0, (\sin x + 2) = 0$

$\Rightarrow (5 \sin x - 3) = 0$

$\Rightarrow \sin x = \frac{3}{5}$

$\Rightarrow x = n\pi + (-1)^n \alpha, \alpha = \sin^{-1}\left(\frac{3}{5}\right)$

Hence, the solution is

$x = \sin^{-1}\left(\frac{3}{5}\right), \pi - \sin^{-1}\left(\frac{3}{5}\right)$

85. Given equation is

$\sin^2 x - \cos x = \frac{1}{4}$

$\Rightarrow 4 \sin^2 x - 4 \cos x - 1 = 0$

$\Rightarrow 4 - 4 \cos^2 x - 4 \cos x - 1 = 0$

$\Rightarrow 3 - 4 \cos^2 x - 4 \cos x = 0$

$\Rightarrow 4 \cos^2 x + 4 \cos x - 3 = 0$

$\Rightarrow \cos x = \frac{-4 \pm 8}{8} = \frac{1}{2}, -\frac{3}{2}$

$\Rightarrow \cos x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$

86. Given equation is

$\tan^2 x - 2 \tan x - 3 = 0$

$\Rightarrow (\tan x - 3)(\tan x + 1) = 0$

$\Rightarrow (\tan x - 3) = 0, (\tan x + 1) = 0$

$\Rightarrow (\tan x - 3) = 0, (\tan x + 1) = 0$

$\Rightarrow \tan x = -1, \tan x = 3$

$\Rightarrow x = n\pi - \frac{\pi}{4}, x = n\pi + \alpha, \alpha = \tan^{-1}(3)$

87. Given equation is

$2 \cos^2 x - \sqrt{3} \sin x + 1 = 0$

$\Rightarrow 2 - 2 \sin^2 x - \sqrt{3} \sin x + 1 = 0$

$\Rightarrow 3 - 2 \sin^2 x - \sqrt{3} \sin x = 0$

$\Rightarrow 2 \sin^2 x + \sqrt{3} \sin x - 3 = 0$

$\Rightarrow \sin x = \frac{-\sqrt{3} \pm \sqrt{27}}{4} = \frac{-\sqrt{3} \pm 3\sqrt{3}}{4}$

$\Rightarrow \sin x = \frac{-4\sqrt{3}}{4}, \frac{2\sqrt{3}}{4}$

$\Rightarrow \sin x = -\sqrt{3}, \frac{\sqrt{3}}{2}$

$\Rightarrow \sin x = \frac{\sqrt{3}}{2}$

$\Rightarrow x = n\pi + (-1)^n \left(\frac{\pi}{3}\right), n \in I$

88. The given equation can be written as

$(\sin x + \sin 5x) + \sin 3x = 0$

$\Rightarrow 2 \sin 3x \cdot \cos 2x + \sin 3x = 0$

$\Rightarrow \sin 3x(2 \cos 2x + 1) = 0$

$\Rightarrow \sin 3x = 0, \cos 2x = -1/2$

$\Rightarrow \sin 3x = 0,\ \cos 2x = \cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow 3x = n\pi,\ 2x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

$\Rightarrow x = \frac{n\pi}{3}, x = n\pi \pm \frac{\pi}{3}, n \in Z$

$\Rightarrow x = 0, \frac{\pi}{3}$

89. Given equation is

$\cos x - \cos 2x = \sin 3x$

$\Rightarrow 2 \sin\left(\frac{3x}{2}\right) \sin\left(\frac{x}{2}\right) = 2 \sin\left(\frac{3x}{2}\right) \cos\left(\frac{3x}{2}\right)$

$\Rightarrow 2 \sin\left(\frac{3x}{2}\right)\left(\sin\left(\frac{x}{2}\right) - \cos\left(\frac{3x}{2}\right)\right) = 0$

$\Rightarrow \sin\left(\frac{3x}{2}\right) = 0, \left(\sin\left(\frac{x}{2}\right) - \cos\left(\frac{3x}{2}\right)\right) = 0$

when $\sin\left(\frac{3x}{2}\right) = 0$

Then $\frac{3x}{2} = n\pi$

$\Rightarrow \quad x = \frac{2n\pi}{3}$

when $\left(\sin\left(\frac{x}{2}\right) - \cos\left(\frac{3x}{2}\right)\right) = 0$

$\Rightarrow \quad \left(\sin\left(\frac{x}{2}\right) = \cos\left(\frac{3x}{2}\right)\right)$

$\Rightarrow \quad \sin\left(\frac{x}{2}\right) = \cos\left(\frac{3x}{2}\right)$

$\Rightarrow \quad \cos\left(\frac{3x}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{x}{2}\right)$

$\Rightarrow \quad \left(\frac{3x}{2}\right) = 2n\pi \pm \left(\frac{\pi}{2} - \frac{x}{2}\right)$

Taking positive sign, we get,

$$2x = 2n\pi + \frac{\pi}{2}$$

$\Rightarrow \quad x = n\pi + \frac{\pi}{4}$

Taking negative sign, we get,

$$x = 2n\pi - \frac{\pi}{2}, n \in I$$

90. Given equation is

$\sin 7x + \sin 4x + \sin x = 0$

$\Rightarrow \quad (\sin 7x + \sin x) + \sin 4x = 0$

$\Rightarrow \quad 2 \sin 4x \cos 3x = \sin 4x = 0$

$\Rightarrow \quad \sin 4x(2 \cos 3x + 1) = 0$

$\Rightarrow \quad \sin 4x = 0, (2 \cos 3x + 1) = 0$

$\Rightarrow \quad \sin 4x = 0, \cos 3x = -\frac{1}{2}$

$\Rightarrow \quad 4x = n\pi, 3x = 2n\pi \pm \frac{2\pi}{3}$

$\Rightarrow \quad x = \frac{n\pi}{4}, x = \frac{2n\pi}{3} \pm \frac{2\pi}{9}, n \in I$

Hence, the solutions are

$$x = 0, \frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}$$

91. Given equation is

$$\cos 3x + \cos 2x = \sin\left(\frac{3x}{2}\right) + \sin\left(\frac{x}{2}\right)$$

$\Rightarrow \quad 2\cos\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right) = 2 \sin x \cos\left(\frac{x}{2}\right)$

$\Rightarrow \quad 2\left(\cos\left(\frac{5x}{2}\right) - \sin x\right)\cos\left(\frac{x}{2}\right) = 0$

$\Rightarrow \quad 2\left(\cos\left(\frac{5x}{2}\right) - \sin x\right) = 0, \cos\left(\frac{x}{2}\right) = 0$

$\Rightarrow \quad \cos\left(\frac{5x}{2}\right) = \sin x, \cos\left(\frac{x}{2}\right) = 0$

$\Rightarrow \quad \cos\left(\frac{5x}{2}\right) = \cos\left(\frac{\pi}{2} - x\right), \cos\left(\frac{x}{2}\right) = 0$

$\Rightarrow \quad \left(\frac{5x}{2}\right) = 2n\pi \pm \left(\frac{\pi}{2} - x\right), \left(\frac{x}{2}\right) = (2n+1)\frac{\pi}{2}$

$\Rightarrow \quad x = \frac{4n\pi}{5} \pm (\pi - 2x), x = (2n+1)\pi$

$\Rightarrow \quad x = \frac{4n\pi}{5} \pm (\pi - 2x), x = (2n+1)\pi$

$\Rightarrow \quad x = \frac{\pi}{3}, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{13\pi}{15}, \frac{17\pi}{15}, \frac{7\pi}{5}, \frac{5\pi}{3}, \frac{29\pi}{15}$

92. Do yourself.

93. Given equation is

$\cos 2x + \cos 4x = 2 \cos x$

$\Rightarrow \quad 2 \cos 3x \cos x = 2 \cos x$

$\Rightarrow \quad (2 \cos 3x -1) \cos x = 0$

$\Rightarrow \quad (2 \cos 3x - 1) = 0, \cos x = 0$

$\Rightarrow \quad \cos 3x = \frac{1}{2}, \cos x = 0$

$\Rightarrow \quad x = \frac{2n\pi}{3} \pm \frac{2\pi}{9}, x = (2n+1)\frac{\pi}{2}$

94. Given equation is

$\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$

$\Rightarrow \quad (1 + \sin 2x) + (\sin x + \cos x) + \cos 2x = 0$

$\Rightarrow \quad (\sin x + \cos x)^2 + (\sin x + \cos x) + (\cos^2 x - \sin^2 x) = 0$

$\Rightarrow \quad (\sin x + \cos x)(2 \cos x + 1) = 0$

$\Rightarrow \quad (\sin x + \cos x) = 0, (2 \cos x + 1) = 0$

$\Rightarrow \quad \tan x = -1, \cos x = -\frac{1}{2}$

$\Rightarrow \quad x = n\pi - \frac{\pi}{4}, x = 2n\pi \pm \frac{2\pi}{3}, n \in I$

95. Do yourself.

96. Given equation is

$\tan 3x + \tan x = 2 \tan 2x$

$\Rightarrow \quad \frac{\sin 4x}{\cos 3x \cos x} = \frac{2 \sin 2x}{\cos 2x}$

$\Rightarrow \quad \frac{2 \sin 2x \cos 2x}{\cos 3x \cos x} = \frac{2 \sin 2x}{\cos 2x}$

$\Rightarrow \quad 2 \sin 2x\left(\frac{\cos 2x}{\cos 3x \cos x} - \frac{1}{\cos 2x}\right) = 0$

$\Rightarrow \quad 2 \sin 2x = 0, \left(\frac{\cos 2x}{\cos 3x \cos x} = \frac{1}{\cos 2x}\right)$

$\Rightarrow \quad \sin 2x = 0, 2 \cos^2 2x = \cos 4x + \cos 2x$

$\Rightarrow \quad \sin 2x = 0, 2 \cos^2 2x = 2 \cos^2 2x - 1 + \cos 2x$

$\Rightarrow$ $\sin 2x = 0, \cos 2x = 1$

$\Rightarrow$ $2x = n\pi, 2x = 2n\pi$

$\Rightarrow$ $x = \frac{n\pi}{2}, x = n\pi, n \in I$

97. Given equation is

$(1 - \tan x)(1 + \sin 2x) = (1 + \tan x)$

$\Rightarrow$ $(1 - \tan x)\left(1 + \frac{2 \tan x}{1 + \tan^2 x}\right) = (1 + \tan x)$

$\Rightarrow$ $(1 - \tan x)(1 + \tan x)^2 = (1 + \tan x)(1 + \tan^2 x)$

$\Rightarrow$ $(1 - \tan^2 x)(1 + \tan x) = (1 + \tan x)(1 + \tan^2 x)$

$\Rightarrow$ $((1 - \tan^2 x) - (1 + \tan^2 x))(1 + \tan x) = 0$

$\Rightarrow$ $\tan^2 x(1 + \tan x) = 0$

$\Rightarrow$ $\tan^2 x = 0, (1 + \tan x) = 0$

$\Rightarrow$ $\tan^2 x = 0, \tan x = 4$

$\Rightarrow$ $x = n\pi, x = n\pi - \frac{\pi}{4}, n \in I$

98. Given equation is

$\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$

$\Rightarrow$ $(\sin 3x + \sin x) - 3 \sin 2x = (\cos 3x + \cos x) - 3 \cos 2x$

$\Rightarrow$ $2 \sin 2x \cos x - 3 \sin 2x = 2 \cos 2x \cos x - 3 \cos 2x$

$\Rightarrow$ $\sin 2x (2 \cos x - 3) = (2 \cos x - 3) \cos 2x$

$\Rightarrow$ $\frac{\sin 2x}{\cos 2x}(2 \cos x - 3) = (2 \cos x - 3)$

$\Rightarrow$ $\frac{\sin 2x}{\cos 2x} = 1$

$\Rightarrow$ $\tan 2x = 1$

$\Rightarrow$ $2x = n\pi + \frac{\pi}{4}$

$\Rightarrow$ $x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in I$

99. The given equation can be written as

$(2 \sin 2x \cdot \sin x)\, 2 \sin 4x - \sin 3x = 0$

$\Rightarrow$ $2(\cos x - \cos 3x) \sin 4x - \sin 3x = 0$

$\Rightarrow$ $2 \sin 4x \cos x - 2\sin 4x \cos 3x - \sin 3x = 0$

$\Rightarrow$ $(\sin 5x + \sin 3x) - (\sin 7x + \sin x) - \sin 3x = 0$

$\Rightarrow$ $(\sin 7x - \sin 5x) + \sin x = 0$

$\Rightarrow$ $\sin x (2 \cos 6x + 1) = 0$

$\Rightarrow$ $\sin x = 0, \cos 6x = -1/2,$

$\Rightarrow$ $\sin x = 0, \ \cos 6x = \cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow$ $x = n\pi, 6x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

$\Rightarrow$ $x = n\pi, x = (3n \pm 1)\frac{\pi}{9}, n \in Z$

100. Do yourself.

101. Do yourself.

102. Given equation is

$\sin 4x \sin 2x = \cos 6x - \cos 2x$

$\Rightarrow$ $\sin 4x \sin 2x = -2 \sin 4x \sin 2x$

$\Rightarrow$ $3 \sin 4x \sin 2x = 0$

$\Rightarrow$ $\sin 4x = 0, \sin 2x = 0$

$\Rightarrow$ $4x = n\pi, 2x = n\pi, n \in I$

$\Rightarrow$ $x = \frac{n\pi}{4}, x = \frac{n\pi}{2}, n \in I$

103. Given equation is

$\sec x \cos 5x + 1 = 0$

$\Rightarrow$ $\cos 5x + \cos x = 0$

$\Rightarrow$ $2 \cos (3x) \cos (2x) = 0$

$\Rightarrow$ $2 \cos (3x) = 0, \cos (2x) = 0$

$\Rightarrow$ $\cos (3x) = 0, \cos (2x) = 0$

$\Rightarrow$ $3x = (2n + 1)\frac{\pi}{2}, 2x = (2n + 1)\frac{\pi}{2}, n \in I$

$\Rightarrow$ $x = (2n + 1)\frac{\pi}{6}, x = (2n + 1)\frac{\pi}{4}, n \in I$

Hence, the solutions are

$x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$

105. Given equation is

$\cos (6x) \cos x = 1$

$\Rightarrow$ $2 \cos (6x) \cos x = -2$

$\Rightarrow$ $\cos 7x + \cos 5x = -2$

It is possible only when

$\cos (7x) = -1, \cos (5x) = -1$

$\Rightarrow$ $x = (2n + 1)\frac{\pi}{7}, x = (2n + 1)\frac{\pi}{5}, n \in I$

106. The given equation can be written as

$2 \sin^2 x - 5 \sin x \cos x - 8 \cos^2 x$

$= -2(\sin^2 x + \cos^2 x)$

$\Rightarrow$ $2 \tan^2 x - 5 \tan x - 8 = -(\tan^2 x + 1)$

$\Rightarrow$ $4 \tan^2 x - 5 \tan x - 6 = 0$

$\Rightarrow$ $(\tan x - 2)(4 \tan x + 3) = 0$

$\Rightarrow$ $\tan x = -2, \tan x = -\frac{3}{4}$

$\Rightarrow$ $x = n\pi + \alpha, x = n\pi + \beta$, where

$\Rightarrow$ $\alpha = \tan^{-1}(2), \beta = \tan^{-1}\left(-\frac{3}{4}\right), n \in Z.$

107. Given equation is

$5 \sin^2 x - 7 \sin x \cos x + 10 \cos^2 x = 4$

$\Rightarrow$ $5 \tan^2 x - 7 \tan x + 10 = 4 \sec^2 x$

$\Rightarrow$ $5 \tan^2 x - 7 \tan x + 10 = 4 + 4 \tan^2 x$

$\Rightarrow$ $\tan^2 x - 7 \tan x + 6 = 0$

$\Rightarrow$ $(\tan x - 1)(\tan x - 6) = 0$

$\Rightarrow$ $\tan x = 1, 6$

$\Rightarrow$ $x = n\pi + \frac{\pi}{4}, x = n\pi + \alpha, \alpha = \tan^{-1}(5)$

108. Given equation is

$$2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -3$$
$$\Rightarrow 2\tan^2 x - 5\tan x - 8 = -3\sec^2 x$$
$$\Rightarrow 2\tan^2 x - 5\tan x - 8 = -3 - 3\tan^2 x$$
$$\Rightarrow 5\tan^2 x - 5\tan x - 5 = 0$$
$$\Rightarrow \tan^2 x - \tan x - 1 = 0$$
$$\Rightarrow \tan x = \frac{1 \pm \sqrt{5}}{2}$$
$$\Rightarrow x = n\pi + \alpha, \alpha = \tan^{-1}\left(\frac{1 \pm \sqrt{5}}{2}\right)$$

109. Given equation is

$$\sin^2 x\cos x + \sin^2 x\cos^2 x + \sin x\cos^3 x = 1$$
$$\Rightarrow \sin x\cos x[\sin^2 x + \sin x\cos x + \cos^2 x] = 1$$
$$\Rightarrow \sin x\cos x[1 + \sin x\cos x] = 1$$
$$\Rightarrow 2\sin x\cos x[2 + 2\sin x\cos x] = 4$$
$$\Rightarrow \sin(2x)(2 + \sin(2x)) = 4$$
$$\Rightarrow \sin^2(2x) + 2\sin(2x) - 4 = 0$$
$$\Rightarrow \sin(2x) = \frac{-2 \pm \sqrt{20}}{2}$$
$$\Rightarrow \sin(2x) = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

It is not possible.
So, it has no solution.

110. The given equation can be written as

$$1 + \frac{2}{\sin x} = -\frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right)$$
$$\Rightarrow 2(\sin x + 2) = -\left(1 + \tan^2\left(\frac{x}{2}\right)\right)\sin x$$
$$\Rightarrow 2\left(\frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} + 2\right) = -\left(1 + \tan^2\frac{x}{2}\right)\cdot\left(\frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right)$$
$$\Rightarrow 2\left(\frac{2t}{1+t^2} + 2\right) = -(1+t^2)\times\left(\frac{2t}{1+t^2}\right),$$

where $t = \tan(x/2)$

$$\Rightarrow t^3 + 2t^2 + 3t + 2 = 0$$
$$\Rightarrow t^3 + t^2 + t^2 + t + 2t + 2 = 0$$
$$\Rightarrow (t+1)(t^2 + t + 2) = 0$$
$$\Rightarrow t + 1 = 0, t^2 + t + 2 \neq 0$$
$$\Rightarrow \tan\left(\frac{x}{2}\right) = -1 = \tan\left(\frac{-\pi}{4}\right)$$
$$\Rightarrow \frac{x}{2} = n\pi - \frac{\pi}{4}, n \in Z$$
$$\Rightarrow x = 2n\pi - \frac{\pi}{2}, n \in Z.$$

111. Given equation is

$$(\cos x - \sin x)(2\tan x + \sec x) + 2 = 0$$
$$\Rightarrow (\cos x - \sin x)(2\sin x + 1) + 2\cos x = 0$$
$$\Rightarrow \left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} - \frac{2\tan(x/2)}{1+\tan^2(x/2)}\right)\left(\frac{4\tan(x/2)}{1+\tan^2(x/2)} + 1\right) + 2\left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right) = 0$$

Put $\tan\left(\frac{x}{2}\right) = t$ and then solve it.

112. Given equation is

$$\frac{\sin^3\frac{x}{2} - \cos^3\frac{x}{2}}{2 + \sin x} = \frac{\cos x}{3}$$
$$\Rightarrow \frac{\left(\sin\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right)\right)\left(1 + \frac{\sin x}{2}\right)}{2 + \sin x} = \frac{\left(\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right)}{3}$$
$$\Rightarrow -\frac{3}{2} = \left(\cos\left(\frac{x}{2}\right) + \sin\frac{x}{2}\right)$$

It is not possible.
So, it has no solution.

113. Given equation is

$$\Rightarrow \cot\left(\frac{x}{2}\right) - \operatorname{cosec}\left(\frac{x}{2}\right) = \cot x$$
$$\frac{\cos\left(\frac{x}{2}\right) - 1}{\sin\left(\frac{x}{2}\right)} = \cot x$$
$$\Rightarrow 2\sin^2\left(\frac{x}{2}\right) = -\sin\left(\frac{x}{2}\right)\cot x$$
$$\Rightarrow \left(2\sin\left(\frac{x}{2}\right) + \cot x\right)\sin\left(\frac{x}{2}\right) = 0$$
$$\Rightarrow \left(2\sin\left(\frac{x}{2}\right) + \cot x\right) = 0, \sin\left(\frac{x}{2}\right) = 0$$

when $\sin\left(\frac{x}{2}\right) = 0$

$$\Rightarrow x = 2n\pi, n \in 1$$

when $\left(2\sin\left(\frac{x}{2}\right) + \cot x\right) = 0$

$$\Rightarrow 2\sin\left(\frac{x}{2}\right) = -\frac{\cos x}{\sin x}$$
$$\Rightarrow 2\sin\left(\frac{x}{2}\right) = -\frac{\left(\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$\Rightarrow \quad 4\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)=0$$

$$\Rightarrow \quad \cos^2\left(\frac{x}{2}\right)+4\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)=0$$

For all real $x$,

$$16\sin^4\left(\frac{x}{2}\right)+4\sin^2\left(\frac{x}{2}\right)\geq 0$$

$$\Rightarrow \quad 4\sin^4\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)=0$$

$$\Rightarrow \quad \sin^2\left(\frac{x}{2}\right)\left(4\sin^2\left(\frac{x}{2}\right)+1\right)=0$$

$$\Rightarrow \quad \sin^2\left(\frac{x}{2}\right)=0, \left(4\sin^2\left(\frac{x}{2}\right)+1\right)=0$$

$$\Rightarrow \quad \sin^2\left(\frac{x}{2}\right)=0$$

$$\Rightarrow \quad \sin\left(\frac{x}{2}\right)=0$$

$\Rightarrow \quad x = 2n\pi, n \in I$

114. Given equation is

$\sin(\theta + \alpha) = k\sin(2\theta)$

$\Rightarrow \quad \sin\theta\cos\alpha + \cos\theta\sin\alpha = k\sin(2\theta)$

$$\Rightarrow \quad \left(\frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)\cos\alpha+\left(\frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)\sin\alpha$$

$$=2k\left(\frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)\left(\frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)$$

$$\Rightarrow \quad \left(\frac{2t}{1+t^2}\right)\cos\alpha+\left(\frac{1-t^2}{1+t^2}\right)\sin\alpha$$

$$=2k\cdot\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)$$

$\Rightarrow \quad 2t(1+t^2)\cos\alpha + (1+t^4)\sin\alpha = 4k\,t(1-t^2)$

$\Rightarrow \quad (\sin\alpha)t^4 - (4k + 2\cos\alpha)t^3$

$+ (4k - 2\cos\alpha)t = \sin\alpha = 0$

Let $t_1$, $t_2$, $t_3$ and $t_4$ be its four roots

$$\Sigma t_1 = \frac{4k+2\cos\alpha}{\sin\alpha} = s_1$$

$$\Sigma t_1t_2 = 0 = s_2$$

$$\Sigma t_1t_2t_3 = \frac{2\cos\alpha - 4k}{\sin\alpha} = s_3$$

$$\Sigma t_1t_2t_3t_4 = -\frac{\sin\alpha}{\sin\alpha} = -1 = s_4$$

Now, $\tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)=\frac{s_1-s_3}{1-s_2+s_4}$

$$\Rightarrow \quad \tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)=\infty$$

$$\Rightarrow \quad \tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)=\tan\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)=n\pi+\left(\frac{\pi}{2}\right)$$

$\Rightarrow \quad (\theta_1+\theta_2+\theta_3+\theta_4) = (2n+1)\pi, n \in I$

115. Let $\sin x + \cos x = t$

$$\Rightarrow \quad \sin x\cdot\cos x = \frac{t^2-1}{2}$$

So, the given equation can be reduces to

$$t - 2\sqrt{2}\left(\frac{t^2-1}{2}\right)=0$$

$$\Rightarrow \quad \sqrt{2}t^2 - t - \sqrt{2} = 0$$

$$\Rightarrow \quad (\sqrt{2}t+1)(t-\sqrt{2})=0$$

$$\Rightarrow \quad t=\sqrt{2}, -\frac{1}{\sqrt{2}}$$

When $\sin x + \cos x = \sqrt{2}$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1$$

$$\Rightarrow \quad \sin\left(x+\frac{\pi}{4}\right)=1=\sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad x+\frac{n\pi}{4}=n\pi+(-1)^n\frac{\pi}{2}, n\in Z$$

When $\sin x + \cos x = -\frac{1}{\sqrt{2}}$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{2}$$

$$\Rightarrow \quad \sin\left(x+\frac{\pi}{4}\right)=\sin\left(-\frac{\pi}{6}\right)$$

$$\Rightarrow \quad \left(x+\frac{\pi}{4}\right)=n\pi+(-1)^n\left(-\frac{\pi}{6}\right)$$

$$\Rightarrow \quad x=n\pi-(-1)^n\frac{\pi}{6}-\frac{\pi}{4}, n\in Z$$

117. Given equation is

$\sin x + \cos x = 1 - \sin x\cos x$ …(i)

Put $\sin x + \cos x = t$

$$\Rightarrow \quad \sin x\cdot\cos x = \frac{t^2-1}{2}$$

Now, equation (i) becomes

$$t = 1 - \frac{t^2 - 1}{2}$$

$\Rightarrow$ $2t = 2 - t^2 + 1$

$\Rightarrow$ $t^2 + 2t - 3 = 0$

$\Rightarrow$ $(t + 3)(t - 1) = 0$

$\Rightarrow$ $(t + 3) = 0, (t - 1) = 0$

$\Rightarrow$ $\sin x + \cos x = 1, \sin x + \cos x = -3$

$\Rightarrow$ $\sin x + \cos x = 1$

$\Rightarrow$ $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $x = n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{4}, n \in I$

118. Given equation is

$$1 + \sin^3 x + \cos^3 x = \frac{3}{2}\sin 2x \quad \ldots\text{(i)}$$

$\Rightarrow$ $1 + (\sin x + \cos x)(1 - \sin x \cos x) = 3 \sin x \cos x$

Put $\sin x + \cos x = t$

$\Rightarrow$ $\sin x \, . \cos x = \frac{t^2 - 1}{2}$

Now, equation (i) becomes

$$1 + t\left(1 - \frac{t^2 - 1}{2}\right) = \frac{3}{2}(t^2 - 1)$$

$\Rightarrow$ $2 + t(3 - t^2) = 3(t^2 - 1)$

$\Rightarrow$ $2 + 3t - t^3 - 3t^2 + 3 = 0$

$\Rightarrow$ $3t - t^3 - 3t^2 + 5 = 0$

$\Rightarrow$ $t^3 + 3t^2 - 3t - 5 = 0$

$\Rightarrow$ $t^3 + t^2 + 2t^2 + 2t - 5t - 5 = 0$

$\Rightarrow$ $t^2(t + 1) + 2t(t + 1) - 5(t + 1) = 0$

$\Rightarrow$ $(t^2 + 2t - 5)(t + 1) = 0$

$\Rightarrow$ $t = -1, t = \frac{-2 \pm \sqrt{24}}{2}$

$\Rightarrow$ $\sin x + \cos x = -1, \frac{-2 \pm \sqrt{24}}{2}$

$\Rightarrow$ $\sin x + \cos x = -1$

$\Rightarrow$ $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{\sqrt{2}}$

$\Rightarrow$ $\sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\Rightarrow$ $x = n\pi + (-1)^n\left(-\frac{\pi}{4}\right) - \frac{\pi}{4}, n \in I$

119. Given equation is

$\sin 2x - 12(\sin x - \cos x) + 12 = 0$

$2 \sin x \cos x - 12(\sin x - \cos x) + 12 = 0$

$\sin x \cos x - 6(\sin x - \cos x) + 6 = 0 \quad \ldots\text{(i)}$

Put $\sin x + \cos x = t$

$\Rightarrow$ $\sin x \cdot \cos x = \frac{t^2 - 1}{2}$

Now, equation (i) becomes

$\Rightarrow$ $\frac{t^2 - 1}{2} - 6t + 6 = 0$

$\Rightarrow$ $t^2 - 1 - 12t + 12 = 0$

$\Rightarrow$ $t^2 - 12t + 1 = 0$

$\Rightarrow$ $(t - 1)(t - 1) = 0$

$\Rightarrow$ $t = 1, 1$

$\Rightarrow$ $\sin x + \cos x = 1$

$\Rightarrow$ $\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $x = n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{\pi}{4}, n \in I$

Hence, the solution are

$$x = 0, \frac{\pi}{2}, 2\pi$$

120. The given equation can be written as

$\sin 6x + \cos 4x = -2$

$\Rightarrow$ $\sin 6x = -1$ and $\cos 4x = -1$

$\Rightarrow$ $\sin 6x = \sin\frac{3\pi}{2}, \cos 4x = \cos \pi$

$\Rightarrow$ $6x = 2n\pi + \frac{3\pi}{2}, 4x = 2n\pi + \pi, n \in Z$

$\Rightarrow$ $x = \frac{n\pi}{3} + \frac{\pi}{4}, x = \frac{n\pi}{2} + \frac{\pi}{4}, n \in Z$

$\Rightarrow$ $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \ldots$

$\ldots x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$

$\Rightarrow$ $x = \frac{\pi}{4}, \frac{5\pi}{4}$

Hence, the general solution will be,

$\Rightarrow$ $x = 2n\pi + \frac{\pi}{4}, 2n\pi + \frac{5\pi}{4}, n \in Z$

$\Rightarrow$ $x = 2n\pi + \frac{\pi}{4}, (2n + 1)\pi + \frac{\pi}{4}, n \in Z$

$\Rightarrow$ $x = m\pi + \frac{\pi}{4}, m \in Z$

122. Given equation is

$\sin^4 x = 1 + \tan^8 x$

It is possible only when

$\sin^4 x = 1, \tan^8 x = 0$

$\Rightarrow$ $\sin^2 x = 1, \tan x = 0$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{2}, x = n\pi, n \in I$

There is no common value which satisfies both the above equations.

Hence, the equation has no solution.

123. Given $\sin^2 x + \cos^2 y = 2\sec^2 z$

Here, LHS $\leq 2$ and RHS $\geq 2$

It is possible only when

$\sin^2 x = 1, \cos^2 y = 1, \sec^2 z = 1$

$\Rightarrow \quad \cos^2 x = 0, \sin^2 y = 0, \cos^2 z = 1$

$\Rightarrow \quad \cos^2 x = 0, \sin^2 y = 0, \sin^2 z = 1$

$\Rightarrow \quad \cos x = 0, \sin y = 0, \sin z = 0$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{2}, y = m\pi, z = k\pi$

where, $n, m, k \in I$

124. Given equation is

$\sin 3x + \cos 2x + 2 = 0$

It is possible only when

$\sin 3x = -1, \cos 2x = -1$

$\Rightarrow \quad 3x = \frac{3\pi}{2}, 2x = \pi$

$\Rightarrow \quad x = \frac{\pi}{2}, x = \frac{\pi}{2}$

Hence, the general solution is

$$x = 2n\pi + \frac{\pi}{2}, n \in I$$

125. Given equation is $\cos 4x + \sin 5x = 2$

It is possible only when

$\cos 4x = 1, \sin 5x = 1$

$\Rightarrow \quad 4x = 2n\pi, 5x = (4n+1)\frac{\pi}{2}$

$\Rightarrow \quad x = \frac{2n\pi}{4}, x = (4n+1)\frac{\pi}{10}$

Thus, $x = \frac{\pi}{2}$ satisfies both

Hence, the solution is

$$x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}, n \in I$$

126. The given equation can be written as

$$8^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cos^4 x+|\cos x|^5+\cdots \text{to}\,\infty} = 8^2$$

$\Rightarrow \quad 1 + |\cos x| + \cos^2 x + |\cos x|^3 + \cos^4 x + |\cos x|^5 + \ldots$ to $\infty = 2$

$\Rightarrow \quad \frac{1}{1-|\cos x|} = 2$

$\Rightarrow \quad \cos x = \pm\frac{1}{2}$

When $\cos x = \frac{1}{2}$

$\Rightarrow \quad \cos x = \cos\left(\frac{\pi}{3}\right)$

$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{3}, n \in Z$

When $\cos x = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow \quad x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

Hence the values of $x$ are $\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$.

127. Given equation is

$$2^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cos^4 x+|\cos x|^5+\cdots \text{to}\,\infty} = 4$$

$\Rightarrow \quad 2^{\frac{1}{1-|\cos x|}} = 4 = 2^2$

$\Rightarrow \quad \frac{1}{1-|\cos x|} = 2$

$\Rightarrow \quad 1 - |\cos x| = \frac{1}{2}$

$\Rightarrow \quad |\cos x| = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \quad \cos x = \pm\frac{1}{2}$

Hence the values of $x$ are $\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$.

128. Given equation is

$1 + \sin\theta + \sin^2\theta + \cdots = 4 + 2\sqrt{3}$

$\Rightarrow \quad \frac{1}{1-\sin\theta} = 4 + 2\sqrt{3}$

$\Rightarrow \quad 1 - \sin\theta = \frac{1}{4+2\sqrt{3}}$

$\Rightarrow \quad \sin\theta = 1 - \frac{1}{4+2\sqrt{3}}$

$\Rightarrow \quad \sin\theta = 1 - \frac{1}{4+2\sqrt{3}} = 1 - \frac{2-\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta = n\pi + (-1)^n\left(\frac{\pi}{3}\right), n \in I$

129. Given equation is

$|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$

$\Rightarrow \quad \left(\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}\right)\log|\cos x| = 0$

$\Rightarrow \quad (2\sin^2 x - 3\sin x + 1)\log|\cos x| = 0$

$\Rightarrow \quad (\sin x - 1)(2\sin x - 1)\log|\cos x| = 0$

$\Rightarrow$ $(\sin x - 1) = 0, (2 \sin x - 1) = 0, \log |\cos x| = 0$

$\Rightarrow$ $\sin x = 1, \sin x = \frac{1}{2}, \log|\cos x| = 0$

$\Rightarrow$ $\sin x = \frac{1}{2}, |\cos x| = 1$

$\Rightarrow$ $\sin x = \frac{1}{2}, \cos = 1, \cos x = -1$

$\Rightarrow$ $x = n\pi + (-1)^n\left(\frac{\pi}{6}\right), x = 2n\pi, x = (2n+1)\pi$

130. Given equation is

$e^{\sin x} - e^{-\sin x} - 4 = 0$

$\Rightarrow$ $t - \frac{1}{t} - 4 = 0, t = e^{\sin x}$

$\Rightarrow$ $t^2 - 4t - 1 = 0$

$\Rightarrow$ $(t-2)^2 = 5$

$\Rightarrow$ $t = 2 \pm \sqrt{5}$

$\Rightarrow$ $e^{\sin x} = 2 \pm \sqrt{5}$

$\Rightarrow$ $\sin x = \log_e(2 \pm \sqrt{5})$

$\Rightarrow$ $\sin x = \log_e(2 + \sqrt{5})$

$\Rightarrow$ $\sin x = \log_e(2 + \sqrt{5}) > 1$

It is not possible

So, it has no solution.

131. We have

$e^{[\sin^2 x + \sin^4 x + \sin^6 x + \cdots \text{to} \infty]\log_e 2}$

$= e^{\left(\frac{\sin^2 x}{1-\cos^2 x}\right)\log_e 2} = e^{\tan^2 x \log_e 2} = 2^{\tan^2 x}$

Let $a = 2^{\tan^2 x}$

Thus, $a^2 - 9a + 8 = 0$

$\Rightarrow$ $(a-1)(a-8) = 0$

$\Rightarrow$ $a = 1, 8$

when $a = 1$, then $2^{\tan^2 x} = 1 = 2^0$

$\Rightarrow$ $2^{\tan^2 x} = 1 = 2^0$

$\Rightarrow$ $\tan^2 x = 0$

$\Rightarrow$ $x = n\pi, n \in I$

when $a = 8$, then $2^{\tan^2 x} = 8 = 2^3$

$\Rightarrow$ $\tan^2 x = 3$

$\Rightarrow$ $\tan x = \sqrt{3}$

Now, $\frac{\cos x}{\cos x + \sin x}$

$= \frac{1}{1 + \tan x} = \frac{1}{\sqrt{3}+1}$

$= \frac{(\sqrt{3}-1)}{2}$

132. Given equation is

$\log_{\cos x} \tan x + \log_{\sin x} \cot x = 0$

$\Rightarrow$ $\log_{\cos x}\left(\frac{\sin x}{\cos x}\right) + \log_{\sin x}\left(\frac{\cos x}{\sin x}\right) = 0$

$\Rightarrow$ $\log_{\cos x}(\sin x) + \log_{\sin x}(\cos x) = 2$

It is possible only when

$\sin x = \cos x$

$\Rightarrow$ $\tan x = 1$

$\Rightarrow$ $x = \frac{\pi}{4}$

133. Given equation is

$3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2\sin^2 x} = 28$

$\Rightarrow$ $3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2 - 2\cos^2 x} = 28$

$\Rightarrow$ $3^{\sin 2x + 2\cos^2 x} + \frac{3^3}{3^{\sin 2x + 2\cos^2 x}} = 28$

$\Rightarrow$ $a + \frac{27}{a} = 28, a = 3^{\sin 2x + 2\cos^2 x}$

$\Rightarrow$ $a^2 - 28a + 27 = 0$

$\Rightarrow$ $(a-27)(a-1) = 0$

$\Rightarrow$ $a = 27, 1$

when $a = 27$, then $3^{\sin 2x + 2\cos^2 x} = 3^3$

$\Rightarrow$ $\sin 2x + 2\cos^2 x = 3$

$\Rightarrow$ $\sin 2x + 1 + \cos 2x = 3$

$\Rightarrow$ $\sin 2x + \cos 2x = 2$

It is not possible.

when $a = 1$, then $3^{\sin 2x + 2\cos^2 x} = 3^0$

$\Rightarrow$ $\sin 2x + 2\cos^2 x = 0$

$\Rightarrow$ $\sin 2x + 1 + \cos 2x = 0$

$\Rightarrow$ $\sin 2x + \cos 2x = -1$

$\Rightarrow$ $\frac{1}{\sqrt{2}}\sin 2x + \frac{1}{\sqrt{2}}\cos 2x = -\frac{1}{\sqrt{2}}$

$\Rightarrow$ $\sin\left(2x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\Rightarrow$ $\left(2x + \frac{\pi}{4}\right) = n\pi + (-1)^n\left(-\frac{\pi}{4}\right)$

$\Rightarrow$ $x = \frac{n\pi}{2} + (-1)^n\left(-\frac{\pi}{8}\right) - \frac{\pi}{8}, n \in I$

134. Do yourself.

135. Given equation is

$2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$

Here, LHS < 2 for $0 < x < \frac{\pi}{2}$

and RHS $\geq 2$

So, it has no solutions.

136. Given equation is

$$2\cos^2\left(\frac{x^2+x}{6}\right) = 2^x + 2^{-x}$$

It is possible only when $x = 0$
Hence, the solution is $x = 0$.

## Level III

1. The given equation is

$$\sec x - \operatorname{cosec} x = \frac{4}{3}$$

$\Rightarrow \quad \frac{1}{\cos x} - \frac{1}{\sin x} = \frac{4}{3}$

$\Rightarrow \quad 3(\sin x - \cos x) = 4\sin x \cos x \qquad \ldots$(i)

Put $(\sin x - \cos x)$

$\Rightarrow \quad 1 - 2\sin x\cos x = t^2$

$\Rightarrow \quad \sin x\cos x = \frac{1-t^2}{2}$

(i) reduces to $3t = 4\left(\frac{1-t^2}{2}\right)$

$\Rightarrow \quad 3t = 2(1 - t^2)$

$\Rightarrow \quad 2t^2 + 3t - 2 = 0$

$\Rightarrow \quad 2t^2 + 4t - t - 2 = 0$

$\Rightarrow \quad 2t(t+2) - (t+2) = 0$

$\Rightarrow \quad (1+2)(2t-1) = 0$

$\Rightarrow \quad t = \frac{1}{2}, -2$

when $t = \frac{1}{2}$

$\Rightarrow \quad \sin x - \cos x = \frac{1}{2}$

$\Rightarrow \quad \left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right) = \frac{1}{2\sqrt{2}}$

$\Rightarrow \quad \sin\left(x - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

$\Rightarrow \quad \left(x - \frac{\pi}{4}\right) = n\pi + (-1)^n \cdot \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

$\Rightarrow \quad x = \left(\frac{n\pi}{4} + n\pi + (-1)^n \cdot \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)\right), n \in I$

when $t = -2$

$\Rightarrow \quad \sin x - \cos x = 2$

It is impossible, since the maximum value of $(\sin x - \cos x)$ is $\sqrt{2}$.

2. The given equation is

$\sin 2x + 12 = 12(\sin x - \cos x)$

Put $(\sin x - \cos x) = t$

$\Rightarrow \quad 1 - 2\sin x\cos x = t^2$

$\Rightarrow \quad 2\sin x\cos x = (1 - t^2)$

(i) reduces to $(1 - t^2) + 12 = 12t$

$\Rightarrow \quad (1 - t^2) = 12(t - 1)$

$\Rightarrow \quad (t+1)(t-1) = -12(t-1)$

$\Rightarrow \quad (t-1)(t+1+12) = 0$

$\Rightarrow \quad (t-1)(t+13) = 0$

$\Rightarrow \quad t = 1, -13$

when $t = 1$, then $\sin x - \cos x = 1$

$\Rightarrow \quad \left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin\left(x - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad \left(x - \frac{\pi}{4}\right) = n\pi + (-1)^n\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad x = n\pi + \frac{\pi}{4} + (-1)^n\left(\frac{\pi}{4}\right), n \in I$

when $t = -13$

$\sin x - \cos x = -13$

It is impossible, since the maximum value of $(\sin x - \cos x)$ is $\sqrt{2}$

3. The given equation is

$|\sec x + \tan x| = |\sec x| + |\tan x|$

$\Rightarrow \quad \sec x \cdot \tan x \geq 0$

$\Rightarrow \quad \frac{\sin x}{\cos^2 x} \geq 0$

$\Rightarrow \quad \sin x \geq 0, \cos x \neq 0$

$\Rightarrow \quad x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\Rightarrow \quad x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Hence, the solution set is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

4. The given equation is

$\Rightarrow \quad \sin\left(\frac{\pi}{2n}\right) + \cos\left(\frac{\pi}{2n}\right) = \frac{\sqrt{n}}{2}$

$\frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{2n}\right) + \frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{2n}\right) = \frac{\sqrt{n}}{2\sqrt{2}}$

$\Rightarrow \quad \cos\left(\frac{\pi}{4} - \frac{\pi}{2n}\right) = \frac{\sqrt{n}}{2\sqrt{2}}$

It is satisfied for $n = 6$ only.

5. The given equation is $\cos 2x + a\sin x = 2a - 7$

$\Rightarrow \quad 1 - 2\sin^2 x + a\sin x = 2a - 7$

$\Rightarrow \quad 2\sin^2 x - a\sin x + (2a - 8) = 0$

$$\Rightarrow \quad \sin x = \frac{a \pm \sqrt{a^2 - 16(a-4)}}{4}$$

$$\Rightarrow \quad \sin x = \frac{a \pm \sqrt{a^2 - 16a + 64}}{4}$$

$$\Rightarrow \quad \sin x = \frac{a \pm \sqrt{(a-8)^2}}{4}$$

$$\Rightarrow \quad \sin x = \frac{a \pm (a-8)}{4}$$

$$\Rightarrow \quad \sin x = \frac{2a-8}{4}, 2$$

$$\Rightarrow \quad \sin x = \frac{a-4}{2}$$

$$\Rightarrow \quad -1 \le \left(\frac{a-4}{2}\right) \le 1$$

$\Rightarrow \quad -2 \le (a-4) \le 3$
$\Rightarrow \quad 2 \le a \le 6$
$\Rightarrow \quad a \in [2, 6]$

6. The given equation is
$\sin^{100} x - \cos^{100} x = 1$
It is possible only when $\sin x = 1$, $\cos x = 0$
Hence, the general solution is

$$x = n\pi + \frac{\pi}{2}, n \in I$$

7. The given equation is

$$\Rightarrow \quad \sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$$

$$\left(\frac{1-\cos 2x}{2}\right)^5 + \left(\frac{1-\cos 2x}{2}\right)^5 = \frac{29}{16}\cos^4 2x$$

$$\Rightarrow \quad \frac{2}{32}(1 + 10\cos^2 2x + 5\cos^4 2x) = \frac{29}{16}\cos^4 2x$$

$\Rightarrow \quad (1 + 10\cos^2 2x + 5\cos^4 2x) = 29\cos^4 2x$
$\Rightarrow \quad 24\cos^4 2x - 10\cos^2 2x - 1 = 0$
$\Rightarrow \quad (2\cos^2 2x - 1)(12\cos^2 2x + 1) = 0$
$\Rightarrow \quad (2\cos^2 2x - 1) = 0$, since $(12\cos^2 2x + 1) \neq 0$

$$\Rightarrow \quad \cos^2 2x = \frac{1}{2}$$

$\Rightarrow \quad 2\cos^2 2x - 1 = 0$
$\Rightarrow \quad \cos 4x = 0$

$$\Rightarrow \quad 4x = (2n+1)\frac{\pi}{2}, n \in I$$

$$\Rightarrow \quad x = (2n+1)\frac{\pi}{8}, n \in I$$

Hence, the solution is $x = (2n+1)\frac{\pi}{8}, n \in I$.

8. The given equation is

$$|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$$

$$\Rightarrow \quad \left(\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}\right)\log|\cos x| = 0$$

$$\Rightarrow \quad \left(\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}\right) = 0, \log|\cos x| = 0$$

when $\left(\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}\right) = 0$

$\Rightarrow \quad (2\sin^2 x - 3\sin x + 1) = 0$
$\Rightarrow \quad (2\sin^2 x - 2\sin x - \sin x + 1) = 0$
$\Rightarrow \quad 2\sin x(\sin x - 1) - (\sin x - 1) = 0$
$\Rightarrow \quad (2\sin x - 1)(\sin x - 1) = 0$
$\Rightarrow \quad (2\sin x - 1) = 0, (\sin x - 1) = 0$

$$\Rightarrow \quad \sin x = \frac{1}{2}, 1$$

$$\Rightarrow \quad \sin x = \frac{1}{2}, \text{ since } |\cos x| = 0$$

$$\Rightarrow \quad x = n\pi + (-1)^n\left(\frac{\pi}{6}\right), n \in I$$

when $\log|\cos x| = 0$
$\Rightarrow \quad \log|\cos x| = \log 1$
$\Rightarrow \quad |\cos x| = 1$
$\Rightarrow \quad \cos x = \pm 1$
when $\cos x = 1$
$\Rightarrow \quad x\ 2n\pi$
when $\cos x = -1$
$x\ (2n+1)\pi$
Hence, the solution is

$$x = 2n\pi, (2n+1)\pi, n\pi + (-1)^n\left(\frac{\pi}{6}\right), n \in I$$

9. It is possible only when

$$\cos(\pi\sqrt{x-4}) = 1 \text{ and } \cos(\pi\sqrt{x}) = 1$$

$x = 4$ and $x = 0$
$x = 0$ does not satisfy the equation simultaneously.
Hence, the solution is $x = 4$
Therefore, the number of solution is 1.

10. The given equation is

$$x^4 - 2x^2\sin^2\left(\frac{\pi}{2}\right)x + 1 = 0$$

$$\left(x^2 - \sin^2\left(\frac{\pi}{2}\right)x\right)^2 + 1 - \sin^4\left(\frac{\pi}{2}\right)x = 0$$

It is possible only when

$$\left(x^2 - \sin^2\left(\frac{\pi}{2}\right)x\right) = 0, 1 - \sin^4\left(\frac{\pi}{2}\right)x = 0$$

when $1 - \sin^4\left(\frac{\pi}{2}\right)x = 0$

$\Rightarrow \quad \sin^4\left(\frac{\pi}{2}\right)x = 1$

$\Rightarrow \quad \sin^4\left(\frac{\pi}{2}\right)x = \sin^2\left(\frac{\pi}{2}\right)$

$\Rightarrow \quad \left(\frac{\pi}{2}\right)x = n\pi \pm \left(\frac{\pi}{2}\right)$

$\Rightarrow \quad x = (2n \pm 1), n \in I$

when $\left(x^2 - \sin^2\left(\frac{\pi}{2}\right)x\right) = 0$

$\Rightarrow \quad x^2 = \sin^2\left(\frac{\pi}{2}\right)x$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x \pm 1$

Hence, the number of solutions is 2.

11. The given equation is

$\cos^4 x + a\cos^2 x + 1 = 0 \quad \ldots(i)$

Let $\cos^2 x = 1$

Then $t \in [0, 1]$

(i) reduces to $t^2 + at + 1 = 0$.

since it has at-least one real root in [0, 1], so $a^2 - 4 \geq 0$ and $1 + a + 1 \leq 0$

$\Rightarrow \quad |a| \geq 2, a \leq -2$

$\Rightarrow \quad a \geq 2, a \leq -2; a \leq -2$

$\Rightarrow \quad a \leq -2$

$\Rightarrow \quad a \in (-\infty, -2]$

12. The given equation is

$\tan^4 x - 2\sec^2 x + b^2 = 0$

$\Rightarrow \quad \tan^2 x = 2(1 + \tan^2 x) + b^2 = 0$

$\Rightarrow \quad \tan^4 x - 2\tan^2 x + 1 = 3 - b^2$

$\Rightarrow \quad (\tan^2 x - 1)^2 = 3 - b^2$

$\Rightarrow \quad (3 - b^2) = (\tan^2 x - 1)^2 \geq 0$

$\Rightarrow \quad (3 - b^2) \geq 0$

$\Rightarrow \quad b^2 \leq 3$

$\Rightarrow \quad |b| \leq \sqrt{3}$

13. The given equation is

$x^2 + 4 + 3\sin(ax + b) = 2x$

$\Rightarrow \quad (x^2 - 2x + 1) + 3 + 3\sin(ax + b) = 0$

$\Rightarrow \quad (x - 1)^2 + 3(1 + \sin(ax + b)) = 0$

It is possible only when

$(x - 1) = 0, (1 + \sin(ax + b)) = 0$

$\Rightarrow \quad x = 1, \sin(ax + b) = -1$

$\Rightarrow \quad \sin(a + b) = -1$

$\Rightarrow \quad (a + b) = (4n - 1)\frac{\pi}{2}, n \in I$

$\Rightarrow \quad (a + b) = \frac{3\pi}{2}, \frac{7\pi}{2}$

14. The given equation is

$|x| + |y| = 4$

$\Rightarrow \quad -4 \leq x, y \leq 4$

$\Rightarrow \quad |x| \leq 4, |y| \leq 4$

Also, $\sin\left(\frac{\pi x^2}{2}\right) = 1$

$\Rightarrow \quad \left(\frac{\pi x^2}{2}\right) = (4n + 1)\frac{\pi}{2}, n \in I$

$\Rightarrow \quad x^2 = (4n + 1)$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x = \pm 1$

Then $|y| = 4 - 1 = 3$

$y = \pm 3$

Thus, the possible ordered pairs are

(1, 3), (1, –3), (–1, 3), (–1, –3).

15. The given equation is

$\log_{|\cos x|}|\sin x| + \log_{|\sin x|}|\cos x| = 2$

It is possible only when

$|\sin x| = |\cos x| \neq 1$

$\Rightarrow \quad |\tan x| = 1$

$\Rightarrow \quad x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \pm\frac{7\pi}{4}$

Hence, the number of values of $x$ is 8.

16. The given equation is

$\tan x + \sec x = 2\cos x$

$\Rightarrow \quad \frac{\sin x + 1}{\cos x} = 2\cos x$

$\Rightarrow \quad \sin x + 1 = 2\cos^2 x$

$\Rightarrow \quad 1 + \sin x = 2(1 - \sin x)(1 + \sin x)$

$\Rightarrow \quad (1 + \sin x)(1 - 2 + 2\sin x) = 0$

$\Rightarrow \quad (1 + \sin x) = 0, (2\sin x - 1) = 0$

$\Rightarrow \quad \sin x = -1, \sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{3\pi}{2}; x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence, the solutions are $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

17. The given equation is

$2\sin^2 x + 5\sin x - 3 = 0$

$\Rightarrow \quad 2\sin^2 x + 6\sin x - \sin x - 3 = 0$

$\Rightarrow \quad 2\sin x(\sin x + 3) - (\sin x + 3) = 0$

$\Rightarrow \quad (\sin x + 3)(2\sin x - 1) = 0$

$\Rightarrow \quad (\sin x + 3) = 0, (2\sin x - 1) = 0$

$\Rightarrow \quad 2(\sin x - 1) = 0$

$\Rightarrow \quad \sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}$

Hence, the number of solutions is 4.

18. The given equation is

$$5\cos(2\theta) + 2\cos^2\left(\frac{\theta}{2}\right) + 1 = 0$$

$\Rightarrow$ $5(2\cos^2\theta - 1) + (1 + \cos\theta) + 1 = 0$
$\Rightarrow$ $10\cos^2\theta + \cos\theta - 3 = 0$
$\Rightarrow$ $10\cos^2\theta + 6\cos\theta - 5\cos\theta - 3 = 0$
$\Rightarrow$ $2\cos\theta(5\cos\theta + 3 - 1)(5\cos\theta + 3) = 0$
$\Rightarrow$ $(2\cos\theta - 1)(5\cos\theta + 3) = 0$
$\Rightarrow$ $\cos\theta = \frac{1}{2}, \cos\theta = -\frac{3}{5}$

When $\cos\theta = \frac{1}{2}$

Then $\theta = -\frac{\pi}{3}, \frac{\pi}{3}$

When $\cos\theta = -\frac{3}{5}$

Then $\theta = \frac{\pi}{2} + \cos^{-1}\left(-\frac{3}{5}\right)$

and $-\frac{\pi}{2} - \cos^{-1}\left(-\frac{3}{5}\right)$

Hence, the solutions are

$$\theta = \pm\frac{\pi}{3}, \frac{\pi}{2} + \cos^{-1}\left(-\frac{3}{5}\right), -\frac{\pi}{2} - \cos^{-1}\left(-\frac{3}{5}\right)$$

19. The given equation is
$2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0$
$\Rightarrow$ $2\sin x - 2\cos 2x - 2\sin x\cos x$
$- \sin 2x + 2\cos x = 0$
$\Rightarrow$ $2\sin x(1 - \cos x) + 4\cos^3 x$
$- 4\cos^2 x - 2\cos x + 2 = 0$
$\Rightarrow$ $2\sin x(1 - \cos x) + 4\cos^2 x(\cos x - 1)$
$-2(\cos x - 1) = 0$
$\Rightarrow$ $(\cos x - 1)(4\cos^2 x - 2 - 2\sin x) = 0$
$\Rightarrow$ $(\cos x - 1)(2\sin^2 x + \sin x - 1) = 0$
$\Rightarrow$ $(\cos x - 1)(\sin x + 1)(2\sin x - 1) = 0$
$\Rightarrow$ $\cos x = 1, \sin x = -1, \sin x = \frac{1}{2}$
$\Rightarrow$ $x = 2n\pi, (4n-1)\frac{\pi}{2}, n\pi + (-1)^n\left(\frac{\pi}{6}\right)$

20. The given equations are
$x\cos^3 y + 3x\cos y\sin^2 y = 14,$
$x\sin^3 y + 3x\cos^2 y\sin y = 13$
Adding and subtracting, we get,
$x(\cos y + \sin y)^3 = 27$ …(i)
$x(\cos y - \sin y)^3 = 1$ …(ii)
Dividing (ii) by (i), we get,

$$\frac{(\cos y + \sin y)^3}{(\cos y - \sin y)^3} = 27$$

$\Rightarrow$ $\frac{(\cos y + \sin y)}{(\cos y - \sin y)} = 3$

$\Rightarrow$ $\frac{1 + \tan y}{1 - \tan y} = 3$

$\Rightarrow$ $\tan y = \frac{1}{2}$

$\Rightarrow$ $y = \tan^{-1}\left(\frac{1}{2}\right)$

Put the value of $y = \tan^{-1}\left(\frac{1}{2}\right)$ into (ii), we get

$\Rightarrow$ $x\left(\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}}\right)^3 = 1$

$x\left(\frac{1}{\sqrt{5}}\right)^3 = 1$

$\Rightarrow$ $x = 5\sqrt{5}$
Hence, the solutions are

$$x = 5\sqrt{5} \text{ and } y = \tan^{-1}\left(\frac{1}{2}\right)$$

21. The given equation is $4\sin^4 x + \cos^4 x = 1$
$\Rightarrow$ $5\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2$
$\Rightarrow$ $4\sin^4 x + \cos^4 x = \sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x$
$\Rightarrow$ $3\sin^4 x - 2\sin^2 x\cos^2 x = 0$
$\Rightarrow$ $3\sin^4 x - 2\sin^2 x + 2\sin^4 x = 0$
$\Rightarrow$ $5\sin^4 x - 2\sin^2 x = 0$
$\Rightarrow$ $\sin^2 x(5\sin^2 x - 2) = 0$
$\Rightarrow$ $\sin^2 x = 0, (5\sin^2 x - 2) = 0$
$\Rightarrow$ $\sin x = 0, \sin^2 x = \frac{2}{5}$
$\Rightarrow$ $x = n\pi, n\pi \pm \alpha$, where $\alpha = \sin^{-1}\left(\sqrt{\frac{2}{5}}\right)$

22. The given equation is
$\sin^4 x + \cos^4 x + \sin 2x + a = 0$
$\Rightarrow$ $1 - 2\sin^2 x\cos^2 x + \sin 2x + a = 0$
$\Rightarrow$ $1 - \frac{1}{2}(4\sin^2 x\cos^2 x) + \sin 2x + a = 0$
$\Rightarrow$ $1 - \frac{1}{2}(\sin 2x)^2 + \sin 2x + a = 0$
$\Rightarrow$ $2 - (\sin 2x)^2 + 2\sin 2x + 2a = 0$
$\Rightarrow$ $(\sin 2x)^2 + 2\sin 2x - 2a = 2a$
$\Rightarrow$ $(\sin 2x - 1)^2 - 3 = 2a$
$\Rightarrow$ $(\sin 2x - 1)^2 = 2a + 3$
$\Rightarrow$ $2a + 3 = (\sin 2x - 1)^2 \geq 0$
$\Rightarrow$ $a \geq -\frac{3}{2}$
Also, $(\sin 2x - 1)^2 \geq 0$
$\Rightarrow$ $(\sin 2x - 1) \geq 0$
$\Rightarrow$ $\sin 2x \geq 1$
$\Rightarrow$ $\sin 2x = 1$

$$\Rightarrow \quad 2x = n\pi + (-1)^n\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad x = \frac{n\pi}{2} + (-1)^n\left(\frac{\pi}{4}\right), n \in I$$

23. The given equation is

$$(\sqrt{3}-1)\sin\theta + (\sqrt{3}+1)\cos\theta = 2$$

$$\Rightarrow \quad \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\sin\theta + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\cos\theta = \frac{1}{\sqrt{2}}$$

$$\sin\theta\cos(75°) + \cos\theta\sin(75°) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(\theta + \frac{5\pi}{12}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(\theta + \frac{5\pi}{12}\right) = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \left(\theta + \frac{5\pi}{12}\right) = n\pi + (-1)^n\left(\frac{\pi}{4}\right), n \in I$$

$$\Rightarrow \quad \theta = \left(n\pi + (-1)^n\left(\frac{\pi}{4}\right) - \frac{5\pi}{12}\right), n \in I$$

24. The given equation is $\sec\theta - \operatorname{cosec}\theta = \frac{4}{3}$

$3(\sin\theta - \cos\theta) = 4\sin\theta\cos\theta$ …(i)

Let $\sin\theta - \cos\theta = t$’

Then $\sin\theta\cos\theta = \frac{1-t^2}{2}$

Equation (i) reduces to

$$\Rightarrow \quad 3t = 4\times\left(\frac{1-t^2}{2}\right) = 2(1-t^2)$$

$\Rightarrow \quad 2t^2 + 3t - 2 = 0$

$\Rightarrow \quad (2t-1)(t+2) = 0$

$\Rightarrow \quad t = \frac{1}{2}, -2$

when $t = -2$, $(\sin\theta - \cos\theta) = -2$

It is not possible.

when $t = \frac{1}{2}$, $\sin\theta - \cos\theta = \frac{1}{2}$

$$\Rightarrow \quad \sin\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}} = \sin\alpha$$

$$\Rightarrow \quad \left(\theta - \frac{\pi}{4}\right) = n\pi + (-1)^n\alpha, \text{ where } \alpha = \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

$$\Rightarrow \quad \theta = \left(n\pi + (-1)^n\alpha + \frac{\pi}{4}\right), n \in I$$

25. Given $3\cos 2\theta = 1$

$$\Rightarrow \quad \cos 2\theta = \frac{1}{3}$$

$$\Rightarrow \quad \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1}{3}$$

$\Rightarrow \quad 1 + \tan^2\theta = 3 - 3\tan^2\theta$

$\Rightarrow \quad 4\tan^2\theta = 2$

$$\Rightarrow \quad \tan^2\theta = \frac{1}{2}$$

Also, it is given that,

$\Rightarrow \quad 32\tan^8\theta = 2\cos^2\alpha - 3\cos\alpha$

$$\Rightarrow \quad 32\left(\frac{1}{2}\right)^4 = 2\cos^2\alpha - 3\cos\alpha$$

$\Rightarrow \quad 2\cos^2\alpha - 3\cos\alpha = 2$

$\Rightarrow \quad 2\cos^2\alpha - 3\cos\alpha - 2 = 0$

$\Rightarrow \quad 2\cos^2\alpha - 4\cos\alpha + \cos\alpha - 2 = 0$

$\Rightarrow \quad 2\cos\alpha(\cos\alpha - 2) + 1(\cos\alpha - 2) = 0$

$\Rightarrow \quad (2\cos\alpha + 1)(\cos\alpha - 2) = 0$

$\Rightarrow \quad (2\cos\alpha + 1) = 0, (\cos\alpha - 2) = 0$

$$\Rightarrow \quad \cos\alpha = -\frac{1}{2}, 2$$

$\Rightarrow \quad \cos\alpha = 2$ is not possible.

Also, when $\cos\alpha = -\frac{1}{2}$

$$\Rightarrow \quad x = 2n\pi \pm \frac{2\pi}{3}$$

26. Given equations are

$5\sin x\cos y = 1$ and $4\tan x = \tan y$

$\Rightarrow \quad 5\sin x\cos y = 1$ …(i)

and $\quad 4\sin x\cos y = \cos x\sin y$ …(ii)

Dividing (i) by (ii), we get,

$\cos x\sin y = \frac{4}{5}$ …(iii)

Adding (i) and (iii), we get,

$\sin(x+y) = 1$

$\Rightarrow \quad (x+y) = (4n+1)\frac{\pi}{2}, n \in I$ …(iv)

Subtracting (iii) from (i), we get,

$$\sin(x-y) = -\frac{3}{5} = \sin\alpha$$

$\Rightarrow \quad (x-y) = m\pi + (-1)^n\alpha, m \in I$ …(v)

From (iv) and (v), we get,

$$2x = (2n+m)\pi + \frac{\pi}{2} + (-1)^m\alpha$$

$$\Rightarrow \quad x = (2n+m)\frac{\pi}{2} + \frac{\pi}{4} + (-1)^m\frac{\alpha}{2}, \alpha = \sin^{-1}\left(-\frac{3}{5}\right)$$

and

$\Rightarrow \quad 2y = (2n - m)\pi + \dfrac{\pi}{2} - (-1)^m \alpha, \alpha = \sin^{-1}\left(-\dfrac{3}{5}\right)$

$y = (2n - m)\dfrac{\pi}{2} + \dfrac{\pi}{4} - (-1)^m \dfrac{\alpha}{2}$

27. Given equations are $(x - y) = \dfrac{\pi}{4}$,

and $\cot x + \cot y = 2$

$\Rightarrow \quad \cos x \sin y + \sin x \cos y = 2 \sin x \sin y$

$\Rightarrow \quad \sin (x + y) = \cos (x - y) - \cos (x + y)$

$\Rightarrow \quad \sin (x + y) + \cos (x + y) = \cos (x - y)$

$\Rightarrow \quad \dfrac{1}{\sqrt{2}}\sin (x + y) + \dfrac{1}{\sqrt{2}}\cos (x + y) = \dfrac{1}{\sqrt{2}}\cos (x - y)$

$\Rightarrow \quad \dfrac{1}{\sqrt{2}}\sin (x + y) + \dfrac{1}{\sqrt{2}}\cos (x + y) = \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin\left(x + y + \dfrac{\pi}{4}\right) = \dfrac{1}{2}$

$\Rightarrow \quad \left(x + y + \dfrac{\pi}{4}\right) = \dfrac{5\pi}{6}$

$\Rightarrow \quad (x + y) = \dfrac{5\pi}{6} - \dfrac{\pi}{4} = \dfrac{7\pi}{12}$

Also, $(x - y) = \dfrac{\pi}{4}$

Thus, $x = \dfrac{5\pi}{12}$ and $y = \dfrac{\pi}{6}$

28. The given equations are

$5^{(\operatorname{cosec}^2 x - \sec^2 y)} = 1 \quad \ldots\text{(i)}$

$2^{(2\operatorname{cosec} x + \sqrt{3}|\sec y|)} = 64 \quad \ldots\text{(ii)}$

From (i), we get

$\operatorname{cosec}^2 x - 3 \sec^2 y = 0$

$\Rightarrow \quad \operatorname{cosec}^2 x = 3 \sec^2 y$

$\Rightarrow \quad \operatorname{cosec} x = \sqrt{3}\,|\sec y| \quad \ldots\text{(iii)}$

Also, from (ii), we get

$2^{(2\operatorname{cosec} x + \sqrt{3}|\sec y|)} = 64 = 2^6$

$\Rightarrow \quad 2 \operatorname{cosec} x + \sqrt{3}\,|\sec y| = 6$

$\Rightarrow \quad 2 \operatorname{cosec} x + \operatorname{cosec} x = 6$, from (iii)

$\Rightarrow \quad 3 \operatorname{cosec} x = 6$

$\Rightarrow \quad \operatorname{cosec} x = 2$

$\Rightarrow \quad \sin x = \dfrac{1}{2}$

$\Rightarrow \quad x = n\pi + (-1)^n \left(\dfrac{\pi}{6}\right), n \in I$

Again, $|\sec y| = \dfrac{2}{\sqrt{3}}$

$\Rightarrow \quad |\cos y| = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad \cos^2 y = \dfrac{3}{4} = \cos^2\left(\dfrac{\pi}{6}\right)$

$\Rightarrow \quad y = n\pi \pm \left(\dfrac{\pi}{6}\right), n \in I$.

Hence, the solutions are

$$\begin{cases} x = n\pi + (-1)^n \left(\dfrac{\pi}{6}\right), n \in I \\ y = n\pi \pm \left(\dfrac{\pi}{6}\right), n \in I \end{cases}$$

## Level IV

1. Given equation is

$$\cot\left(\frac{x}{2}\right) - \operatorname{cosec}\left(\frac{x}{2}\right) = \cot x$$

$\Rightarrow \quad \dfrac{\cos (x/2) - 1}{\sin (x/2)} = \cot x$

$\Rightarrow \quad -\dfrac{2 \sin^2 (x/4)}{\sin (x/2)} = \cot x$

$\Rightarrow \quad -\dfrac{2 \sin^2 (x/4)}{2 \sin (x/4) \cos (x/4)} = \cot x$

$\Rightarrow \quad \tan (x/4) + \cot x = 0$

$\Rightarrow \quad \dfrac{\sin (x/4)}{\cos (x/4)} + \dfrac{\cos x}{\sin x} = 0$

$\Rightarrow \quad \cos\left(x - \dfrac{x}{4}\right) = 0$

$\Rightarrow \quad \cos\left(\dfrac{3x}{4}\right) = 0$

$\Rightarrow \quad \dfrac{3x}{4} = (2n + 1)\dfrac{\pi}{2}$

$\Rightarrow \quad x = (4n + 2)\dfrac{\pi}{3}, n \in I$

2. Given equation is

$$8 \cos x \cdot \cos 2x \cdot \cos 4x = \frac{\sin 6x}{\sin x}$$

$\Rightarrow \quad 4 \sin 2x \cos 2x \cos 4x = \sin 6x$

$\Rightarrow \quad 2 \sin 4x \cos 4x = \sin 6x$

$\Rightarrow \quad \sin 8x - \sin 6x = 0$

$\Rightarrow \quad 2 \cos (7x) \sin x = 0$

$\Rightarrow \quad \cos (7x) = 0, \sin x = 0$

$\Rightarrow \quad (7x) = (2n+1)\dfrac{\pi}{2}, x = n\pi$

$\Rightarrow \quad x = (2n+1)\dfrac{\pi}{14}, x = n\pi, n \in I$

2. Given equation is

$\Rightarrow \quad \dfrac{\tan x}{\tan 2x} + \dfrac{\tan 2x}{\tan x} + 2 = 0$

$\Rightarrow \quad (\tan x + \tan 2x)^2 = 0$

$\Rightarrow \quad (\tan x + \tan 2x) = 0$

$\Rightarrow \quad \sin(2x + x) = 0$

$\Rightarrow \quad \sin(3x) = 0$

$\Rightarrow \quad 3x = n\pi$

$\Rightarrow \quad x = \dfrac{n\pi}{3}, n \in I$

4. Given equation is

$\cos x \cos(6x) = -1$

$\Rightarrow \quad 2\cos(6x)\cos x = -2$

$\Rightarrow \quad \cos(7x) + \cos(5x) = -2$

It is possible only when

$\cos(7x) = -1, \cos(5x) = -1$

$\Rightarrow \quad 7x = (2k+1)\pi, 5x = (2m+1)\pi$

$\Rightarrow \quad x = (2k+1)\dfrac{\pi}{7}, x = (2m+1)\dfrac{\pi}{5}$

when $k = 3$ and $m = 2$, then common value of $x$ is $\pi$

Hence, the general solution is

$x = 2n\pi + \pi = (2n+1)\pi, n \in I$

5. Given equation is

$\cos(4x) + \sin(5x) = 2$

It is possible only when

$\cos(4x) = 1, \sin(5x) = 1$

$\Rightarrow \quad 4x = 2k\pi, 5x = (4m+1)\dfrac{\pi}{2}$

$\Rightarrow \quad x = \dfrac{k\pi}{2}, x = (4m+1)\dfrac{\pi}{10}, k, m \in I$

when $k = 1$ and $m = 1$, then the common value of $x$ is $\dfrac{\pi}{2}$.

Hence, the general solution is

$$x = \left(2n\pi + \frac{\pi}{2}\right) = (4n+1)\frac{\pi}{2}, n \in I$$

6. Given equation is

$(1 + \sin 2x) + 5(\sin x + \cos x) = 0$

$\Rightarrow \quad (\sin x + \cos x)^2 + 5(\sin x + \cos x) = 0$

$\Rightarrow \quad ((\sin x + \cos x) + 5)(\sin x + \cos x) = 0$

$\Rightarrow \quad ((\sin x + \cos x) + 5) = 0, (\sin x + \cos x) = 0$

$\Rightarrow \quad (\sin x + \cos x) = 0$

$\Rightarrow \quad \tan x = 1$

$x = n\pi - \dfrac{\pi}{4}, n \in I$

7. Given equation is

$\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

$\Rightarrow \quad (\sin 3x + \sin x) + \sin 2x = (\cos 3x + \cos x) + \cos 2x$

$\Rightarrow \quad 2\sin 2x \cos x + \sin 2x = 2\cos 2x \cos x + \cos 2x$

$\Rightarrow \quad \sin 2x(2\cos x + 1) = \cos 2x(2\cos x + 1)$

$\Rightarrow \quad (\sin 2x - \cos 2x)(2\cos x + 1) = 0$

$\Rightarrow \quad (\sin 2x - \cos 2x) = 0, (2\cos x + 1) = 0$

when $(\sin 2x - \cos 2x) = 0$

$\Rightarrow \quad \tan 2x = 1$

$\Rightarrow \quad 2x = n\pi + \dfrac{\pi}{4}$

$\Rightarrow \quad x = \dfrac{n\pi}{2} + \dfrac{\pi}{8}$

when $2\cos x + 1 = 0$

$\Rightarrow \quad \cos x = -\dfrac{1}{2}$

$\Rightarrow \quad x = 2n\pi \pm \dfrac{2\pi}{3}$

Hence, the solution is

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}, \frac{2\pi}{3}, \frac{4\pi}{3}$$

8. Given equation is

$$\frac{\sin^3 x}{\cos x} + \frac{\cos^3 x}{\sin x} - 2\sin x \cos x = 2$$

$\Rightarrow \quad \dfrac{\sin^4 x + \cos^4 x}{\sin x \cos x} = 2\sin x \cos x + 2$

$\Rightarrow \quad 1 - 2\sin^2 x\cos^2 x = 2\sin^2 x \cos^2 x + \sin(2x)$

$\Rightarrow \quad 4\sin^2 x \cos^2 x + \sin(2x) - 1 = 0$

$\Rightarrow \quad \sin^2 2x + \sin(2x) - 1 = 0$

$\Rightarrow \quad \sin(2x) = \dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad \sin(2x) = \dfrac{-1 + \sqrt{5}}{2}$

$\Rightarrow \quad \sin(2x) = \sin\alpha, \alpha = \sin^{-1}\left(\dfrac{-1+\sqrt{5}}{2}\right)$

$\Rightarrow \quad (2x) = n\pi + (-1)^n \alpha$

$\Rightarrow \quad x = \dfrac{n\pi}{2} + (-1)^n \dfrac{\alpha}{2}$

9. Given equation is

$\sin^2 4x + \cos^2 x = 2\sin 4x \cos^4 x$

$\Rightarrow \quad \sin^2 4x - 2\sin 4x \cos^4 x + \cos^2 x = 0$

$\Rightarrow \quad (\sin^2 4x - \cos^4 x)^2 + \cos^2 x - \cos^8 x = 0$

$\Rightarrow \quad (\sin^2 4x - \cos^4 x)^2 + \cos^2 x(1 - \cos^6 x) = 0$

It is possible only when

$(\sin^2 4x - \cos^4 x) = 0, \cos^2 x(1 - \cos^6 x) = 0$

Now, $\cos x = 0, \cos^2 x = 1$

when $\cos x = 0$ then $x = (2n+1)\frac{\pi}{2}$

So, $\sin 4\left(n + \frac{1}{2}\right)\pi = 0$

which is true

when $\cos^2 x = 1$, then $x = n\pi$

which is not satisfied the equation

$\sin(4x) - \cos^4 x = 0$

Hence, the solution is $x = (2n+1)\frac{\pi}{2}$.

10. Given equation is

$$\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \cos x$$

$\Rightarrow$ $1 - 2\sin^2 x \cos^2 x = \frac{7}{4}\sin(2x)$

$\Rightarrow$ $4 - 8\sin^2 x \cos^2 x = 7\sin(2x)$

$\Rightarrow$ $4 - 2\sin^2 2x - 7\sin(2x) = 0$

$\Rightarrow$ $2\sin^2 2x + 7\sin(2x) - 4 = 0$

$\Rightarrow$ $2\sin^2 2x + 8\sin(2x) - \sin(2x) - 3 = 0$

$\Rightarrow$ $(2\sin(2x) + 1)(\sin(2x) + 4) = 0$

$\Rightarrow$ $(2\sin(2x) + 1) = 0, (\sin(2x) + 4) = 0$

$\Rightarrow$ $(2\sin(2x) + 1) = 0$

$\Rightarrow$ $\sin(2x) = -\frac{1}{2}$

$\Rightarrow$ $\sin(2x) = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow$ $2x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

$\Rightarrow$ $x = \frac{n\pi}{2} + (-1)^n\left(-\frac{\pi}{12}\right), n \in I$

11. Given equation is

$$\sin^4 x + \cos^4 x = \cos(4x) + \frac{1}{2}$$

$\Rightarrow$ $1 - 2\sin^2 x \cos^2 x = \cos(4x) + \frac{1}{2}$

$\Rightarrow$ $2 - 4\sin^2 x \cos^2 x = 2\cos(4x) + 1$

$\Rightarrow$ $2 - \sin^2 2x = 2\cos(4x) + 1$

$\Rightarrow$ $2\cos(4x) + \sin^2 2x = 1$

$\Rightarrow$ $2(1 - 2\sin^2 2x) + \sin^2(2x) = 1$

$\Rightarrow$ $3\sin^2(2x) = 1$

$\Rightarrow$ $\sin^2(2x) = \frac{1}{3}$

$\Rightarrow$ $\sin^2(2x) = \frac{1}{3} = \sin^2\alpha$

$\Rightarrow$ $2x = n\pi \pm \alpha$

$\Rightarrow$ $x = \frac{n\pi}{2} \pm \frac{\alpha}{2}, n \in I, \alpha = \sin^{-1}\left(\frac{1}{3}\right)$

12. Given equation is

$$\sin^4 x + \sin^4\left(x + \frac{\pi}{4}\right) = \frac{1}{4}$$

$\Rightarrow$ $4\sin^4 x + 4\sin^4\left(x + \frac{\pi}{4}\right) = 1$

$\Rightarrow$ $(2\sin^2 x)^2 + \left(2\sin^2\left(x + \frac{\pi}{4}\right)\right)^2 = 1$

$\Rightarrow$ $(1 - \cos(2x))^2 + \left(1 - \cos\left(2x + \frac{\pi}{2}\right)\right)^2 = 1$

$\Rightarrow$ $(1 - \cos(2x))^2 + (1 + \sin(2x))^2 = 1$

$\Rightarrow$ $1 - 2\cos(2x) + 1 + 2\sin(2x) = 0$

$\Rightarrow$ $2 - 2(\cos(2x) - \sin(2x)) = 0$

$\Rightarrow$ $(\cos(2x) - \sin(2x)) = 1$

$\Rightarrow$ $\left(\frac{1}{\sqrt{2}}\cos(2x) - \frac{1}{\sqrt{2}}\sin(2x)\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\left(2x + \frac{\pi}{4}\right) = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow$ $x = n\pi, x = n\pi - \frac{\pi}{4}, n \in I$

13. We have $a = \cos\left(x + \frac{\pi}{3}\right) + \cos x$

$$= \cos x \cos\left(\frac{\pi}{3}\right) - \sin x \sin\left(\frac{\pi}{3}\right) + \cos x$$

$$= \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x + \cos x$$

$$= \frac{3}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$$

The equation will provide us a real solutions if

$$-\sqrt{\frac{9}{4} + \frac{3}{4}} \le a \le \sqrt{\frac{9}{4} + \frac{3}{4}}$$

$\Rightarrow$ $-\sqrt{3} \le a \le \sqrt{3}$

14. Let $f(x) = \cos x - x + \frac{1}{2}$

Now, $f(0) = 1 + \frac{1}{2} = \frac{3}{2} > 0$

and $f\left(\frac{\pi}{2}\right) = 0 - \frac{\pi}{2} + \frac{1}{2} = \frac{1}{2} - \frac{\pi}{2} < 0$

By intermediate value theorem there is a root lies in $\left(0, \frac{\pi}{2}\right)$.

Hence, the number of roots is 1.

15. Now, $\cos(xy)\tan(xy) = xy$

$\Rightarrow \quad \sin(xy) = xy$

It is possible only when $xy = 0$

$\Rightarrow \quad x = 1$ and $y = 0$

Thus, the solution is $(1, 0)$

Hence, the number of integral ordered pairs is 1.

16. Given equation is

$\sin^{2016} x - \cos^{2016} x = 1$

$\Rightarrow \quad \sin^{2016} x = \cos^{2016} x + 1$

It is possible only when

$\sin^{2016} x = 1, \cos^{2016} x = 0$

$\Rightarrow \quad \sin x = 1, \cos x = 0$

Hence, the solution is

$$x = 2n\pi + \frac{\pi}{2}, n \in I$$

Thus, the number of solutions is 1.

17. Given equation is

$x^2 + 2x\sin(xy) + 1 = 0$

$\Rightarrow \quad (x\sin(xy))^2 + (1 - \sin^2(xy)) = 0$

$\Rightarrow \quad (x\sin(xy))^2 + \cos^2(xy) = 0$

It is possible only when,

$(x + \sin(xy))^2 = 0, \cos^2(xy) = 0$

$\Rightarrow \quad (x + \sin(xy)) = 0, \cos(xy) = 0$

$\Rightarrow \quad \cos(xy) = 0$

$\Rightarrow \quad xy = (2n+1)\dfrac{\pi}{2}, n \in I$

when $x = 1$, $n = 0$, then $y = \dfrac{\pi}{2}$

when $x = -1$, $n = 1$, then $y = \dfrac{3\pi}{2}$

Hence, the number of ordered pairs are

$$\left(1, \frac{\pi}{2}\right), \left(-1, \frac{3\pi}{2}\right)$$

18. Given equation is

$\sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x$

$\Rightarrow \quad \sin(5x)\cos(3x) = 2\sin(3x)\cos(3x)\cos(2x)$

$\Rightarrow \quad (\sin(5x) - 2\sin(3x)\cos(3x)\cos(2x)$

$\Rightarrow \quad (\sin(5x) - \sin(5x) - \sin(x))\cos(3x) = 0$

$\Rightarrow \quad \sin(x)\cos(3x) = 0$

$\Rightarrow \quad \sin(x) = 0, \cos(3x) = 0$

$$x = n\pi, x = (2n+1)\frac{\pi}{6}, n \in I$$

$$x = 0, \pi, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

Hence, the number of solutions is 5.

19. Given equation is

$\cos 3x \cdot \tan 5x = \sin 7x$

$\Rightarrow \quad \cos(3x)\sin(5x) = \sin(7x)\cos(5x)$

$\Rightarrow \quad 2\cos(3x)\sin(5x) = 2\sin(7x)\cos(5x)$

$\Rightarrow \quad \sin(8x) + \sin(2x) = \sin(12x) + \sin(2x)$

$\Rightarrow \quad \sin(8x) = \sin(12x)$

$\Rightarrow \quad \sin(12x) - \sin(8x) = 0$

$\Rightarrow \quad 2\cos(10x)\sin(2x) = 0$

$\Rightarrow \quad \cos(10x) = 0, \sin(2x) = 0$

$\Rightarrow \quad 10x = (2n+1)\dfrac{\pi}{2}, 2x = n\pi$

$\Rightarrow \quad x = (2n+1)\dfrac{\pi}{20}, x = \dfrac{n\pi}{2}, n \in I$

$\Rightarrow \quad x = 0, \dfrac{\pi}{20}$

Hence, the number of solutions is 2.

20. Given equation is

$2\tan x - \lambda(1 + \tan^2 x) = 0$

$\Rightarrow \quad \lambda\tan^2 x - 2\tan x + \lambda = 0$

Let it has two roots, say, $\tan B$ and $\tan C$

Now, $\tan B + \tan C = \dfrac{2}{\lambda}$

$\Rightarrow \quad \tan B \cdot \tan C = 1$

Now, $\tan(B + C) = \dfrac{\tan B + \tan C}{1 - \tan B \tan C}$

$\Rightarrow \quad \tan(\pi - A) = \infty$

$\Rightarrow \quad (\pi - A) = \dfrac{\pi}{2}$

$\Rightarrow \quad A = \dfrac{\pi}{2}$

21. Given equation is

$\cos^4 x - (a + 2)\cos^2 x - (a + 3) = 0$

$\Rightarrow \quad \cos^4 x - 2\cos^2 x - 3 = a(1 + \cos^2 x)$

$\Rightarrow \quad (\cos^2 x - 3)(\cos^2 x + 1) = a(1 + \cos^2 x)$

$\Rightarrow \quad (\cos^2 x - 3) = 0$

$\Rightarrow \quad a + 3 = \cos^2 x$

Clearly, $0 \le a + 3 \le 1$

$\Rightarrow \quad -3 \le a \le -2$

22. Given equation is

$$\sin x + \sin\frac{\pi}{8}(\sqrt{(1-\cos x)^2 + \sin^2 x}) = 0$$

$\Rightarrow \quad \sin x + \sin\left(\dfrac{\pi}{8}\right)(\sqrt{2(1-\cos x)}) = 0$

$\Rightarrow \quad \sin x = -\sin\left(\dfrac{\pi}{8}\right)\sqrt{2(1-\cos x)}$

$\Rightarrow \quad \sin^2 x = 2\sin^2\left(\dfrac{\pi}{8}\right)(1-\cos x)$

$\Rightarrow \quad \sin^2 x = \left(1 - \dfrac{1}{\sqrt{2}}\right)(1-\cos x)$

$\Rightarrow \quad 4\sin^2\left(\dfrac{x}{2}\right)\cos^2\left(\dfrac{x}{2}\right) = \left(1 - \dfrac{1}{\sqrt{2}}\right)2\sin^2\left(\dfrac{x}{2}\right)$

$\Rightarrow \quad 2\cos^2\left(\dfrac{x}{2}\right) = \left(1 - \dfrac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad 1+\cos x=\left(1-\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad \cos x=-\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos x=\cos\left(\frac{3\pi}{4}\right)$

$\Rightarrow \quad x=2n\pi\pm\frac{3\pi}{4}, n\in I$

Hence, the solution is $x=\frac{13\pi}{4}$.

23. Given equation is

$\sin^4 x-(k+2)\sin^2 x-(k+3)=0$

$\Rightarrow \quad \sin^4 x-2\sin^2 x-3=k(\sin^2 x+1)$

$\Rightarrow \quad k=\frac{\sin^4 x-2\sin^2 x-3}{(\sin^2 x+1)}$

$\Rightarrow \quad k=\frac{(\sin^2 x+1)(\sin^2 x-3)}{(\sin^2 x+1)}$

$\Rightarrow \quad k=(\sin^2 x-3)$

$\Rightarrow \quad k+3=\sin^2 x$

$\Rightarrow \quad 0\le k+3\le 1$

$\Rightarrow \quad -3\le k\le -2$

24. Given equation is

$4.16^{\sin^2 x}=2^{6\sin x}$

$\Rightarrow \quad 4.4^{2\sin^2 x}=4^{3\sin x}$

$\Rightarrow \quad 4^{1+2\sin^2 x}=4^{3\sin x}$

$\Rightarrow \quad 1+2\sin^2 x=3\sin x$

$\Rightarrow \quad 2\sin^2 x-3\sin x+1=0$

$\Rightarrow \quad 2\sin^2 x-2\sin x-\sin x+1=0$

$\Rightarrow \quad 2\sin x(\sin x-1)-\sin x-1)=0$

$\Rightarrow \quad (2\sin x-1)(\sin x-1)=0$

$\Rightarrow \quad \sin x=\frac{1}{2}, 1$

$\Rightarrow \quad x=\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$

Thus, the number of principal solutions is 3.

25. Given equation is

$\sec x=1+\cos x+\cos^2 x+\cos^3 x+\dots$

$\Rightarrow \quad \sec x=\frac{1}{1-\cos x}$

$\Rightarrow \quad \frac{1}{\cos x}=\frac{1}{1-\cos x}$

$\Rightarrow \quad 2\cos x=1$

$\Rightarrow \quad \cos x=\frac{1}{2}$

$\Rightarrow \quad x=2n\pi\pm\frac{\pi}{3}, n\in I$

## Integer Type Questions

1. Given equation is

$3\sin^2 x-7\sin x+2=0$

$\Rightarrow \quad 3\sin^2 x-6\sin x-\sin x+2=0$

$\Rightarrow \quad 3\sin x(\sin x-2)-(\sin x-2)=0$

$\Rightarrow \quad (3\sin x-1)(\sin x-2)=0$

$\Rightarrow \quad \sin x=\frac{1}{3}, 2$

$\Rightarrow \quad \sin x=\frac{1}{3}$

Hence, the number of real solutions is 6.

2. Given equation is

$2\cos x+3\sin x=k+1$

$\Rightarrow \quad -\sqrt{13}\le(k+1)\le\sqrt{13}$

$\Rightarrow \quad -\sqrt{13}-1\le k\le\sqrt{13}-1$

$\Rightarrow \quad k=-4, -3, -2, -1, 0, 1, 2$

Hence, the number of integral values of $k$ is 7.

3. Given equation is

$$\begin{vmatrix}\sin x & \cos x & \cos x\\ \cos x & \sin x & \cos x\\ \cos x & \cos x & \sin x\end{vmatrix}=0$$

$$\begin{vmatrix}\sin x+2\cos x & \cos x & \cos x\\ \sin x+2\cos x & \sin x & \cos x\\ \sin x+2\cos x & \cos x & \sin x\end{vmatrix}=0$$

$(C_1\to C_1+C_2+C_3)$

$$(\sin x+2\cos x)\begin{vmatrix}1 & \cos x & \cos x\\ 1 & \sin x & \cos x\\ 1 & \cos x & \sin x\end{vmatrix}=0$$

$$(\sin x+2\cos x)\begin{vmatrix}1 & \cos x & \cos x\\ 0 & \sin x-\cos x & 0\\ 0 & 0 & \sin x-\cos x\end{vmatrix}=0$$

$(\sin x+2\cos x)(\sin x-\cos x)^2=0$

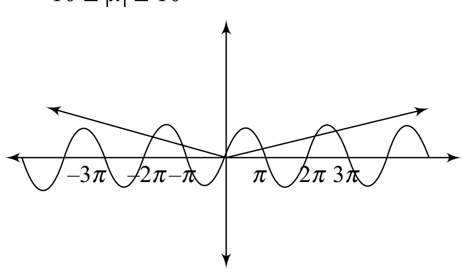
$\tan x=1, -2$

So, there is only one solution, $x=\frac{\pi}{4}$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

4. Given equation is
$\sin x + \sin y = \sin(x+y)$
$$2\sin\left(\frac{x+y}{2}\right)\left(\cos\left(\frac{x-y}{2}\right)-\cos\left(\frac{x+y}{2}\right)\right)=0$$
$$4\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)=0$$
$$\sin\left(\frac{x+y}{2}\right)=0, \sin\left(\frac{x}{2}\right)=0, \sin\left(\frac{y}{2}\right)=0$$
$x+y=0, x=0, y=0$
It is also given that $|x|+|x|=1$
when x = 0, then $|y|=1 \Rightarrow y=\pm 1$
when y = 0, then $|x|=1 \Rightarrow x=\pm 1$
when $y=-x$, then $|x|=\frac{1}{2} \Rightarrow x=\pm\frac{1}{2}$ and then $y=\mp\frac{1}{2}$
Hence, the pairs of solutions are
$$(0,1), (0,-1), (1,0), (-1,0), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)$$
Thus, the number of pairs is 6.

5. Given expression is
$$f(x)=\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$
$$=\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$
$$=\begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$
$$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$
$= 4\sin 2x + (1 + \sin^2 x + \cos^2 x)$
$= 4\sin 2x + 2$
So, the maximum value is 6.

6 We have, $-1 \le \sin x \le 1$
$$\Rightarrow \quad -1 \le \frac{|x|}{10} \le 1$$
$\Rightarrow \quad -10 \le |x| \le 10$




Clearly, the number of solutions is 6.

7. Given equation is
$\tan x + \cos x = 2\operatorname{cosec} x$
$$\Rightarrow \quad \frac{1}{\sin x \cos x} = \frac{2}{\sin x}$$
$$\Rightarrow \quad \cos x = \frac{1}{2}$$
$$\Rightarrow \quad x = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$$
Thus, the number of solutions is 4.

8. Given equation is
$$\cos x \cdot \cos 2x \cdot \cos 3x = \frac{1}{4}$$
$2(\cos 3x \cos x)\cos 2x = 1$
$2(\cos 4x + \cos 2x)\cos 2x = 1$
$2(\cos 4x)\cos 2x + 2\cos^2 2x = 1$
$2(2\cos^2 2x - 1)\cos 2x + 2\cos^2 2x = 1$
$4(\cos^3 2x) + 2\cos^2 2x - 2\cos 2x = 1$
$2\cos^2 2x(2\cos 2x + 1) = (2\cos 2x + 1)$
$(2\cos^2 2x - 1)(2\cos 2x + 1) = 0$
$4\cos 4x(2\cos 2x + 1) = 0$
$4\cos 4x = 0, (2\cos 2x + 1) = 0$
$$\cos 4x = 0, \cos 2x = -\frac{1}{2}$$
$$4x = (4n+1)\frac{\pi}{2}, 2x = 2n\pi \pm \frac{\pi}{3}$$
$$x = (4n+1)\frac{\pi}{8}, x = n\pi \pm \frac{\pi}{6}, n \in I$$
$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{\pi}{6}, \frac{5\pi}{6}$$
Hence, the number of solutions is 4.

9. Given equation is $\sin x$, $\cos y = 1$
It is possible only when
$\sin x = 1, \cos y = 1$
$$\Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2} \text{ and } y = 0, 2\pi$$
Also, when $\sin x = -1, \cos y = -1$
$$\Rightarrow \quad x = \frac{3\pi}{2}, y = \pi$$
Hence, the number of ordered pairs is 5
i.e. $\left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 2\pi\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{3\pi}{2}, \pi\right), \left(\frac{3\pi}{2}, 2\pi\right)$

10. When $\cot x$ is positive
The equation becomes
$$\cot x = \cot x + \frac{1}{\sin x}$$
$\operatorname{cosec} x = 0$
It is not possible.

When cot $x$ is negative the given equation becomes

$$\Rightarrow \quad -\cot x = \cot x + \frac{1}{\sin x}$$

$$\Rightarrow \quad 2\cot x + \frac{1}{\sin x} = 0$$

$$\Rightarrow \quad 2\cos x + 1 = 0$$

$$\Rightarrow \quad \cos x = -\frac{1}{2}$$

$$\Rightarrow \quad x = 2n\pi \pm \frac{2\pi}{3}, n \in I$$

$$\Rightarrow \quad x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Hence, the number of solutions is 2.

11. Given equation is

$$\tan(4x)\tan x = 1$$
$$\cos(4x)\cos x - \sin(4x)\sin x = 0$$
$$\cos(5x) = 0$$
$$5x = (2n+1)\frac{\pi}{2}$$
$$x = (2n+1)\frac{\pi}{10}, n \in I$$
$$x = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$$

Hence, the number of solutions is 5.

12. Given equation is

$$\sin x(\sin x + \cos x) = n$$
$$\Rightarrow \quad n = \sin x(\sin x + \cos x)$$
$$= \sin^2 x + \sin x \cos x$$
$$= \frac{1-\cos(2x)}{2} + \frac{\sin(2x)}{2}$$
$$\Rightarrow \quad \sin(2x) - \cos(2x) = 2n - 1$$
$$\Rightarrow \quad -\sqrt{2} \le 2n - 1 \le \sqrt{2}$$
$$\Rightarrow \quad \frac{1-\sqrt{2}}{2} \le n \le \frac{1+\sqrt{2}}{2}$$
$$\Rightarrow \quad n = 0, 1$$

Hence, the number of integral values of $n$ is 2.

13. Given equation is

$$\sin\{x\} = \cos\{x\}$$
$$\Rightarrow \quad \tan\{x\} = 1$$
$$\Rightarrow \quad x = \frac{\pi}{4}, 1+\frac{\pi}{4}, 2+\frac{\pi}{4}, 3+\frac{\pi}{4}, 4+\frac{\pi}{4}, 5+\frac{\pi}{4}$$

Hence, the number of solutions is 6

14. Given equation is

$$(\sqrt{3}+1)^{2x} + (\sqrt{3}-1)^{2x} = 2^{3x}$$
$$\Rightarrow \quad (\sqrt{3}+1)^{2x} + (\sqrt{3}-1)^{2x} = (2\sqrt{2})^{2x}$$
$$\Rightarrow \quad \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^{2x} + \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^{2x} = 1$$
$$\Rightarrow \quad \left(\sin\left(\frac{5\pi}{12}\right)\right)^{2x} + \left(\cos\left(\frac{5\pi}{12}\right)\right)^{2x} = 1$$

It is satisfied only when $x = 1$

Thus, the number of solutions is 1.

15. Here, $1 \le |\sin(2x)| + |\cos(2x)| \le \sqrt{2}$

and $|\sin(y)| \le 1$

It is possible only when $|\sin(y)| = 1$

$$\Rightarrow \quad \sin(y) = \pm 1$$
$$\Rightarrow \quad y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$$

Thus, the number of values of $y$ is 4.

## Previous Years' JEE-Advanced Examinations

1. The given equation is

$$\sin x + \cos x = 1$$
$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$$
$$\Rightarrow \quad \cos\left(x - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$
$$\Rightarrow \quad \left(x - \frac{\pi}{4}\right) = \left(2n\pi \pm \frac{\pi}{4}\right)$$
$$\Rightarrow \quad x = \left(2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4}\right)$$
$$\Rightarrow \quad x = 2n\pi, 2n\pi + \frac{\pi}{2}$$

But $x = 2n\pi + \frac{\pi}{2}$ does not satisfy the given equation.

Therefore, the solution is $x = 2n\pi$, $n \in I$

2. We have $\cos x = \sin 3x$

$$\cos x = \sin 3x = \cos\left(\frac{\pi}{2} - 3x\right)$$
$$\Rightarrow \quad x = 2n\pi \pm \left(\frac{\pi}{2} - 3x\right)$$

Taking positive sign, we get,

$$x = 2n\pi + \left(\frac{\pi}{2} - 3x\right)$$
$$\Rightarrow \quad 4x = 2n\pi + \frac{\pi}{2}$$
$$\Rightarrow \quad x = \frac{2n\pi}{4} + \frac{\pi}{8}$$
$$\Rightarrow \quad x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in I$$

Taking negative sign, we get,

$$\Rightarrow \quad x = 2n\pi - \left(\frac{\pi}{2} - 3x\right)$$

$$\Rightarrow \quad -2x = 2n\pi - \frac{\pi}{2}$$

$$\Rightarrow \quad x = -n\pi + \frac{\pi}{4}, n \in I$$

As $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $x = \frac{\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{4}$

Thus, the point of intersections of two curves are

$$\left(\frac{\pi}{8}, \cos\left(\frac{\pi}{8}\right)\right), \left(-\frac{3\pi}{8}, \cos\left(\frac{3\pi}{8}\right)\right), \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$$

3. The given equation is

$$4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$$
$$\Rightarrow \quad 4(1 - \sin^2 x)\sin x - 2\sin^2 x = 3\sin x$$
$$\Rightarrow \quad 4\sin x - 4\sin^3 x - 2\sin^2 x = 3\sin x$$
$$\Rightarrow \quad \sin x - 4\sin^3 x - 2\sin^2 x = 0$$
$$\Rightarrow \quad 4\sin^3 x + 2\sin^2 x - \sin x = 0$$
$$\Rightarrow \quad \sin x\,(4\sin^2 x = 2\sin x - 1) = 0$$
$$\Rightarrow \quad \sin x = 0, (4\sin^2 x + 2\sin x - 1) = 0$$
$$\Rightarrow \quad \sin x = 0, \sin x = \frac{-2 \pm 2\sqrt{5}}{8}$$
$$\Rightarrow \quad \sin x = 0, \sin x = \frac{-1 \pm \sqrt{5}}{4}$$
$$\Rightarrow \quad x = n\pi, x = n\pi + (-1)^n \sin^{-1}\left(\frac{-1 \pm \sqrt{5}}{4}\right)$$

where $n \in I$

4. The given equation is

$$\sin^2\theta - 2\sin^2\theta - 1 = 0$$
$$\Rightarrow \quad (\sin^2\theta - 1)^2 = 2$$
$$\Rightarrow \quad (\sin^2\theta - 1) = \pm\sqrt{2}$$
$$\Rightarrow \quad \sin^2\theta = (1 \pm \sqrt{2})$$
$$\Rightarrow \quad \sin^2\theta = (1 + \sqrt{2}), (1 - \sqrt{2})$$

since $(1 + \sqrt{2}) > 1$ and $(1 - \sqrt{2}) < 0$, so

there is no value of $\theta$ satisfying the given equation.

5. No questions asked in 1985.

6. We have $\cos x + \cos y = \frac{3}{2}$

$$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{3}{2}$$
$$\Rightarrow \quad 2\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{x-y}{2}\right) = \frac{3}{2}$$
$$\Rightarrow \quad 2 \times \frac{1}{2} \times \cos\left(\frac{x-y}{2}\right) = \frac{3}{2}$$
$$\Rightarrow \quad \cos\left(\frac{x-y}{2}\right) = \frac{3}{2}$$

It is not possible, so the solution set is

$$x = \varphi$$

7. The given inequation is

$$2\sin^2 x - 3\sin x + 1 \ge 0$$
$$\Rightarrow \quad (2\sin x - 1)(\sin x - 1) \ge 0$$
$$\Rightarrow \quad \sin x \le \frac{1}{2} \text{ and } \sin x \ge 1$$
$$\Rightarrow \quad \sin x \le \frac{1}{2} \text{ and } \sin x = 1$$
$$x \in \left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, \pi\right] \text{ and } x = \frac{\pi}{2}$$

Hence, the solution set is

$$x \in \left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, \pi\right] \cup \left\{\frac{\pi}{2}\right\}$$

8. As we know that $\tan x \ge x$

So there is no root between $\left(0, \frac{\pi}{2}\right)$

$$\left(\frac{\pi}{2}, \pi\right) \text{ and } \left(\frac{3\pi}{2}, 2\pi\right)$$

But there is a root in $\left(\pi, \frac{3\pi}{2}\right)$

9. The given equation is

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x = \cos 3x$

$$\Rightarrow \quad (\sin x + \sin 3x) - 3\sin 2x$$
$$(\cos x + \cos 3x) - 3\cos 2x$$
$$\Rightarrow \quad 2\sin 2x\cos x - 3\sin 2x$$
$$= 2\cos 2x\cos x - 3\cos 2x$$
$$\Rightarrow \quad (2\cos x - 3)(\sin 2x - \cos 2x) = 0$$
$$\Rightarrow \quad (\sin 2x - \cos 2x) = 0, (\because 2\cos x - 3 \ne 0)$$
$$\tan 2x = 1$$
$$\Rightarrow \quad \tan 2x = 1 = \tan\left(\frac{\pi}{4}\right)$$
$$\Rightarrow \quad 2x = \left(n\pi + \frac{\pi}{4}\right), n \in I$$
$$\Rightarrow \quad x = \left(\frac{n\pi}{2} + \frac{\pi}{8}\right), n \in I$$

10. No questions asked in between 1990 to 1992.

11. The given equation is

$$\tan x + \sec x = 2\cos x$$
$$\Rightarrow \quad \sin x + 1 = 2\cos^2 x$$
$$\Rightarrow \quad \sin x + 1 = 2(1 - \sin^2 x)$$
$$\Rightarrow \quad (\sin x + 1) = 2(1 - \sin x)(1 + \sin x)$$
$$\Rightarrow \quad (\sin x + 1)(1 - 2(1 - \sin x)) = 0$$
$$\Rightarrow \quad \sin x = -1, \sin x = \frac{1}{2}$$
$$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

But $x = \frac{3\pi}{2}$ does not satisfy the given equation

Hence, the number of solutions is 2.

12. The given equation can be written as
$\tan(x + 100°) \cot x = \tan(x + 50°) \tan(x - 50°)$

$\Rightarrow \quad \frac{\sin(x+100°)\cos x}{\cos(x+100°)\sin x} = \frac{\sin(x+50°)\sin(x-50°)}{\cos(x+50°)\cos(x-50°)}$

Applying compenendo and dividendo, we get,

$\Rightarrow \frac{\sin(x+100°)\cos x + \cos(x+100°)\sin x}{\sin(x+100°)\cos x - \cos(x+100°)\sin x}$
$= \frac{\sin(x+50°)\sin(x-50°) + \cos(x+50°)\cos(x-50°)}{\sin(x+50°)\sin(x-50°) - \cos(x+50°)\cos(x-50°)}$

$\Rightarrow \quad \frac{\sin(x+100°+x)}{\sin(x+100°-x)} = \frac{\cos(x+50°-x+50°)}{-\cos(x+50°+x-50°)}$

$\Rightarrow \quad \sin(2x + 100°)\cos 2x = -\sin(100°)\cos(100°)$

$\Rightarrow \quad 2\sin(2x + 100°)\cos 2x = -2\sin(100°)\cos(100°)$

$\Rightarrow \quad \sin(4x + 100°) + \sin(100°) = -\sin(200°)$

$\Rightarrow \quad \sin(4x + 100°) = -(\sin(200°) + \sin(100°))$

$\Rightarrow \quad \sin(4x + 100°) = -2\sin(150°)\cos(50°)$

$\Rightarrow \quad \sin(4x + 100°) = -2 \times \frac{1}{2} \times \sin(40°)$

$\Rightarrow \quad \sin(4x + 100°) = -\sin(40°) = \sin(220°)$

$\Rightarrow \quad (4x + 100°) = (220°)$

$\Rightarrow \quad 4x = 120°$

$\Rightarrow \quad x = 30°$

Hence, the result.

13. Let $\theta = \frac{\pi}{2n}$

The given equation reduces to

$$\sin\theta + \cos\theta = \frac{\sqrt{n}}{2}$$

$\Rightarrow \quad (\sin\theta + \cos\theta)^2 = \frac{n}{4}$

$\Rightarrow \quad 1 + 2\sin\theta\cos\theta = \frac{n}{4}$

$\Rightarrow \quad \sin 2\theta = \frac{n}{4} - 1 = \left(\frac{n-4}{4}\right)$

As per choices, $n \geq 4$

$\Rightarrow \quad 0 < 2\theta < \frac{\pi}{2}$

$\Rightarrow \quad 0 < \sin 2\theta < 1$

$\Rightarrow \quad 0 < \left(\frac{n-4}{4}\right) < 1$

$\Rightarrow \quad 0 < (n - 4) < 4$

$\Rightarrow \quad 4 < n < 8$

14. The given in-equations are

$2\sin^2 x + 3\sin x - 2 \geq 0$ and $x^2 - x - 2 < 0$

$\Rightarrow \quad (2\sin x - 1)(\sin x + 2) \geq 0$

$\Rightarrow \quad (2\sin - 1)$ ³ 0 ($\because \sin x + 2 > 0$, ®$x \in R$)

$\Rightarrow \quad \sin x \geq \frac{1}{2}$

$\Rightarrow \quad \frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ …(i)

Also $x^2 - x - 2 < 0$

$\Rightarrow \quad (x - 2)(x + 1) < 0$

$\Rightarrow \quad -1 < x < 2$ …(ii)

From (i) and (ii), we get

$$x \in \left(\frac{\pi}{6}, 2\right)$$

15. The given equation is
$\cos(p\sin x) = \sin(p\cos x)$

$\Rightarrow \quad \cos(p\sin x) = \cos\left(\frac{\pi}{2} - p\cos x\right)$

$\Rightarrow \quad (p\sin x) = 2n\pi \pm \left(\frac{\pi}{2} - p\cos x\right)$

$\Rightarrow \quad p(\sin x + \cos x) = 2n\pi \pm \frac{\pi}{2}$

$\Rightarrow \quad \sqrt{2}p\sin\left(x + \frac{\pi}{4}\right) = 2n\pi \pm \frac{\pi}{2}$

As we know that, $-1 \leq \sin\left(x + \frac{\pi}{4}\right) \leq 1$

$$-\sqrt{2}p \leq \sqrt{2}p\sin\left(x + \frac{\pi}{4}\right) \leq \sqrt{2}p$$

$\Rightarrow \quad -\sqrt{2}p \leq 2n\pi \pm \frac{\pi}{2} \leq \sqrt{2}p$

$\Rightarrow \quad \sqrt{2}p \geq 2n\pi \pm \frac{\pi}{2}$

$\Rightarrow \quad \sqrt{2}p \geq 2n\pi + \frac{\pi}{2}, 2n\pi - \frac{\pi}{2}$

As we require smallest +ve value of $p$, so we consider,
$\sqrt{2}p = \frac{\pi}{2}$

$\Rightarrow \quad p = \frac{\pi}{2\sqrt{2}}$

For this value of $p$, $x = \frac{\pi}{4}$ is a solution of the given equation.

16. The given equation can be written as

$$(1 - \tan^2\theta)(1 + \tan^2\theta) + 2^{\tan^2\theta} = 0$$

$\Rightarrow \quad (1 - \tan^2\theta)(1 + \tan^2\theta) + 2^{\tan^2\theta} = 0$

$\Rightarrow \quad (1 - \tan^4\theta) + 2^{\tan^2\theta} = 0$

$\Rightarrow \quad 2^{\tan^2\theta} = (\tan^4\theta - 1)$

$\Rightarrow \quad 2^x = (x^2 - 1)$, where $x = \tan^2\theta$

It is true for x = 3

Thus, $\tan^2\theta = 3$

$\Rightarrow \quad \tan^2\theta = (\sqrt{3})^2$

$\Rightarrow \quad \tan\theta = \pm(\sqrt{3})$

$\Rightarrow \quad \theta = \pm\left(\frac{\pi}{3}\right)$

17. The given equation is

$\tan^2\theta + \sec 2\theta = 1$

$\Rightarrow \quad \tan^2\theta + \frac{1+\tan^2\theta}{1-\tan^2\theta} = 1$

$\Rightarrow \quad x + \frac{1+x}{1-x} = 1$, where $x = \tan^2\theta$

$\Rightarrow \quad x - x^2 + 1 + x = 1 - x$

$\Rightarrow \quad x^2 - 3x = 0$

$\Rightarrow \quad x = 0, 3$

$\Rightarrow \quad \tan^2\theta = 0, 3$

$\tan\theta = 0$ and $\tan\theta = \pm\sqrt{3}$

$\Rightarrow \quad \theta = n\pi, \theta = n\pi \pm \frac{\pi}{3}, n \in I$

18 The given equation is $\cos^7 x + \sin^4 x = 1$

$\cos^7 x = 1 - \sin^4 x$

$\Rightarrow \quad \cos^7 x = (1 - \sin^2 x)(1 + \sin^2 x)$

$\Rightarrow \quad \cos^7 x = \cos^2 x\,(2 - \cos^2 x)$

$\Rightarrow \quad \cos^2 x\,(\cos^5 x - (2 - \cos^2 x)) = 0$

$\Rightarrow \quad \cos^2 x = 0, (\cos^5 x - (2 - \cos^2 x)) = 0$

$\Rightarrow \quad \cos^2 x = 0, \cos^5 x + \cos^2 x = 2$

$\Rightarrow \quad \cos x = 0, \cos^5 x + \cos^2 x = 2$

$\Rightarrow \quad \cos x = 0, \cos x = 1$

$\Rightarrow \quad x = \pm\frac{\pi}{2}, 0$

Hence, the real roots are $\left\{\pm\frac{\pi}{2}, 0\right\}$

19. The given equation is

$\Rightarrow \quad 3\sin^2 x - 7\sin x + 2 = 0$

$\Rightarrow \quad (3\sin x - 1)(\sin x - 2) = 0$

$\Rightarrow \quad \sin x = \frac{1}{3}, 2$

$\Rightarrow \quad \sin x = \frac{1}{3}$

There are 2 solutions in its period $2\pi$

So, the number of solutions is 6

20. We have $\sin(n\theta) = \sum_{r=0}^{n} b_r \sin^r\theta$

$\Rightarrow \quad \sin(n\theta) = [b_0 + b_1\sin\theta + b_2\sin^2\theta + b_3\sin^3\theta + \ldots + b_n\sin^n\theta]$

Put $\theta = 0$, then $b_0 = 0$

Thus,

$\sin(n\theta) = b_1\sin\theta + b_2\sin^2\theta + \ldots + b_n\sin^n\theta$

$\Rightarrow \quad \frac{\sin(n\theta)}{\sin\theta} = b_1 + b_2\sin\theta + \cdots + b_n\sin^{n-1}\theta$

Taking limit $\theta \to 0$, we get, $b_1 = n$

Therefore, $b_0 = 0$, $b_1 = n$

22. We have $-\sqrt{74} \le 7\cos x + 5\sin x \le \sqrt{74}$

$\Rightarrow \quad -\sqrt{74} \le (2k+1) \le \sqrt{74}$

$\Rightarrow \quad -\sqrt{74} - 1 \le 2k \le \sqrt{74} - 1$

$\Rightarrow \quad -8 - 1 \le 2k \le 8 - 1 \; (\because \sqrt{74} < 9)$

$\Rightarrow \quad -9 \le 2k \le 7$

$\Rightarrow \quad -4.5 \le k \le 3.5$

$\Rightarrow \quad k = -4, -3, -2, -1, 0, 1, 2, 3$

Thus, the number of integral values of $k$ is 8.

23. No questions asked in between 2003 to 2004.

24. Given $-\pi \le a, b \le \pi$

$\Rightarrow \quad -\pi \le a \le \pi, -\pi \le b \le \pi$

$\Rightarrow \quad -\pi \le a \le \pi, -\pi \le -b \le \pi$

$\Rightarrow \quad -2\pi \le a - b \le 2\pi$

Given $\cos(a - b) = 1$

$a = b$

Also, $\cos(a + b) = \frac{1}{e}$

$\Rightarrow \quad \cos(2a) = \frac{1}{e}$

It has one solution in its period $\pi$

So it has 4 solutions in $[0, 4\pi]$.

25. Let $y = \frac{5x^2 - 2x + 1}{3x^2 - 2x - 1}$

$\Rightarrow \quad (3y - 5)x^2 - 2(y - 1)x - (y + 1) = 0$

As $x$ is real, so $(y - 1)^2 + (3y - 5)(y + 1) \ge 0$

$\Rightarrow \quad y^2 - 2y + 1 + 3y^2 - 2y - 5 \ge 0$

$\Rightarrow \quad 4y^2 - 4y - 4 \ge 0$

$\Rightarrow \quad y^2 - y - 1 \ge 0$

$\Rightarrow \quad \left(y - \left(\frac{1-\sqrt{5}}{2}\right)\right)\left(y - \left(\frac{1+\sqrt{5}}{2}\right)\right) \ge 0$

$\Rightarrow \quad y \le \left(\frac{1-\sqrt{5}}{2}\right), y \ge \left(\frac{1+\sqrt{5}}{2}\right)$

$\Rightarrow \quad 2\sin t \le \left(\frac{1-\sqrt{5}}{2}\right), 2\sin t \ge \left(\frac{1+\sqrt{5}}{2}\right)$

$\Rightarrow \quad \sin t \le \left(\frac{1-\sqrt{5}}{4}\right), \sin t \ge \left(\frac{1+\sqrt{5}}{4}\right)$

$\Rightarrow \quad \sin t \le \sin\left(-\frac{\pi}{10}\right), \sin t \ge \sin\left(\frac{3\pi}{10}\right)$

As $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, we get,

$\Rightarrow \quad -\frac{\pi}{2} \le t \le -\frac{\pi}{10}$ and $\frac{3\pi}{10} \le t \le \frac{\pi}{2}$

Thus, $t \in \left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$

26. The given in-equation is

$2\sin^2\theta - 5\tan\theta + 2 > 0$

$\Rightarrow \quad (2\sin\theta - 1)(\sin\theta - 2) > 0$

$\Rightarrow \quad (\sin\theta - 1) < 0$

$\Rightarrow \quad \sin\theta < \frac{1}{2}$

$\Rightarrow \quad \theta \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$

27. The given equations are

$2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$

Now, $2\sin^2\theta - (1 - 2\sin^2\theta) = 0$

$\Rightarrow \quad 4\sin^2\theta = 1$

$\Rightarrow \quad \sin^2\theta = \left(\frac{1}{2}\right)^2 = \sin^2\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{6}$

$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Also, $2\cos^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad 2 - 2\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad 2\sin^2\theta + 3\sin\theta - 2 = 0$

$\Rightarrow \quad (2\sin\theta - 1)(\sin\theta + 2) = 0$

$\Rightarrow \quad \sin\theta = \frac{1}{2}, 2$

$\Rightarrow \quad \sin\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence, the solutions are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Therefore, the number of solutions is 2.

28. The given equation is $\sin\theta = \cos\varphi$

$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\varphi$

$\Rightarrow \quad \left(\frac{\pi}{2} - \theta\right) = 2n\pi \pm \varphi, n \in I$

$\Rightarrow \quad -2n\pi = \left(\theta \pm \varphi - \frac{\pi}{2}\right)$

$\Rightarrow \quad \left(\theta \pm \varphi - \frac{\pi}{2}\right) = -2n\pi$

$\Rightarrow \quad \frac{1}{\pi}\left(\theta \pm \varphi - \frac{\pi}{2}\right) = -2n$

Thus, $\frac{1}{\pi}\left(\theta \pm \varphi - \frac{\pi}{2}\right)$ is an even integer.

29. We have

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

$$\Rightarrow \quad \sum_{m=1}^{6} \frac{1}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} = 4\sqrt{2}$$

$$\Rightarrow \quad \sum_{m=1}^{6} \frac{\sin\left(\left(\theta + \frac{m\pi}{4}\right) - \left(\theta + \frac{(m-1)\pi}{4}\right)\right)}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} = 4$$

$$\Rightarrow \quad \sum_{m=1}^{6}\left(\cot\left(\theta + \left(\frac{m-1}{4}\right)\pi\right) - \cot(\theta + m\pi)\right) = 4$$

$$\Rightarrow \quad \left(\cot\theta - \cot\left(\theta + \frac{6\pi}{4}\right)\right) = 4$$

$\Rightarrow \quad (\cot\theta + \tan\theta) = 4$

$\Rightarrow \quad \frac{1}{\sin\theta\cos\theta} = 4$

$\Rightarrow \quad \frac{1}{2\sin\theta\cos\theta} = 2$

$\Rightarrow \quad \frac{1}{\sin 2\theta} = 2$

$\Rightarrow \quad \sin 2\theta = \frac{1}{2}$

$\Rightarrow \quad 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow \quad \theta = \frac{\pi}{12}, \frac{5\pi}{12}$

Hence, the solutions are $\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}$.

30. We have $\tan\theta = \cos 5\theta$

$\frac{\sin\theta}{\cos\theta} = \frac{\cos 5\theta}{\sin 5\theta}$

$\Rightarrow \quad 2\cos 5\theta\cos 5 = 2\sin 5\theta\sin 5$

$\Rightarrow \quad \cos 6\theta + \cos\theta = \cos 4\theta - \cos 6\theta$

$\Rightarrow \quad 2\cos 6\theta = 0$

$\Rightarrow \quad \cos 6\theta = 0$

$\Rightarrow \quad 6\theta = \pm\frac{5\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{\pi}{2}$

$\Rightarrow \quad \theta = \pm\frac{5\pi}{12}, \pm\frac{3\pi}{12}, \pm\frac{\pi}{12}$

$\Rightarrow \quad \theta = \pm\frac{5\pi}{12}, \pm\frac{\pi}{4}, \pm\frac{\pi}{12}$

Also, $\sin(2\theta) = \cos(4\theta)$

$\Rightarrow \quad \cos(4\theta) = \cos\left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow \quad 4\theta = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta\right)$

Taking positive sign, we get,

$4\theta = 2n\pi + \left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow \quad 6\theta = 2n\pi + \frac{\pi}{2}$

$\Rightarrow \quad \theta = \frac{n\pi}{3} + \frac{\pi}{12}$

$\Rightarrow \quad \theta = \frac{\pi}{12}, \frac{5\pi}{12}, -\frac{\pi}{4}$

Taking negative sign, we get,

$4\theta = 2n\pi - \left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow \quad 2\theta = 2n\pi - \frac{\pi}{2}$

$\Rightarrow \quad \theta = n\pi - \frac{\pi}{4}$

$\Rightarrow \quad \theta = -\frac{\pi}{4}$

Hence, the solutions are $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, -\frac{\pi}{4}$

31. Let $\frac{\pi}{n} = \theta$

Then the given equation becomes

$\frac{1}{\sin\theta} = \frac{1}{\sin(2\theta)} + \frac{1}{\sin(3\theta)}$

$\Rightarrow \quad \frac{1}{\sin\theta} - \frac{1}{\sin(3\theta)} = \frac{1}{\sin(2\theta)}$

$\Rightarrow \quad \frac{\sin 3\theta - \sin\theta}{\sin\theta \sin(3\theta)} = \frac{1}{\sin(2\theta)}$

$\Rightarrow \quad \frac{2\cos 2\theta \sin\theta}{\sin\theta \sin 3\theta} = \frac{1}{\sin 2\theta}$

$\Rightarrow \quad \frac{2\cos 2\theta}{\sin 3\theta} = \frac{1}{\sin 2\theta}$

$\Rightarrow \quad \sin 4\theta = \sin 3\theta$

$\Rightarrow \quad \sin 4\theta = \sin(\pi - 30)$

$\Rightarrow \quad 4\theta = \pi - 3\theta$

$\Rightarrow \quad 7\theta = \pi$

$\Rightarrow \quad \theta = \frac{\pi}{7}$

$\Rightarrow \quad \frac{\pi}{n} = \frac{\pi}{7}$

$\Rightarrow \quad n = 7$

Hence, the integral value of $n$ is 7.

32. No questions asked in between 2012 to 2013.

33. The given equation is

$\sin x + 2\sin 2x - \sin 3x = 3$

$\Rightarrow \quad \sin x + 4\sin x\cos x - 3\sin x + 4\sin^3 x = 3$

$\Rightarrow \quad \sin x\,[-2 + 4\cos x + 4(1 - \cos^2 x)] = 3$

$\Rightarrow \quad \sin\,[2 - (4\cos 2x - 4\cos x + 1) + 1] = 3$

$\Rightarrow \quad \sin x\,[3 - (2\cos x - 1)^2] = 3$

It is possible only when

$\sin x = 1$ and $(2\cos x - 1) = 0$

$\Rightarrow \quad x = \frac{\pi}{2}$ and $x = \frac{\pi}{3}$

Thus, the values of $x$ does not satisfy the given equation.

34. The given equation can be written as

$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$

$\Rightarrow \quad \frac{5}{4}\cos^2 2x + (1 - 2\sin^2 x\cos^2 x)$

$+ (1 - \sin^2 x\cos^2 x) = 2$

$\Rightarrow \quad \frac{5}{4}\cos^2 2x + \left(1 - \frac{1}{2}\sin^2 2x\right) + \left(1 - \frac{3}{4}\sin^2 2x\right) = 2$

$\Rightarrow \quad \frac{5}{4}\cos^2 2x - \frac{5}{4}\sin^2 2x = 0$

$\Rightarrow \quad \cos^2 2x - \sin^2 2x = 0$

$\Rightarrow \quad \cos^2 2x = \sin^2 2x$

$\Rightarrow \quad \tan^2 2x = 1$

$\Rightarrow \quad 2x = n\pi \pm \frac{\pi}{4}, n \in I$

$\Rightarrow \quad x = \frac{n\pi}{2} \pm \frac{\pi}{8}, n \in I$

Hence, the solutions are

$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

CHAPTER

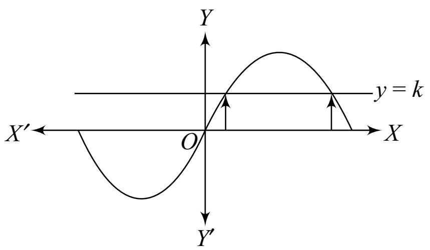
# 3 Trigonometric In-Equation

## 1. Trigonometric Inequalities

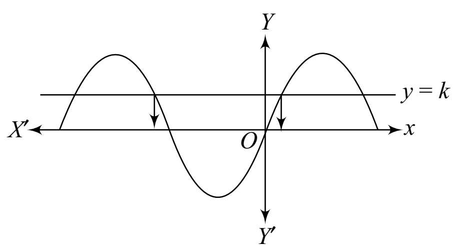
Suppose we have to solve $f(x) > k$ or $f(x) < k$

When we solve the inequation, we often use the graphs of the functions $y = f(x)$ and $y = k$.

Then, the solution of the inequality $f(x) > k$ is the value of $x$, for which the point $(x, f(x))$ of the graph of $y = f(x)$ lies above the straight line $y = k$.

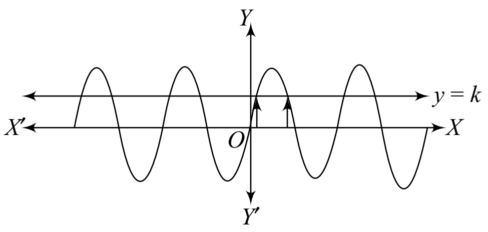



Similarly, when we solve $f(x) < k$, then the solution of the inequalition $f(x) < k$ is the values of $x$ for which the point $(x, f(x))$ of the graph of $y = f(x)$ lies below the straight line $y = k$.



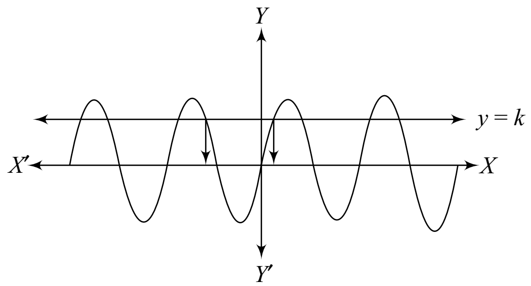

**Type I: An in-equation is of the form sin $x > k$**

**Rule:** Find the smallest values of $x$ satisfies the given inequation and then add $2n\pi$ with that values of $x$



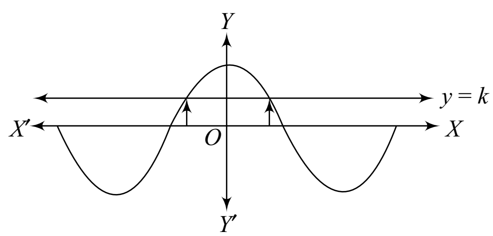

**Type II: An in-equation is of the form sin $x < k$**

**Rule:** Find the smallest values of $x$ satisfies the given in-equation and then add $2n\pi$ with that values of $x$.



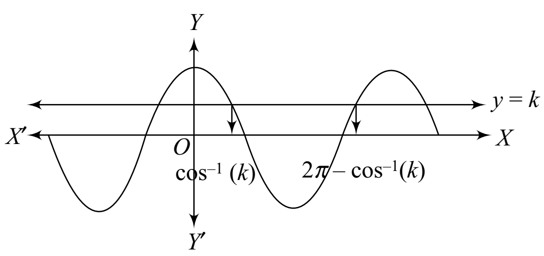

**Type III: An in-equation is of the form cos $x > k$**

**Rule:** First we find the smallest interval for which $x$ satisfies the given in-equation and then add $2n\pi$ with each values of $x$.



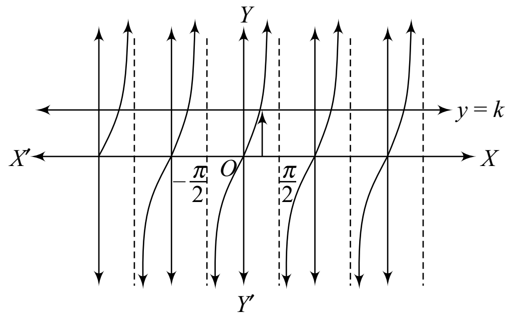

**Type IV: An in-equation is of the form cos $x < k$**

**Rule:** First we find the smallest interval for which $x$ satisfies the given inequation and then add $2n\pi$ with each values of $x$.



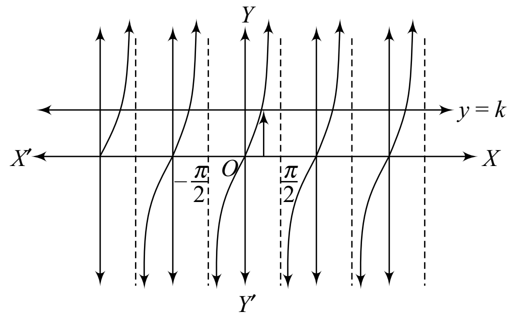
**Type V: An in-equation is of the form $\tan x > k$**

**Rule:** First we find the smallest interval for which $x$ satisfies the given in-equation and then add $n\pi$ with each values of $x$.




**Type VI: An in-equation is of the form $\tan x < k$**

**Rule:** First we find the smallest interval for which $x$ satisfies the given in equation and then add $n\pi$ with each values of $x$.




## Exercises

### Level I

**(Problems Based on Fundamentals)**

#### Type 1

1. Solve $\sin x > 1/2$
2. Solve $\sin x > 1/3$
3. Solve $\sin x \geq 1$
4. Solve $\sin x > 0$

#### Type 2

5. Solve $\sin x < 1/2$
6. Solve $\sin x < 1/5$
7. Solve $\sin x \leq \frac{\sqrt{3}}{2}$

#### Type 3

8. Solve $\cos x > \frac{1}{\sqrt{2}}$
9. Solve $\cos x \geq \frac{1}{2}$

#### Type 4

10. Solve $\cos x < \frac{1}{3}$
11. Solve $\cos x \leq \frac{\sqrt{3}}{2}$

#### Type 5

12. Solve $\tan x > 1$
13. Solve $\tan x \geq \frac{1}{\sqrt{3}}$
14. Solve $\tan x > 2$

#### Type 6

15. Solve $\tan x < 1$
16. Solve $\tan x \leq \sqrt{3}$

#### Mixed Problems

17. Solve $\sin x > \cos x$
18. Solve $\cos x > \sin x$
19. Solve $-\frac{1}{2} \leq \cos x < \frac{1}{\sqrt{2}}$
20. Solve $|\sin x + \cos x| = |\sin x| + |\cos x|$
21. Solve $\sin x \sin 2x < \sin 3x \sin 4x$, $\forall x \in \left(0, \frac{\pi}{2}\right)$
22. Solve $\cos x - \sin x - \cos 2x > 0$, $\forall x \in (0, 2\pi)$
23. Solve $\frac{5}{4}\sin^2 x + \frac{1}{4}\sin^2 2x > \cos 2x$
24. Solve $6\sin^2 x - \sin x \cos x - \cos^2 x > 2$
25. Solve $\sin^6 x + \cos^6 x > \frac{13}{16}$
26. Solve $\cos^3 x \cdot \cos 3x - \sin^3 x \sin 3x > \frac{5}{8}$
27. Solve the inequality $\sin^6 x + \cos^6 x > \frac{13}{16}$
28. Solve $\frac{5}{4}\sin^2 x + \frac{1}{4}\sin^2 2x > \cos 2x$
29. Solve $\sin 3x \sin 4x > \sin x \sin 2x \ \forall x \in \left(0, \frac{\pi}{2}\right)$
30. Solve $|\sin x + \cos x| = |\sin x| + |\cos x$
31. Solve $|\sec x + \tan x| = |\sec x| + |\tan x$
32. Solve for $x$: $\sin^2 x + \sin x - 2 < 0$ and $x^2 - 3x + 2 < 0$

33. Solve for $x$: $2\sin^2 x + \sin x - 1 < 0$ and $x^2 + x - 2 < 0$
34. Solve for $x$: $\tan^2 x - 5\tan x + 6 > 0$ and $x^2 - 16 \le 0$
35. Solve for $x$: $\frac{x-1}{5-x} < 0$ and $\tan^2 x + \tan x - 6 < 0$
36. Solve for $x$: $[\sin x] = 0$, where [,] = GIF

## Level II

### (Mixed Problems)

1. $\sin(3x - 1) > 0$
2. $\cos(2x - 3) < 0$
3. $|\sin x| \le \frac{1}{2}$
4. $|\cos x| \le \frac{1}{\sqrt{2}}$
5. $|\sin 2x + \cos 2x| = |\sin 2x| + |\cos 2x|$
6. $2\cos^2 x + \cos x < 1$
7. $4\sin^2 x - 1 \le 0$
8. $4\cos^2 x - 3 \ge 0$
9. $|\sin x| > |\cos x|$
10. $|\cos x| > |\sin x|$
11. $\sin x + \cos x > 1$
12. $\sin x + \cos x < 1$
13. $\sqrt{3}\sin x + \cos x > 1$
14. $\sin x - \sqrt{3}\cos x < 1$
15. $\sin^2 x + \sin x - 2 < 0$
16. $\sin^2 x + 3\sin x + 2 < 0$
17. $\cos^2 x - \cos x > 0$
18. $\sin x + \cos x - \cos 2x > 0$
19. $x^2 + x - 2 < 0$ and $\sin x > \frac{1}{2}$
20. $x^2 - 1 \le 0$ and $\cos x < \frac{1}{2}$
21. $4x^2 - 1 \le 0$ and $\tan x \ge \frac{1}{\sqrt{3}}$
22. $x^2 - 3x + 2 < 0$ and $(\sin x)^2 - \sin x > 0$

## Level III

### (Problems for JEE Advanced)

Q. Solve for $x$:

1. $\frac{\sin x + \cos x}{\sin x - \cos x} > \sqrt{3}$
2. $|\sin x| > |\cos x|$
3. $\cot x + \frac{\sin x}{\cos x - 2} \ge 0$
4. $\sin x + \cos x > \sqrt{2}\cos 2x$
5. $4\sin x \sin 2x \sin 3x > \sin 4x$
6. $\frac{\cos^2 2x}{\cos^2 x} \ge 3\tan x$
7. $\frac{\cos x + 2\cos^2 x + \cos 3x}{\cos x + 2\cos^2 x - 1} > 1$
8. $2(\sqrt{2} - 1)\sin x - 2\cos 2x + \sqrt{2}(\sqrt{2} - 1) < 0$
9. $\sin 2x > \sqrt{2}\sin^2 x + (2 - \sqrt{2})\cos^2 x$, $x \in (0, 2\pi)$
10. $1 + \log_4 \sin x + 2\log_{16} \cos x > 0$

## Comprehensive Link Passages

**Passage I**

If $x_1, x_2, x_3 \in R$, then

$$\frac{f(x_1) + f(x_2) + f(x_3)}{3} \le f\left(\frac{x_1 + x_2 + x_3}{3}\right)$$

Find

1. The value of $\sin\alpha + \sin\beta + \sin\gamma$ is, where $\alpha + \beta + \gamma = 180°$
   (a) $\le 1$ (b) $\le 3$ (c) $\le \frac{3\sqrt{3}}{2}$ (d) $\le \frac{3}{2}$
2. The value of $\cos\alpha + \cos\beta + \cos\gamma$ is, where $\alpha + \beta + \gamma = 180°$
   (a) $\le 2$ (b) $\le \frac{3}{2}$ (c) $\le 3$ (d) $\le \frac{\sqrt{3}}{2}$
3. The value of $\cot\alpha + \cot\beta + \cot\gamma$ is, where $\alpha + \beta + \gamma = 180°$
   (a) $\ge 1$ (b) $\ge \sqrt{3}$ (c) $\ge 3/2$ (d) $\ge \frac{\sqrt{3}}{2}$
4. The value of $\cot\alpha\cot\beta\cot\gamma$ is, where $\alpha + \beta + \gamma = 180°$
   (a) $\le \frac{1}{3}$ (b) $\le \frac{1}{3\sqrt{3}}$ (c) $\le \frac{1}{2\sqrt{3}}$ (d) $\le \frac{3}{2}$
5. The value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is, where $\alpha + \beta + \gamma = 180°$
   (a) $\le 9/4$ (b) $\le 3/4$ (c) $\le 3/2$ (d) $\le 1/2$
6. The value of $\sin\alpha\sin\beta \,.\, \sin\gamma$ is, where $\alpha + \beta + \gamma = 180°$
   (a) $\le \frac{\sqrt{3}}{4}$ (b) $\le \frac{3\sqrt{3}}{8}$ (c) $\le \frac{1}{2\sqrt{3}}$ (d) $\le \frac{1}{3\sqrt{2}}$
7. The value of $\cot^2\alpha + \cot^2\beta + \cot^2\gamma$ is, where $\alpha + \beta + \gamma = 180°$
   (a) $\ge 1$ (b) $\ge 3$ (c) $\ge \sqrt{3}$ (d) $\ge \sqrt{3}/2$

**Passage II**

If $|f(x) + g(x)| = |f(x)| + |g(x)|$, then $f(x) \,.\, g(x) \ge 0$

On the basis of the above information answer the following questions.

1. If $|\sec x + \tan x| = |\sec x| + |\tan x|$
   $\forall x \in [0, 2\pi]$, then $x$ does not satisfy the equation is
   (a) 0 (b) $\pi$ (c) $\frac{\pi}{2}$ (d) $2\pi$
2. If $|x - 1| + |x - 3| = 2$, then $x$ is
   (a) $x > 1$ (b) $x > 3$ (c) $x < 1$ (d) $1 < x < 3$

3. If $|\sin x + \cos x| = |\sin x| + |\cos x|$, $\forall x \in [0, 2\pi]$ then the solution set is

(a) $\left[0, \frac{\pi}{2}\right]$

(b) $\left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right] \cup \{2\pi\}$

(c) $\left[\pi, \frac{3\pi}{2}\right] \cup \{2\pi\}$

(d) $[0, 2\pi]$

## Matrix Match
## (For JEE-Advanced Examination Only)

1. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | The number of solutions of $\sin x > \frac{1}{2}$ in $(0, 2\pi)$ is | (P) | 6 |
| (B) | The number of solutions of $\lvert\tan x\rvert \le 1$ in $(-\pi, \pi)$ is | (Q) | 0 |
| (C) | The number of solutions of $\lvert\cos x\rvert > 1$ in $(0, 2013\pi)$ is | (R) | 4 |
| (D) | The number of solutions of $\sin x + \cos x = \lvert\sin x\rvert + \lvert\cos x\rvert$ in $(0, 2\pi)$ is | (S) | 2 |

2. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | If $\sin x \cdot \cos^3 x > \cos x \cdot \sin^3 x$, $x \in [0, 2\pi]$ Then $x$ is | (P) | $\left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$ |
| (B) | If $4\sin^2 x - 8\sin x + 3 \le 0$, $x \in [0, 2\pi]$ then $x$ is | (Q) | $\left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$ |
| (C) | If $\lvert\tan x\rvert \le 1$, $x \in [-\pi, \pi]$ then $x$ is | (R) | $\left(0, \frac{\pi}{4}\right)$ |
| (D) | If $\cos x - \sin x \ge 1$, $x \in [0, 2\pi]$ then $x$ is | (S) | $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ |

## Assertion and Reason

**Codes:**

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true but R is false.

(D) A is false but R is true.

1. Assertion (A): The value of $\tan 3\alpha \cdot \cot \alpha$ cannot lie between 3 and 1/3.
   Reason (R): In a triangle $ABC$, the maximum value of $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$ is $\frac{1}{8}$.
2. Assertion (A): The minimum value of $a^2 \tan^2 \theta + b^2 \cot^2 \theta$ is $2ab$.
   Reason (R): For positive real numbers AM $\ge$ GM
3. Assertion (A): The minimum value of $\frac{a^2}{\cos^2 x} + \frac{b^2}{\sin^2 x}$ is $(a + b)^2$
   Reason (R): The maximum value of
   $3\sin^2 x + 4\cos^2 x$ is 4
4. Assertion (A): For all $\theta \in \left[0, \frac{\pi}{2}\right]$,
   $\cos(\sin\theta) > \sin(\cos\theta)$
   Reason (R): In a triangle $ABC$, the maximum value of $\frac{\sin A + \sin B + \sin C}{\cot A + \cot B + \cot C}$ is $\frac{3}{2}$.
5. Assertion (A): $\cot^{-1} x \ge 2 \Rightarrow x \in (-\infty, 2]$
   Reason (R): $\cot^{-1} x$ is a decreasing function.

# Answers

## Level II

1. $x \in \left(\frac{2n\pi + 1}{3}, \frac{(2n\pi + 1)\pi + 1}{3}\right), n \in I$
2. $x \in \left((4n+1)\frac{\pi}{4} + \frac{3}{2}, (4n+3)\frac{\pi}{4} + \frac{3}{2}\right), n \in I$
3. $x \in \left(n\pi - \frac{\pi}{6}, n\pi + \frac{\pi}{6}\right), n \in I$
4. $x \in \left(n\pi - \frac{3\pi}{4}, n\pi - \frac{\pi}{4}\right), n \in I$
6. $x \in \left(n\pi - \pi, n\pi - \frac{\pi}{3}\right), n \in I$
7. $x \in \left(n\pi - \frac{\pi}{6}, n\pi + \frac{\pi}{6}\right), n \in I$
8. $x \in \left(n\pi - \frac{\pi}{6}, n\pi + \frac{\pi}{6}\right), n \in I$

9. $x \in \bigcup_{n\in I} \left\{ \left(2n\pi - \frac{\pi}{2}, n\pi - \frac{\pi}{4}\right) \cup \left(n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{2}\right) \right\}$

10. $x \in \left(n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{4}\right), n \in I$

11. $x \in (2n\pi, (2n+1)\pi), n \in I$

12. $x \in \left((2n-1)\pi, 2n\pi + \frac{\pi}{2}\right), n \in I$

13. $x \in \left(2n\pi, 2n\pi + \frac{2\pi}{3}\right), n \in I$

14. $x \in \left(2n\pi - \frac{5\pi}{6}, 2n\pi + \frac{\pi}{2}\right), n \in I$

15. $x \in \left(2n\pi - \frac{3\pi}{2}, 2n\pi + \frac{\pi}{2}\right), n \in I$

16. $x = \varphi$

17. $x \in \left(2n\pi + \frac{\pi}{12}, 2n\pi + \frac{5\pi}{12}\right), n \in I$

18. $x \in \left(0, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$

19. $\frac{\pi}{6} < x < 1$

20. $x = \varphi$

21. $x \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

22. $x \in (\pi, 2\pi)$

## Level III

1. $\bigcup_{n\in I} \left(n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{12}\right)$

2. $\bigcup_{n\in I} \left(n\pi + \frac{\pi}{4}, n\pi + \frac{3\pi}{4}\right)$

3. $\bigcup_{n\in I} \left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right) \cup \left(2n\pi + \frac{\pi}{3}, (2n+1)\pi\right)$

4. $\bigcup_{n\in I} \left(2n\pi + \frac{\pi}{12}, 2n\pi + \frac{3\pi}{4}\right) \cup \left(2n\pi + \frac{17\pi}{12}, 2n\pi + \frac{7\pi}{4}\right)$

5. $\bigcup_{n\in I} \left(n\pi - \frac{\pi}{8}, n\pi\right) \cup \left(n\pi + \frac{\pi}{2}, n\pi + \frac{5\pi}{8}\right)$
$\cup \left(n\pi + \frac{\pi}{8}, n\pi + \frac{3\pi}{8}\right)$

6. $\bigcup_{n\in I} \left(n\pi - \frac{7\pi}{12}, n\pi - \frac{\pi}{2}\right) \cup \left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{12}\right)$

7. $\bigcup_{n\in I} \left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right)$

8. $\bigcup_{n\in I} \left(2n\pi - \frac{\pi}{4}, 2n\pi + \frac{\pi}{6}\right) \cup \left(2n\pi + \frac{5\pi}{6}, 2n\pi + \frac{5\pi}{4}\right)$

9. $\left(\tan^{-1}(\sqrt{2}-1), \frac{\pi}{4}\right) \cup \left(\pi + \tan^{-1}(\sqrt{2}-1), \frac{5\pi}{4}\right)$

10. $\bigcup_{n\in I} \left(2n\pi + \frac{\pi}{12}, 2n\pi + \frac{5\pi}{12}\right)$

### COMPREHENSIVE LINK PASSAGES

Passage I:

1. (c) 2. (b) 3. (b) 4. (b)
5. (a) 6. (b) 7. (a)

Passage II:

1. (c) 2. (d) 3. (b)

### MATRIX MATCH

1. (A) → (S) ; (B) → (S) ; (C) → (Q) ; (D) → (R)
2. (A) → (R) ; (B) → (S) ; (C) → (P);(D) → (Q)

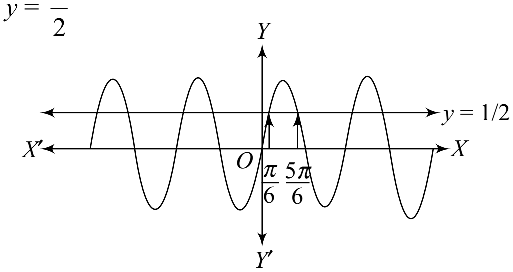
### ASSERTION AND REASON

1. (B) 2. (A) 3. (B) 4. (B) 5. (A)

# Hints and Solutions

## Level I

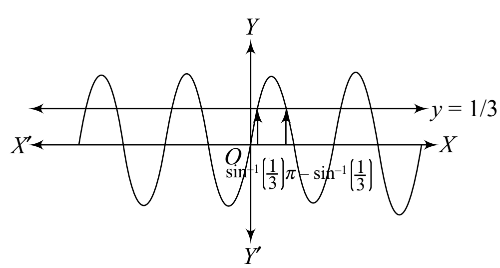
1. Here, we should construct the graph of $y = \sin x$ and $y = \frac{1}{2}$




Hence, the solution set is

$$x = \bigcup_{n\in I} \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$$

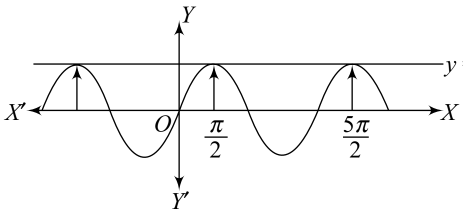
2. Here, we should construct the graph of $y = \sin x$ and $y = \sin^{-1}\left(\frac{1}{3}\right)$



Hence, the solution set is

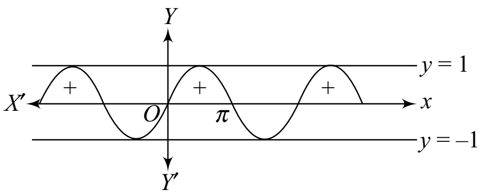
$$x = \bigcup_{n \in I} \left(2n\pi + \sin^{-1}\left(\frac{1}{3}\right), (2n+1)\pi - \sin^{-1}\left(\frac{1}{3}\right)\right)$$

3. We have $\sin x \geq 1$



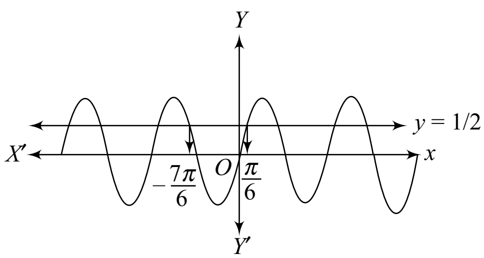

$$\Rightarrow \quad x = (4n+1)\frac{\pi}{2}, \; n \in I$$

4. We have $\sin x > 0$




$$\Rightarrow \quad x = \bigcup_{n \in I} (2\,n\pi, (2n+1)\,\pi)$$

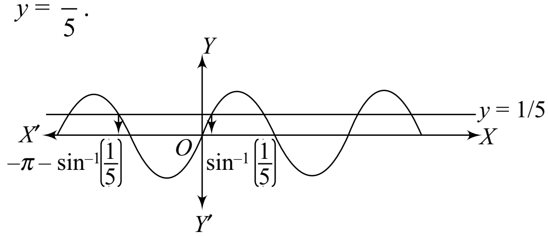
5. Here we should draw the graph of $y = \sin x$ and $y = \dfrac{1}{2}$.




Hence, the solution set is

$$x = \bigcup_{n \in I} \left(2\,n\pi - \frac{7\pi}{6}, 2n\pi + \frac{\pi}{6}\right)$$

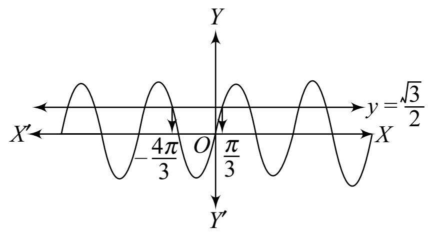
6. Here we should draw the graphs of $y = \sin x$ and $y = \dfrac{1}{5}$.




Hence, the solution set is

$$x = \bigcup_{n \in I} \begin{pmatrix} (2n-1)\pi - \sin^{-1}\left(\dfrac{1}{5}\right), \\ 2n\pi + \sin^{-1}\left(\dfrac{1}{5}\right) \end{pmatrix}$$

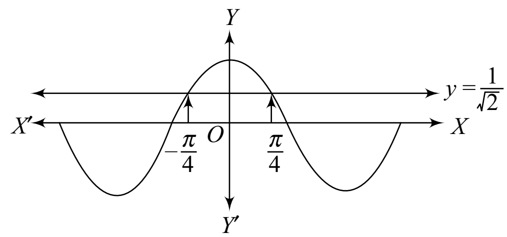
7. Here we should draw the graphs of $y = \sin x$ and $y = \dfrac{\sqrt{3}}{2}$.




Hence the solution set is

$$x = \bigcup_{n \in I} \left[2n\pi - \frac{4\pi}{3}, 2n\pi + \frac{\pi}{3}\right]$$

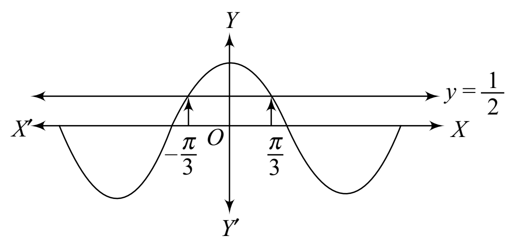
8. Here we should draw the graphs of $y = \cos x$ and $y = \dfrac{1}{\sqrt{2}}$.




Hence, the solution set is

$$x = \bigcup_{n \in I} \left(2n\pi - \frac{\pi}{4}, 2n\pi + \frac{\pi}{4}\right)$$

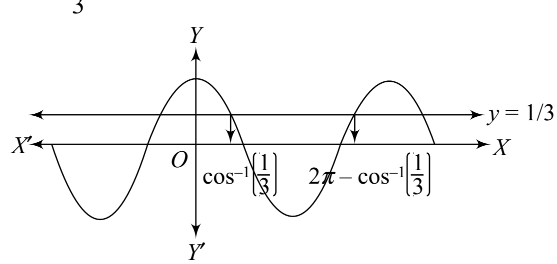
9. Here we should draw the graphs of $y = \cos x$ and $y = \dfrac{1}{2}$




Hence, the solution set is

$$x = \bigcup_{n \in I} \left[2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right]$$

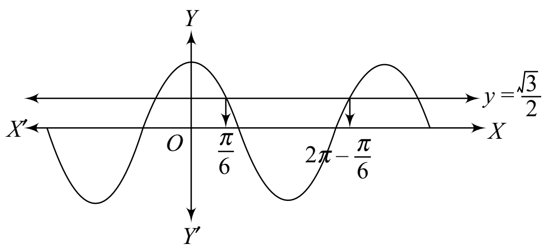
10. Here we should draw the graphs of $y = \cos x$ and $y = \dfrac{1}{3}$.



Hence, the solution set is

$$x=\bigcup_{n\in I}\left(\begin{array}{l}2n\pi+\cos^{-1}\left(\dfrac{1}{3}\right),\\ 2(n+1)\pi-\cos^{-1}\left(\dfrac{1}{3}\right)\end{array}\right)$$

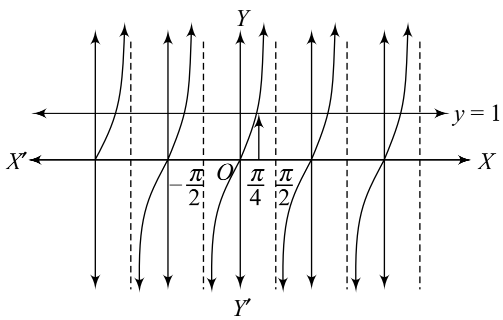
11. Here we should draw the graphs of $y = \cos x$ and $y=\dfrac{\sqrt{3}}{2}$.




Hence, the solution set is

$$x=\bigcup_{n\in I}\left[2n\pi+\frac{\pi}{6},\,2(n+1)\pi-\frac{\pi}{6}\right]$$

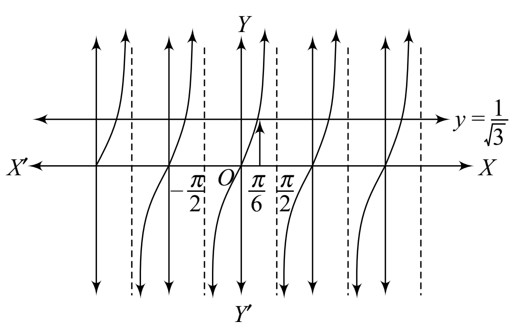
12. Here we should draw the graphs of $y = \tan x$ and $y = 1$.




Hence, the solution set is

$$x=\bigcup_{n\in I}\left(n\pi+\frac{\pi}{4},\,n\pi+\frac{\pi}{2}\right)$$

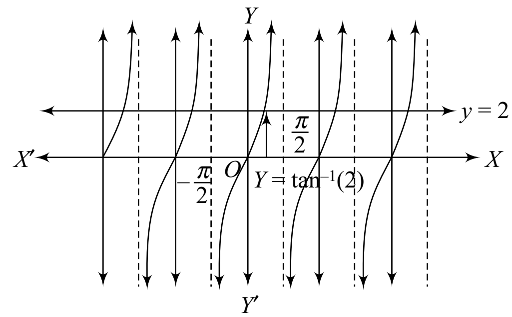
13. Here we should draw the graphs of $y = \tan x$ and $y=\dfrac{1}{\sqrt{3}}$.




Hence, the solution set is

$$x=\bigcup_{n\in I}\left[n\pi+\frac{\pi}{6},\,n\pi+\frac{\pi}{2}\right)$$

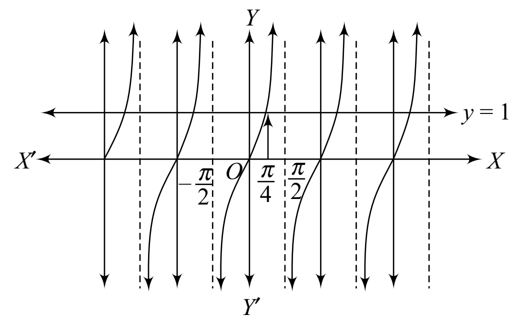
14. Here we should draw the graphs of $y = \tan x$ and $y = 2$.




Hence, the solution set is

$$x=\bigcup_{n\in I}\left(n\pi+\tan^{-1}(2),\,n\pi+\frac{\pi}{2}\right)$$

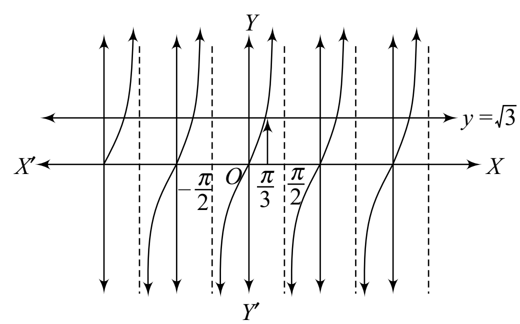
15. Here we should draw the graphs of $y = \tan x$ and $y = 1$.




Hence, the solution set is

$$x=\bigcup_{n\in I}\left(n\pi-\frac{\pi}{2},\,n\pi+\frac{\pi}{4}\right)$$

16. Here we should draw the graphs of $y = \tan x$ and $y=\sqrt{3}$.




Hence, the solution set is

$$x=\bigcup_{n\in I}\left[n\pi-\frac{\pi}{2},\,n\pi+\frac{\pi}{3}\right]$$

17. We have $\sin x > \cos x$

$\Rightarrow\quad \sin x-\cos x>0$

$\Rightarrow\quad \dfrac{1}{\sqrt{2}}\sin x-\dfrac{1}{\sqrt{2}}\cos x>0$

$\Rightarrow\quad \sin\left(x-\dfrac{p}{4}\right)>0$

$\Rightarrow \quad x \in \left(2n\pi + \frac{\pi}{4}, (2n+1)\pi + \frac{\pi}{4}\right)$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left(2n\pi + \frac{\pi}{4}, (2n+1)\pi + \frac{\pi}{4}\right)$$

18. We have $\cos x > \sin x$

$\Rightarrow \quad \cos x - \sin x > 0$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x > 0$

$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad x \in \left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right)$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right)$$

19. We have $\cos x < \frac{1}{\sqrt{2}}$ and $\cos x \geq -\frac{1}{2}$

$\Rightarrow \quad x \in \left(2n\pi + \frac{\pi}{4}, 2n\pi + \frac{7\pi}{4}\right)$

and $\quad x \in \left[2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right]$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left(2n\pi + \frac{\pi}{4}, 2n\pi + \frac{7\pi}{4}\right)$$

$$\cup\left[2n\pi + \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right]$$

20. We have $|\sin x + \cos x| = |\sin x| + |\cos x|$

As we know that, if

$|f(x) + g(x)| = |f(x)| + |g(x)|$

then $f(x)g(x) \geq 0$

Thus, $\sin x \cos x \geq 0$

$\Rightarrow \quad \sin 2x \geq 0$

$\Rightarrow \quad x \in \left[n\pi, n\pi + \frac{\pi}{2}\right]$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left[n\pi, n\pi + \frac{\pi}{2}\right]$$

21. We have $\sin x \sin 2x < \sin 3x \sin 4x$

$\Rightarrow \quad 2\sin x \sin 2x < 2\sin 3x \sin 4x$

$\Rightarrow \quad \cos x - \cos 3x < \cos x - \cos 7x$

$\Rightarrow \quad \cos 3x > \cos 7x$

$\Rightarrow \quad \cos 3x - \cos 7x > 0$

$\Rightarrow \quad 2\sin 5x \sin 2x > 0$

$\Rightarrow \quad \sin 5x > 0$ (since $\sin 2x$ is +ve for $0 < x < \pi/2$)

$\Rightarrow \quad 0 < 5x < \pi$

$\Rightarrow \quad 0 < x < \frac{\pi}{5}$

Hence, the solution set is

$$x = \left(0, \frac{\pi}{5}\right) \cup \left(0, \frac{\pi}{2}\right)$$

22. We have $\cos x - \sin x - \cos 2x > 0$

$\Rightarrow \quad (\cos x - \sin x) - (\cos^2 x - \sin^2 x) > 0$

$\Rightarrow \quad (\cos x - \sin x)(1 - \cos x - \sin x) > 0.$

$\Rightarrow \quad (\sin x - \cos x)(\sin x + \cos x - 1) > 0$

$\Rightarrow \quad \sin\left(x - \frac{\pi}{4}\right) \cdot (\sin x + \cos x - 1) > 0$

Hence, the solution set is

$$x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$$

23. We have $\frac{5}{4}\sin^2 x + \frac{1}{4}\sin^2 2x > \cos 2x$

$\Rightarrow \quad 5(2\sin^2 x) + 2(\sin^2 2x) > 8\cos 2x$

$\Rightarrow \quad 5(1 - \cos 2x) + 2(1 - \cos^2 2x) > 8\cos 2x$

$\Rightarrow \quad 5 - 5\cos 2x + 2 - 2\cos^2 2x - 8\cos 2x > 0$

$\Rightarrow \quad 2\cos^2 2x + 13\cos 2x - 7 < 0$

$\Rightarrow \quad 2\cos^2 2x + 14\cos 2x - \cos 2x - 7 < 0$

$\Rightarrow \quad 2\cos 2x(\cos 2x + 7) - (\cos 2x + 7) < 0$

$\Rightarrow \quad (\cos 2x + 7)(2\cos 2x - 1) < 0$

$\Rightarrow \quad 2\cos 2x - 1 < 0$

$\Rightarrow \quad \cos 2x < 1/2$

$\Rightarrow \quad x \in \left(n\pi + \frac{\pi}{6}, n\pi + \frac{5\pi}{6}\right)$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left(n\pi + \frac{\pi}{6}, n\pi + \frac{5\pi}{6}\right)$$

24. We have $6\sin^2 x - \sin x \cos x - \cos^2 x > 2$

$\Rightarrow \quad 6\sin^2 x - \sin x \cos x - \cos^2 x > 2(\sin^2 x + \cos^2 x)$

$\Rightarrow \quad 4\sin^2 x - \sin x \cos x - 3\cos^2 x > 0$

$\Rightarrow \quad 4\tan^2 x - \tan x - 3 > 0$

$\Rightarrow \quad 4\tan^2 x - 4\tan x + 3\tan x - 3 > 0$

$\Rightarrow \quad 4\tan x(\tan x - 1) + 3(\tan x - 1) > 0$

$\Rightarrow \quad (\tan x - 1)(4\tan x + 3) > 0$

$\Rightarrow \quad \tan x < -3/4$ and $\tan x > 1$

$\Rightarrow \quad x \in \left(n\pi - \frac{\pi}{2}, n\pi - \tan^{-1}\left(\frac{3}{4}\right)\right)$

and $\quad x \in \left(n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{2}\right)$

Hence, the solution set is

$$x = \left(n\pi - \frac{\pi}{2}, n\pi - \tan^{-1}\left(\frac{3}{4}\right)\right)$$

$$\cup\left(n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{2}\right), n \in I$$

25. We have $\sin^6 x + \cos^6 x > \frac{13}{16}$

$\Rightarrow \quad (1 - 3\sin^2 x \cos^2 x) > \frac{13}{16}$

$\Rightarrow \quad \left(1 - \frac{3}{4}(2\sin^2 x)(2\cos^2 x)\right) > \frac{13}{16}$

$\Rightarrow \quad \left(1 - \frac{3}{4}(1 - \cos 2x)(1 + \cos 2x)\right) > \frac{13}{16}$

$\Rightarrow \quad \left(1 - \frac{3}{4}(1 - \cos^2 2x)\right) > \frac{13}{16}$

$\Rightarrow \quad \left(1 - \frac{3}{4}\sin^2 2x\right) > \frac{13}{16}$

$\Rightarrow \quad \left(1 - \frac{3}{8}(2\sin^2 2x)\right) > \frac{13}{16}$

$\Rightarrow \quad \left(1 - \frac{3}{8}(1 - \cos 4x)\right) > \frac{13}{16}$

$\Rightarrow \quad \left(\frac{5}{8} + \frac{3}{8}\cos 4x\right) > \frac{13}{16}$

$\Rightarrow \quad \frac{3}{8}\cos 4x > \frac{3}{16}$

$\Rightarrow \quad \cos 4x > 1/2$

$\Rightarrow \quad 4x \in \left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right)$

$\Rightarrow \quad x \in \left(\frac{n\pi}{2} - \frac{\pi}{12}, \frac{n\pi}{2} + \frac{\pi}{12}\right)$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left(\frac{n\pi}{2} - \frac{\pi}{12}, \frac{n\pi}{2} + \frac{\pi}{12}\right)$$

26. The given in-equation is

$$\cos^3 x \cdot \cos 3x - \sin^3 x \sin 3x > \frac{5}{8}$$

$\Rightarrow \quad (\cos 3x + 3\cos x)\cos 3x$

$\quad - (3\sin x - \sin 3x)\sin 3x > \frac{5}{2}$

$\Rightarrow \quad \sin^2 3x + \cos^2 3x$

$\quad + 3(\cos 3x \cos x - \sin 3x \sin x) > \frac{5}{2}$

$\Rightarrow \quad 3\cos 4x + 1 > \frac{5}{2}$

$\Rightarrow \quad \cos 4x > \frac{1}{2}$

$\Rightarrow \quad 2n\pi - \frac{\pi}{3} < 4x < 2n\pi + \frac{\pi}{3}, n \in I$

$\Rightarrow \quad \frac{n\pi}{2} - \frac{\pi}{12} < x < \frac{n\pi}{2} + \frac{\pi}{12}, n \in I$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left(\left(\frac{n\pi}{2} - \frac{\pi}{12}, \frac{n\pi}{2} + \frac{\pi}{12}\right)\right)$$

27. The given in-equation is

$$\sin^6 x + \cos^6 x > \frac{13}{16}$$

$\Rightarrow \quad (\sin^2 x + \cos^2 x)^3$

$\quad -3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x) > \frac{13}{16}$

$\Rightarrow \quad 1 - 3\sin^2 x \cos^2 x > \frac{13}{16}$

$\Rightarrow \quad 1 - \frac{3}{4}(\sin^2 2x) > \frac{13}{16}$

$\Rightarrow \quad 1 - \frac{3}{8}(2\sin^2 4x) > \frac{13}{16}$

$\Rightarrow \quad 1 - \frac{3}{8}(1 - \cos 4x) > \frac{13}{16}$

$\Rightarrow \quad \frac{5}{8} + \frac{3}{8}\cos 4x > \frac{13}{16}$

$\Rightarrow \quad \frac{3}{8}\cos 4x > \frac{3}{16}$

$\Rightarrow \quad \cos 4x > \frac{1}{2}$

$\Rightarrow \quad 2n\pi - \frac{\pi}{3} < 4x < 2n\pi + \frac{\pi}{3}, n \in I$

$\Rightarrow \quad \frac{n\pi}{2} - \frac{\pi}{12} < x < \frac{n\pi}{2} + \frac{\pi}{12}, n \in I$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left(\frac{n\pi}{2} - \frac{\pi}{12}, \frac{n\pi}{2} + \frac{\pi}{12}\right)$$

28. The given in-equation is

$$\frac{5}{4}\sin^2 x + \frac{1}{4}\sin^2 2x > \cos 2x$$

$\Rightarrow \quad 5\sin^2 x + \sin^2 2x > 4\cos 2x$

$\Rightarrow \quad 5(2\sin^2 x) + 2(\sin^2 2x) > 4\cos 2x$

$\Rightarrow \quad 5(1 - \cos 2x) + 2(1 - \cos^2 2x) > 8\cos 2x$

$\Rightarrow \quad 2\cos^2 2x + 13\cos 2x - 7 < 0$

$\Rightarrow \quad 2\cos^2 2x + 14\cos 2x - \cos 2x - 7 < 0$

$\Rightarrow \quad 2\cos 2x(\cos 2x + 7) - 1(\cos 2x + 7) < 0$

$\Rightarrow \quad (2\cos 2x - 1)(\cos 2x + 7) < 0$

$\Rightarrow \quad -7 < \cos 2x < \frac{1}{2}$

$\Rightarrow \quad \cos 2x < \frac{1}{2}$

$\Rightarrow \quad 2n\pi - \frac{\pi}{3} < 2x < 2n\pi + \frac{\pi}{3}, n \in I$

$\Rightarrow \quad n\pi - \frac{\pi}{6} < x < n\pi + \frac{\pi}{6}, n \in I$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left(n\pi - \frac{\pi}{6}, n\pi - \frac{\pi}{6}\right)$$

29. The given in-equation is
$\sin 3x \sin 4x > \sin x \sin 2x$
$\Rightarrow \quad 2 \sin 3x \sin 4x > 2 \sin x \sin 2x$
$\Rightarrow \quad \cos x - \cos 7x > \cos x - \cos 3x$
$\Rightarrow \quad -\cos 7x > -\cos 3x$
$\Rightarrow \quad \cos 7x < \cos 3x$
$\Rightarrow \quad \cos 7x - \cos 3x < 0$
$\Rightarrow \quad -2 \sin 5x \sin 2x < 0$
$\Rightarrow \quad \sin 5x \sin 2x > 0$
$\Rightarrow \quad \sin 5x > 0$, since $\sin 2x$ is positive in $\left(0, \frac{\pi}{2}\right)$
$\Rightarrow \quad 0 < 5x < \pi$
$\Rightarrow \quad 0 < x < \frac{\pi}{5}$
Hence, the solution set is
$$x = \left(0, \frac{\pi}{5}\right)$$

30. We have $|\sin x + \cos x| = |\sin x| + |\cos x|$
$\Rightarrow \quad \sin x \cos x \geq 0$
$\Rightarrow \quad 2 \sin x \cos x \geq 0$
$\Rightarrow \quad \sin x \geq 0$
$\Rightarrow \quad 2n\pi \leq 2x \leq 2n\pi + \pi, n \in 1$
$\Rightarrow \quad n\pi \leq x \leq n\pi + \frac{\pi}{2}, n \in I$
Hence, the solution set is
$$x = \bigcup_{n \in I}\left(n\pi, n\pi + \frac{\pi}{2}\right)$$

31. We have $|\sec x + \tan x| = |\sec x| + |\tan x|$
$\Rightarrow \quad \sec x \,.\, \tan x \geq 0$
$\Rightarrow \quad \frac{\sin x}{\cos^2 x} \geq 0$
$\Rightarrow \quad \sin x \geq 0, \cos^2 x \neq 0$
$\Rightarrow \quad \sin x \geq 0, x \neq (2n+1)\frac{\pi}{2}, n \in I$
$\Rightarrow \quad 2n\pi \leq x \leq 2n\pi + \pi, x \neq (2n+1)\frac{\pi}{2}, n \in I$
Hence, the solution set is
$$x = \bigcup_{n \in I}(2n\pi, (2n+1)\pi) - (2n+1)\frac{\pi}{2}$$

32. We have $\sin^2 x + \sin x - 2 < 0$
$\Rightarrow \quad (\sin x + 2)(\sin x - 1) < 0$
$\Rightarrow \quad -2 < \sin x < 1$
$\Rightarrow \quad \sin x < 1$
$\Rightarrow \quad -\frac{3\pi}{2} < x < \frac{\pi}{2}$
Also, $x^2 - 3x + 2 < 0$
$\Rightarrow \quad (x - 1)(x - 2) < 0$
$\Rightarrow \quad 1 < x < 2$
Hence, the solution set is $1 < x < \frac{\pi}{2}$

33. We have $2 \sin^2 x + \sin x - 1 < 0$
$\Rightarrow \quad (2 \sin x - 1)(\sin x + 1) < 0$
$\Rightarrow \quad -1 < \sin x < 2$
$\Rightarrow \quad -\frac{\pi}{2} < x < \frac{\pi}{6}$
Also, $x^2 + x - 2 < 0$
$\Rightarrow \quad (x + 2)(x - 1) < 0$
$\Rightarrow \quad -2 < x < 1$
Hence, the solution set is, $-\frac{\pi}{2} < x < \frac{\pi}{6}$

34. We have $\tan^2 x - 5 \tan x + 6 > 0$
$\Rightarrow \quad (\tan x - 2)(\tan x - 3) > 0$
$\Rightarrow \quad \tan x < 2$ and $\tan x > 3$
$\Rightarrow \quad x < \tan^{-1}(2)$ and $x > \tan^{-1}(3)$
Also, $x^2 - 16 \leq 0$
$\Rightarrow \quad (x + 4)(x - 4) \leq 0$
$\Rightarrow \quad -4 \leq x \leq 4$
Hence, the solution set is
$x \in (-4, \tan^{-1}(2)) \cup (\tan^{-1}(3), 4)$

35. We have $\tan^2 x + \tan x - 6 < 0$
$\Rightarrow \quad (\tan x + 3)(\tan x - 2) < 0$
$\Rightarrow \quad -3 < \tan x < 2$
$\Rightarrow \quad \tan^{-1}(-3) < x < \tan^{-1}(2)$
Also, $\frac{x-1}{5-x} < 0$
$\Rightarrow \quad \frac{x-1}{x-5} > 0$
$\Rightarrow \quad x < 1$ and $x > 5$
Hence, the solution set is $x \in (1, \tan^{-1}(2))$

36. We have $[\sin x] = 0$
$\Rightarrow \quad 0 \leq \sin x < 1$
Case-I: When $\sin x \geq 0$
$\Rightarrow \quad 2n\pi \leq x \leq (2n + 1)\pi, n \in I$
Case-II: When $\sin x < 1$
$\Rightarrow \quad 2n\pi - \frac{3\pi}{2} < x < 2n\pi + \frac{\pi}{2}$
$\Rightarrow \quad x \in \left((4n-3)\frac{\pi}{2}, \left((4n+1)\frac{\pi}{2}\right)\right)$
Hence, the solution set is

$$x = \bigcup_{n \in I} \left( \begin{array}{l} [2n\pi, (2n+1)\,\pi] \\ \cup \left( (4n-3)\dfrac{\pi}{2}, (4n+1)\,\dfrac{\pi}{2} \right) \end{array} \right)$$

## Level II

1. $\sin(3x-1) > 0$
   $\Rightarrow \quad 2n\,\pi < (3x-1) < (2n+1)\,\pi$
   $\Rightarrow \quad \dfrac{2n\pi+1}{3} < x < \dfrac{(2n+1)\pi+1}{3},\ n \in I$
2. $\cos(2x-3) < 0$
   $\Rightarrow \quad 2n\pi + \dfrac{\pi}{2} < (2x-3) < 2n\pi + \dfrac{3\pi}{2}$
   $\Rightarrow \quad 2n\pi + \dfrac{\pi}{4} + \dfrac{3}{2} < x < n\pi + \dfrac{3\pi}{4} + \dfrac{3}{2}$
3. $|\sin x| \le \dfrac{1}{2}$
   $\Rightarrow \quad n\pi - \dfrac{\pi}{6} \le x \le n\pi + \dfrac{\pi}{6},\ n \in I$
4. $|\cos x| \le \dfrac{1}{\sqrt{2}}$
   $\Rightarrow \quad n\pi - \dfrac{3\pi}{4} \le x \le n\pi + \dfrac{\pi}{4},\ n \in I$
5. $|\sin 2x + \cos 2x| = |\sin 2x| + |\cos 2x|$
   $\Rightarrow \quad \sin(2x)\cos(2x) \ge 0$
   $\Rightarrow \quad 2\sin(2x)\cos(2x) \ge 0$
   $\Rightarrow \quad \sin(4x) \ge 0$
   $\Rightarrow \quad 2n\,\pi \le 4x \le (2n+1)\pi,$
   $\Rightarrow \quad \dfrac{2n\pi}{4} \le x \le \dfrac{(2n+1)\,\pi}{4},\ n \in I$
6. $2\cos^2 x + \cos x < 1$
   $\Rightarrow \quad 2\cos^2 x + 2\cos x - \cos x - 1 < 0$
   $\Rightarrow \quad 2\cos x(\cos x + 1) - (\cos x + 1) < 0$
   $\Rightarrow \quad (2\cos x - 1)(\cos x + 1) < 0$
   $\Rightarrow \quad -1 < \cos x < \dfrac{1}{2}$
   $\Rightarrow \quad (2n+1)\,\pi < x < 2n\pi + \dfrac{\pi}{6},\ n \in I$
7. $4\sin^2 x - 1 \le 0$
   $\Rightarrow \quad \sin^2 x \le \dfrac{1}{4}$
   $\Rightarrow \quad |\sin x| \le \dfrac{1}{2}$
   $\Rightarrow \quad n\pi - \dfrac{\pi}{6} \le x \le n\pi + \dfrac{\pi}{6},\ n \in I$
8. $4\cos^2 x - 3 \ge 0$
   $\Rightarrow \quad \cos^2 x \ge \dfrac{3}{4}$
   $\Rightarrow \quad |\cos x| \ge \dfrac{\sqrt{3}}{2}$
   $\Rightarrow \quad n\pi - \dfrac{\pi}{6} \le x \le n\pi + \dfrac{\pi}{6},\ n \in I$
9. $|\sin x| > |\cos x|$
   $\Rightarrow \quad |\tan x| > 1$
   $\Rightarrow \quad x \in \bigcup_{n \in I} \left\{ \left( n\pi - \dfrac{\pi}{2}, n\pi - \dfrac{\pi}{4} \right) \cup \left( n\pi + \dfrac{\pi}{4}, n\pi + \dfrac{\pi}{2} \right) \right\}$
10. $|\cos x| > |\sin x|$
    $\Rightarrow \quad |\tan x| < 1$
    $\Rightarrow \quad x \in \left( n\pi - \dfrac{\pi}{4}, n\pi - \dfrac{\pi}{4} \right)$
11. $\sin x + \cos x > 1$
    $\Rightarrow \quad \sin\left( x + \dfrac{\pi}{4} \right) > \dfrac{1}{\sqrt{2}}$
    $\Rightarrow \quad 2n\pi + \dfrac{\pi}{4} < \left( x + \dfrac{\pi}{4} \right) < 2\,n\pi + \dfrac{3\pi}{4}$
    $\Rightarrow \quad 2n\,\pi < x < 2\,n\pi + \dfrac{\pi}{2},\ n \in I$
12. $\sin x - \cos x < 1$
    $\Rightarrow \quad \sin\left( x - \dfrac{\pi}{4} \right) < \dfrac{1}{\sqrt{2}}$
    $\Rightarrow \quad 2n\pi - \dfrac{5\pi}{4} < \left( x - \dfrac{\pi}{4} \right) < 2\,n\pi + \dfrac{\pi}{4}$
    $\Rightarrow \quad (2n-1)\pi < x < 2n\,\pi + \dfrac{\pi}{2},\ n \in I$
13. $\sqrt{3}\sin x + \cos x > 1$
    $\Rightarrow \quad \sin\left( x + \dfrac{\pi}{6} \right) > \dfrac{1}{2}$
    $\Rightarrow \quad 2n\pi + \dfrac{\pi}{6} < \left( x + \dfrac{\pi}{6} \right) < 2n\,\pi + \dfrac{5\pi}{6}$
    $\Rightarrow \quad 2n\,\pi < x < (2n+1)\pi,\ n \in I$
14. $\sin x - \sqrt{3}\cos x < 1$
    $\Rightarrow \quad \sin\left( x - \dfrac{\pi}{3} \right) < \dfrac{1}{2}$
    $\Rightarrow \quad 2n\,\pi + \dfrac{\pi}{6} < \left( x - \dfrac{\pi}{3} \right) < 2n\,\pi + \dfrac{5\pi}{6}$
    $\Rightarrow \quad 2n\,\pi + \dfrac{\pi}{2} < x < (2n+1)\pi,\ n \in I$
15. $\sin^2 x + \sin x - 2 < 0$
    $\Rightarrow \quad (\sin x + 2)(\sin x - 1) < 0$
    $\Rightarrow \quad -2 < \sin x < 1$

$\Rightarrow \quad -1 < \sin x < 1$

$\Rightarrow \quad 2n\pi - \dfrac{\pi}{2} < x < 2n\pi + \dfrac{\pi}{2}, n \in I$

16. $\sin^2 x + 3\sin x + 2 < 0$

$\Rightarrow \quad (\sin x + 1)(\sin x + 2) < 0$

$\Rightarrow \quad -2 < \sin x < 1$

$\Rightarrow \quad x = \varphi$

17. $\cos^2 x - \cos x > 0$

$\Rightarrow \quad \cos x(\cos x - 1) > 0$

$\Rightarrow \quad \cos x < 0, \cos x > 1$

$\Rightarrow \quad \cos x < 0$

$\Rightarrow \quad 2n\pi + \dfrac{\pi}{2} < x < 2n\pi + \dfrac{3\pi}{2}, n \in I$

18. $\sin x + \cos x - \cos 2x > 0$

$\Rightarrow \quad (\sin x - \cos x) - (\cos^2 x - \sin^2 x) > 0$

$\Rightarrow \quad (\sin x + \cos x)(1 - \cos x + \sin x) > 0$

$\Rightarrow \quad (\sin x + \cos x)(\sin x - \cos x + 1) > 0$

**Case I:**

$(\sin x + \cos x) > 0, (\sin x - \cos x + 1) > 0$

$\Rightarrow \quad \sin\left(x + \dfrac{\pi}{4}\right) > 0, \sin\left(x - \dfrac{\pi}{4}\right) > -\dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad 0 < \left(x + \dfrac{\pi}{4}\right) < \pi, -\dfrac{\pi}{4} < \left(x - \dfrac{\pi}{4}\right) < \dfrac{\pi}{2}$

$\Rightarrow \quad -\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}, 0 < x < \dfrac{3\pi}{4}$

$\Rightarrow \quad 0 < x < \dfrac{3\pi}{4}$

$\Rightarrow \quad x \in \left(0, \dfrac{3\pi}{4}\right)$

**Case II:**

$\sin x + \cos x < 0, \sin x - \cos x + 1 < 0$

$\Rightarrow \quad \sin\left(x + \dfrac{\pi}{4}\right) < 0, \sin\left(x - \dfrac{\pi}{4}\right) < -\dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \pi < \left(x + \dfrac{\pi}{4}\right) < 2\pi, \dfrac{5\pi}{4} < \left(x - \dfrac{\pi}{4}\right) < \dfrac{7\pi}{4}$

$\Rightarrow \quad \dfrac{3\pi}{4} < x < \dfrac{7\pi}{4}, \dfrac{3\pi}{2} < x < 2\pi$

$\Rightarrow \quad \dfrac{3\pi}{4} < x < \dfrac{7\pi}{4}$

$\Rightarrow \quad x \in \left(\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right)$

Hence, the solution set is

$$x \in \left(0, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$$

19. $x^2 + x - 2 < 0$ and $\sin x > \dfrac{1}{2}$

when $x^2 + x - 2 < 0$

$\Rightarrow \quad (x + 2)(x - 1) < 0$

$\Rightarrow \quad -2 < x < 1$

when $\sin x > \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{\pi}{6} < x < \dfrac{5\pi}{6}$

Hence, the solution set is $\dfrac{\pi}{6} < x < 1$

i.e., $\quad x \in \left(\dfrac{\pi}{6}, 1\right)$

20. $x^2 - 1 \le 0$ and $\cos x < \dfrac{1}{2}$

when $x^2 - 1 \le 0$

$\Rightarrow \quad (x - 1)(x + 1) \le 0$

$\Rightarrow \quad -1 \le x \le 1$

when $\cos x < \dfrac{1}{2}$

$\Rightarrow \quad -\dfrac{11\pi}{6} < x < -\dfrac{\pi}{3}$

Hence, the solution is $x = \varphi$

21. $4x^2 - 1 \ge 0$ and $\tan x \ge \dfrac{1}{\sqrt{3}}$

when $4x^2 - 1 \ge 0$

$\Rightarrow \quad (2x - 1)(2x + 1) \ge 0$

$\Rightarrow \quad x \le -\dfrac{1}{2}$ and $x \ge \dfrac{1}{2}$

when $\tan x \ge \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \dfrac{\pi}{6} \le x \le \dfrac{\pi}{2}$

Hence, the solution set is $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$

22. $x^2 - 3x + 2 < 0$ and $(\sin x)^2 - \sin x > 0$

when $x^2 - 3x + 2 < 0$

$\Rightarrow \quad (x - 1)(x - 2) < 0$

$\Rightarrow \quad 1 < x < 2$

when $(\sin x)^2 - \sin x > 0$

$\Rightarrow \quad \sin x(\sin x - 1) > 0$

$\Rightarrow \quad \sin x < 0, \sin x > 1$

$\Rightarrow \quad \sin x < 0$

$\Rightarrow \quad \pi < x < 2\pi$

Hence, the solution set is $x = \varphi$

## Level III

1. $\dfrac{\sin x+\cos x}{\sin x-\cos x}>\sqrt{3}$

$\Rightarrow \quad \tan\left(\dfrac{\pi}{4}+x\right)<-\sqrt{3}$

$\Rightarrow \quad n\pi-\dfrac{\pi}{2}<\left(\dfrac{\pi}{4}+x\right)<n\pi-\dfrac{\pi}{3}$

$\Rightarrow \quad n\pi-\dfrac{3\pi}{4}<x<n\pi-\dfrac{7\pi}{12}, n\in I$

2. $|\sin x|>|\cos x|$

$\Rightarrow \quad |\tan x|>1$

$\Rightarrow \quad x\in\bigcup_{n\in I}\left(n\pi-\dfrac{\pi}{2}, n\pi-\dfrac{\pi}{4}\right)\cup\left(n\pi+\dfrac{\pi}{4}, n\pi+\dfrac{\pi}{2}\right)$

3. $\cot x+\dfrac{\sin x}{\cos x-2}\geq 0$

$\Rightarrow \quad \dfrac{\cos^2 x-2\cos x+\sin^2 x}{\sin x(\cos x-2)}\geq 0$

$\Rightarrow \quad \dfrac{1-2\cos x}{\sin x(\cos x-2)}\geq 0$

$\Rightarrow \quad \dfrac{1-2\cos x}{\sin x}\leq 0$, since $(\cos x-2)<0$

$\Rightarrow \quad \dfrac{1-2\cos x}{\sin x}\leq 0$

$\Rightarrow \quad \dfrac{2\cos x-1}{\sin x}\geq 0$

**Case I:**

when $(2\cos x-1)\geq 0$ and $\sin x\geq 0$

$\Rightarrow \quad \cos x\geq\dfrac{1}{2}$ and $\sin x\geq 0$

$\Rightarrow \quad x\in\left[0,\dfrac{\pi}{3}\right]\cup\left[\dfrac{11\pi}{6}, 2\pi\right], 0<x<\pi$

Hence, the solution is $x\in\left(0,\dfrac{\pi}{3}\right]$

**Case II:**

when $(2\cos x-1)\leq 0$ and $\sin x\leq 0$

$\Rightarrow \quad \cos x\leq\dfrac{1}{2}, \sin x<0$

$\Rightarrow \quad \dfrac{\pi}{3}\leq x\leq\dfrac{5\pi}{3}, \pi<x<2\pi$

Hence, the solution is $\pi<x\leq\dfrac{5\pi}{3}$.

Therefore, the required solution set is

$x\in\left(0,\dfrac{\pi}{3}\right]\cup\left(\pi,\dfrac{5\pi}{3}\right]$

4. $\sin x+\cos x>\sqrt{2}\cos 2x$

$\Rightarrow \quad \sin x+\cos x>\sqrt{2}(\cos^2 x-\sin^2 x)$

$\Rightarrow \quad (\sin x+\cos x)(1-\sqrt{2}(\cos x-\sin x))>0$

$\Rightarrow \quad (\sin x+\cos x)\sqrt{2}(\cos x-\sin x)-1)<0$

$\Rightarrow \quad (\sin x+\cos x)\left((\cos x-\sin x)-\dfrac{1}{\sqrt{2}}\right)<0$

**Case I:**

$(\sin x+\cos x)>0, \left((\cos x-\sin x)-\dfrac{1}{\sqrt{2}}\right)<0$

$\Rightarrow \quad \sin\left(x+\dfrac{\pi}{4}\right)>0, \left(\cos\left(x+\dfrac{\pi}{4}\right)-\dfrac{1}{\sqrt{2}}\right)<0$

$\Rightarrow \quad \pi<\left(x+\dfrac{\pi}{4}\right)<2\pi, \dfrac{\pi}{4}<\left(x+\dfrac{\pi}{4}\right)<\dfrac{7\pi}{4}$

$\Rightarrow \quad \dfrac{3\pi}{4}<x<\dfrac{7\pi}{4}, 0<x<\dfrac{3\pi}{2}$

$\Rightarrow \quad \dfrac{3\pi}{4}<x<\dfrac{7\pi}{4}$

Hence, the solution set is $\sin x+\cos x\in\left(\dfrac{3\pi}{4},\dfrac{7\pi}{4}\right)$

**Case II:**

$\sin\left(x+\dfrac{\pi}{4}\right)<0, \cos\left(x+\dfrac{\pi}{4}\right)>\dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \pi<\left(x+\dfrac{\pi}{4}\right)<2\pi, \dfrac{7\pi}{4}<\left(x+\dfrac{\pi}{4}\right)<\dfrac{9\pi}{4}$

$\Rightarrow \quad \dfrac{3\pi}{4}<x<\dfrac{7\pi}{4}, \dfrac{3\pi}{2}<x<2\pi$

Hence, the solution set is

$\dfrac{3\pi}{2}<x<\dfrac{7\pi}{4}$

Therefore, the required solution set is

$x\in\left(\dfrac{3\pi}{2},\dfrac{7\pi}{4}\right)$

5. $4\sin x\sin 2x\sin 3x>\sin 4x$

$\Rightarrow \quad 4\sin x\sin 2x\sin 3x>2\sin 2x\cos 2x$

$\Rightarrow \quad 2\sin x\sin 2x\sin 3x>\sin 2x\cos 2x$

$\Rightarrow \quad \sin 2x(2\sin x\sin 3x-\cos 2x)>0$

$\Rightarrow \quad \sin 2x(\cos 2x-\cos 4x-\cos 2x)>0$

$\Rightarrow \quad \sin 2x(-\cos 4x)>0$

$\Rightarrow \quad \sin 2x(\cos 4x)<0$

**Case I:**

when $\sin 2x>0, (\cos 4x)<0$

$\Rightarrow \quad 0<2x<\pi, \dfrac{\pi}{2}<4x<\dfrac{3\pi}{2}$

$$\Rightarrow \quad 0 < x < \frac{\pi}{2}, \frac{\pi}{8} < x < \frac{3\pi}{8}$$

Hence, the solution set is $\frac{\pi}{8} < x < \frac{3\pi}{8}$.

**Case II:**

When $\sin 2x < 0$, $\cos 4x > 0$

$$\Rightarrow \quad \pi < 2x < 2\pi, \frac{3\pi}{2} < 4x < \frac{5\pi}{2}$$

$$\Rightarrow \quad \frac{\pi}{2} < x < \pi, \frac{3\pi}{8} < x < \frac{5\pi}{8}$$

Hence, the solution set is $\frac{\pi}{2} < x < \frac{5\pi}{8}$

Therefore, the required solution is

$$x \in \left(\frac{\pi}{8}, \frac{3\pi}{8}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{8}\right)$$

6. $\dfrac{\cos^2 2x}{\cos^2 x} \geq 3 \tan x$

$$\Rightarrow \quad \left(\frac{\cos^2 2x}{\cos^2 x}\right) \geq \frac{3\sin x}{\cos x}$$

$$\Rightarrow \quad \left(\frac{\cos^2 2x - 3\sin x \cos x}{\cos^2 x}\right) \geq 0$$

$$\Rightarrow \quad \left(\frac{2\cos^2 2x - 3\sin 2x}{\cos^2 x}\right) \geq 0$$

$$\Rightarrow \quad 2\cos^2 2x - 3\sin 2x \geq 0, x \neq (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad 2\cos^2 2x - 3\sin 2x \geq 0$$

$$\Rightarrow \quad 2 - 2\sin^2 2x - 3\sin 2x \geq 0$$

$$\Rightarrow \quad 2\sin^2 2x + 3\sin 2x - 2 \leq 0$$

$$\Rightarrow \quad 2\sin^2 2x + 4\sin 2x - \sin 2x - 2 \leq 0$$

$$\Rightarrow \quad 2\sin 2x(\sin 2x + 2) - (\sin 2x + 2) \leq 0$$

$$\Rightarrow \quad (2\sin 2x - 1)(\sin 2x + 2) \leq 0$$

$$\Rightarrow \quad -2 \leq \sin 2x \leq \frac{1}{2}$$

$$\Rightarrow \quad -1 \leq \sin 2x \leq \frac{1}{2}$$

$$\Rightarrow \quad 2n\pi - \frac{\pi}{2} \leq 2x \leq 2n\pi + \frac{\pi}{6}$$

$$\Rightarrow \quad n\pi - \frac{\pi}{4} \leq x \leq n\pi + \frac{\pi}{12}, n \in I$$

Hence, the solution set is

$$\left[n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{12}\right] - \left\{(2n+1)\frac{\pi}{2}\right\}, n \in I$$

7. $\dfrac{\cos x + 2\cos^2 x + \cos 3x}{\cos x + 2\cos^2 x - 1} > 1$

$$\Rightarrow \quad \frac{\cos x + 2\cos^2 x + 4\cos^3 x - 3\cos x}{2\cos^2 x + \cos x - 1} > 1$$

$$\Rightarrow \quad \frac{\cos x + 2\cos^2 x + 4\cos^3 x - 3\cos x}{2\cos^2 x + \cos x - 1} - 1 > 0$$

$$\Rightarrow \quad \frac{\cos x + 2\cos^2 x + 4\cos^3 x - 3\cos x}{2\cos^2 x + \cos x - 1} > 1$$

$$\Rightarrow \quad \frac{2\cos x(2\cos^2 x + \cos x - 1)}{2\cos^2 x + \cos x - 1} > 1$$

$$\Rightarrow \quad 2\cos x > 1$$

$$\Rightarrow \quad \cos x > \frac{1}{2}$$

$$\Rightarrow \quad 2n\pi - \frac{\pi}{3} < x < 2n\pi + \frac{\pi}{3}, n \in I$$

8. $2(\sqrt{2} - 1)\sin x - 2\cos 2x + \sqrt{2}(\sqrt{2} - 1) < 0$

$$\Rightarrow \quad 2(\sqrt{2} - 1)\sin x - 2(1 - 2\sin^2 x) + \sqrt{2}(\sqrt{2} - 1) < 0$$

$$\Rightarrow \quad (\sqrt{2} - 1)\sin x - (1 - 2\sin^2 x) + \frac{(\sqrt{2} - 1)}{\sqrt{2}} < 0$$

$$\Rightarrow \quad 2\sin^2 x + (\sqrt{2} - 1)\sin x + \frac{(\sqrt{2} - 1)}{\sqrt{2}} - 1 < 0$$

$$\Rightarrow \quad 2\sqrt{2}\sin^2 x + \sqrt{2}(\sqrt{2} - 1)\sin x - 1 < 0$$

$$\Rightarrow \quad 2\sqrt{2}\sin^2 x + (2 - \sqrt{2})\sin x - 1 < 0$$

$$\Rightarrow \quad 2\sqrt{2}\sin^2 x + 2\sin x - \sqrt{2}\sin x - 1 < 0$$

$$\Rightarrow \quad 2\sin x(\sqrt{2}\sin x + 1) - (\sqrt{2}\sin x + 1) < 0$$

$$\Rightarrow \quad (2\sin x - 1)(\sqrt{2}\sin x + 1) < 0$$

$$\Rightarrow \quad -\frac{1}{\sqrt{2}} < \sin x < \frac{1}{2}$$

$$\Rightarrow \quad 2n\pi - \frac{\pi}{4} < x < 2n\pi + \frac{\pi}{6}, n \in \mathrm{I}$$

9. $\sin 2x > \sqrt{2}\sin^2 x + (2 - \sqrt{2})\cos^2 x$, $x \in (0, 2\pi)$

$$\Rightarrow \quad \sqrt{2}\sin^2 x - 2\sin x\cos x + (2 - \sqrt{2})\cos^2 x < 0$$

$$\Rightarrow \quad \sqrt{2}\tan^2 x - 2\tan x + (2 - \sqrt{2}) < 0$$

$$\Rightarrow \quad \tan^2 x - \sqrt{2}\tan x + (\sqrt{2} - 1) < 0$$

$$\Rightarrow \quad \tan x = \frac{\sqrt{2} \pm \sqrt{2 - 4(\sqrt{2} - 1)}}{2} < 0$$

$$\Rightarrow \quad \tan x = \frac{\sqrt{2} \pm \sqrt{6 - 4\sqrt{2}}}{2} < 0$$

$$\Rightarrow \quad \tan x = \frac{1 \pm (2 - \sqrt{2})}{\sqrt{2}} < 0$$

$\Rightarrow \quad \tan x = \frac{3-\sqrt{2}}{\sqrt{2}}, \frac{\sqrt{2}-1}{\sqrt{2}} < 0$

$\Rightarrow \quad \frac{\sqrt{2}-1}{\sqrt{2}} < \tan x < \frac{3-\sqrt{2}}{\sqrt{2}}$

$\Rightarrow \quad n\pi + \alpha < x < n\pi + \beta, \alpha = \frac{\sqrt{2}-1}{\sqrt{2}}, \tan\beta = \frac{3-\sqrt{2}}{\sqrt{2}}$

10. $1 + \log_4 \sin x + 2 \log_{16} \cos x > 0$

$\Rightarrow \quad \log_4 \sin x + 2 \log_{4^2} \cos x + 1 > 0$

$\Rightarrow \quad \log_4 \sin x + \log_4 \cos x + 1 > 0$

$\Rightarrow \quad \log_2 \sin x + \log_2 \cos x + 2 > 0$

$\Rightarrow \quad \log_2 \left(\frac{\sin 2x}{2}\right) + 2 > 0$

$\Rightarrow \quad \log_2 \sin 2x + 1 > 0$

$\Rightarrow \quad \log_2 \sin 2x > -1$

$\Rightarrow \quad \sin 2x > \frac{1}{2}$

$\Rightarrow \quad 2n\pi + \frac{\pi}{6} < 2x < 2n\pi + \frac{5\pi}{6}, n \in I$

$\Rightarrow \quad n\pi + \frac{\pi}{12} < x < n\pi + \frac{5\pi}{12}, n \in I$

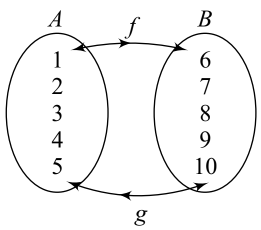
CHAPTER

# 4 Inverse Trigonometric Functions

## INVERSE FUNCTION

### CONCEPT BOOSTER

### 4.1 INTRODUCTION TO INVERSE FUNCTION




Let $f: X \to Y$ be a bijective function.

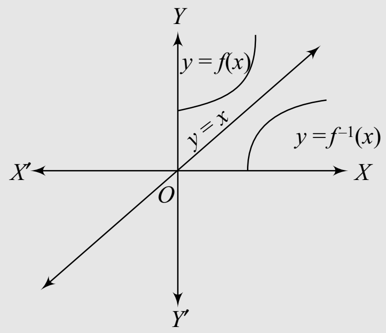
If we can make another function $g$ from $Y$ to $X$, then we shall say that $g$ is the inverse of $f$.

i.e., $g = f^{-1} \neq \dfrac{1}{f}$

Thus, $f^{-1}(f(x)) = x$

**Note**

(i) The inverse of a function exists only when the function $f$ is bijective.
(ii) If the inverse of a function is exists, then it is called an invertible function.
(iii) The inverse of a bijective function is unique.
(iv) Geometrically $f^{-1}(x)$ is the image of $f(x)$ with respect to the line $y = x$.
(v) Another way also we can say that $f^{-1}(x)$ is the symmetrical with respect to the line $y = x$.
(vi) A function $f(x)$ is said to be involution if for all $x$ for which $f(x)$ and $f(f(x))$ are defined such that $f(f(x)) = x$.




(vii) If $f$ is an invertible function, then $(f^{-1})^{-1} = f$.
(viii) If $f: A \to B$ be a one one function, then $f^{-1}of = I_A$ and $fof^{-1} = I_B$, where $I_A$ and $I_B$ are the identity functions of the sets A and B respectively.
(ix) Let $f: A \to B$, $g: B \to C$ be two invertible functions, then $gof$ is also invertible with $(gof)^{-1} = (f^{-1}og^{-1})$.

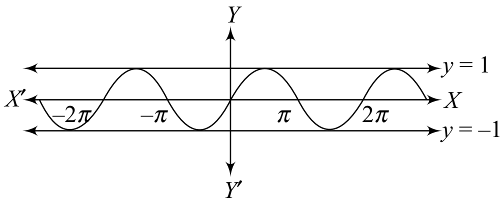
### Rule to Find out the Inverse of a Function

(i) First, we check the given function is bijective or not.
(ii) If the function is bijective, then inverse exists, otherwise not.
(iii) Find $x$ in terms of $y$
(iv) And then replace $y$ by $x$, then we get inverse of $f$. i.e., $f^{-1}(x)$.

### 4.2 INVERSE TRIGONOMETRIC FUNCTIONS

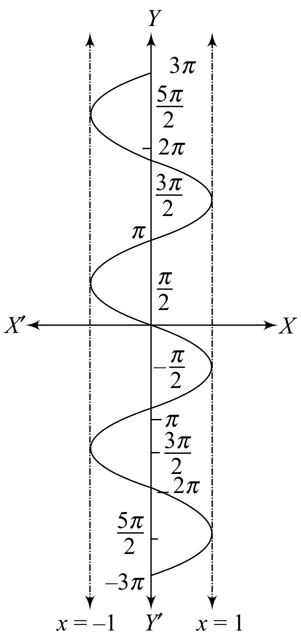
We know that sine function is defined only for every real number and the range of sine function is [–1, 1]. Thus, the graph of $f(x) = \sin(x)$ is as follows

**Graph of $f(x) = \sin(x)$:**




From the graph, we can say that, it will be one one and onto only when we considered it in some particular intervals like $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, and so on. If we consider the whole function, then it is not one one as well as onto.

Also, when we think the inverse function, then domain and range are interchanged. So the graph of this function is as follows.




As a whole, inverse of this function does not exist. Its inverse exists only when, we restrict its range.

So the intervals are $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, and so on.

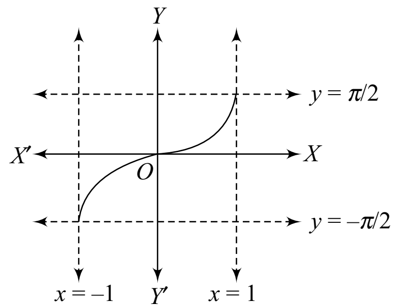
In the conventional mathematics, we consider it in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Thus, sin inverse function is defined as

$$\sin^{-1} : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Therefore, a function $f : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined as $f(x) = \sin^{-1}x$.

So the graph of $f(x) = \sin^{-1}x$ is




Thus, $D_f = [-1, 1]$ and $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

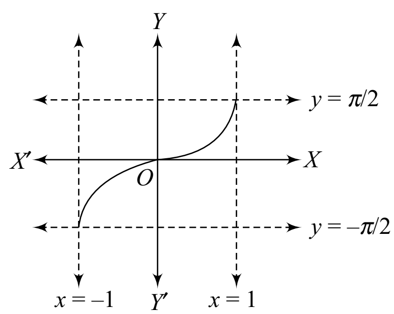
Now, we shall discuss the graphs of other inverse trigonometric functions and their characteristics.

## 4.3 Graphs of Inverse Trigonometric Functions

(i) $\sin^{-1}x$:

A function $f : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined as $f(x) = \sin^{-1}x = \text{arc} \sin x$

Graph of $f(x) = \sin^{-1}x$.



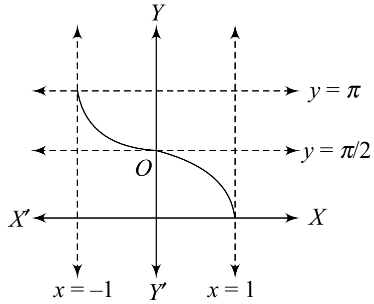

### Characteristics of ARC Sine Function

1. $D_f = [-1, 1]$
2. $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
3. It is not a periodic function.
4. It is an odd function.
   since, $\sin^{-1}(-x) = -\sin^{-1}x$
5. It is a strictly increasing function.
6. It is a one one function.
7. For $0 < x < \frac{\pi}{2}$, $\sin x < x < \sin^{-1}x$.

(ii) $\cos^{-1}x$:

A function $f : [-1, 1] \rightarrow [0, \pi]$
is defined as $f(x) = \cos^{-1}x = \text{arc} \cos x$

**Graph of** $f(x) = \cos^{-1}x$




## Characteristics of ARC Cosine Function

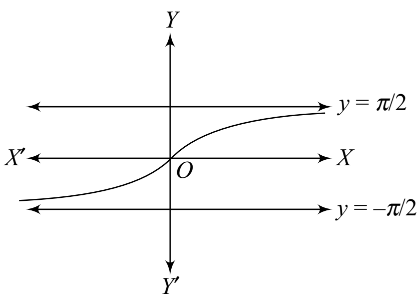
1. $D_f = [-1, 1]$
2. $[0, \pi]$
3. It is not a periodic function.
4. It is neither even nor odd function since, $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$
5. It is a strictly decreasing function.
6. It is a one one function.
7. For $0 < x < \dfrac{\pi}{2}$,

   $\cos^{-1}x < x < \cos x$

(iii) $\tan^{-1}x$:

A function $f : R \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is defined as $f(x) = \tan^{-1}x$.

**Graph of** $f(x) = \tan^{-1}x$:



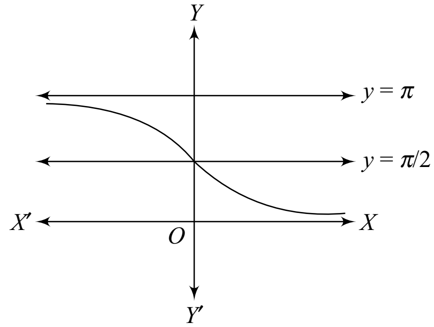

## Characteristics of ARC Tangent Function

1. $D_f = R$
2. $R_f = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
3. It is not a periodic function
4. It is an odd function.
   Since, $\tan^{-1}(-x) = -\tan^{-1}x$
5. It is a strictly increasing function.
6. It is a one one function.
7. For $0 < x < \dfrac{\pi}{2}$, $\tan^{-1}x < x < \tan x$.

(iv) $\cot^{-1}x$:

A function $f: R \to (0, \pi)$ is defined as $f(x) = \cot^{-1}x$.

**Graph of** $f(x) = \cot^{-1}x$




## Characteristics of ARC Co-tangent Function

1. $D_f = R$
2. $R_f = (0, \pi)$
3. It is not a periodic function.
4. It is neither even nor odd function since, $\cot^{-1}(-x) = \pi - \cot^{-1}x$
5. It is a strictly decreasing function.
6. It is a one one function.
7. For $0 < x < \dfrac{\pi}{2}$,

   $\cot x < x < \cot^{-1}x$

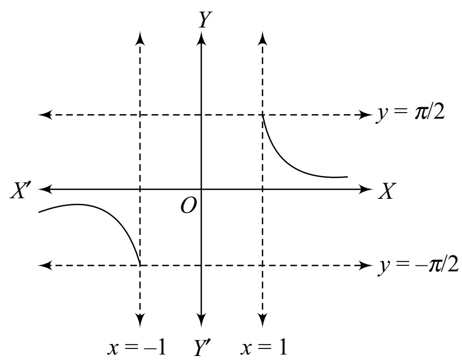
(v) $\text{cosec}^{-1}x$:

A function

$$f : (-\infty, -1] \cup [1, \infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

is defined as $f(x) = \text{cosec}^{-1}x$.

Graph of $f(x) = \text{cosec}^{-1}x$.




## Characteristics of ARC Co-secant Function

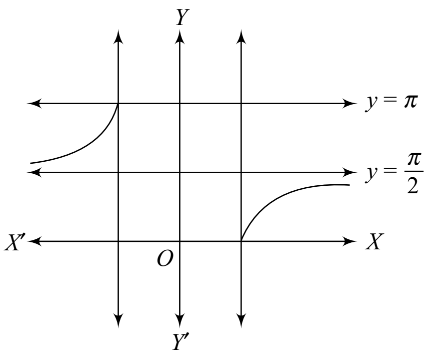
1. $D_f = (-\infty, -1] \cup [1, \infty)$
2. $R_f = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$
3. It is an odd function, since

   $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$
4. It is a non periodic function.
5. It is a one one function.
6. It is a strictly decreasing function with respect to its domain.

7. For $0 < x < \frac{\pi}{2}$, $\text{cosec}^{-1} x < x < \text{cosec } x$

(v) $\sec^{-1}x$: A function $f:(-\infty, -1] \cup [1, \infty) \rightarrow [0, \pi] - \left\{\frac{\pi}{2}\right\}$ is defined as $f(x) = \sec^{-1} x$

Graph of $f(x) = \sec^{-1} x$




## Characteristics of ARC Secant Function

1. $D_f = (-\infty, -1] \cup [1, \infty]$
2. $R_f = [0, \pi] - \left\{\frac{\pi}{2}\right\}$
3. It is neither an even function nor an odd odd function, since $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$
4. It is a non periodic function.
5. It is a one one function.
6. It is strictly decreasing function with respect to its domain.
7. For $0 < x < \frac{\pi}{2}$, $\sec^{-1} x < x < \sec x$

## 4.4 Constant Property

(i) $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}, \forall x \in [-1, 1]$

(ii) $\tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}, \forall x \in R$

(iii) $\text{cosec}^{-1}(x) + \sec^{-1}(x) = \frac{\pi}{2}, \forall x \in R - (-1, 1)$

## 4.5 Conversion of Inverse Trigonometric Functions

**Case I:** When $x > 0$

| Functions | Principal values |
|---|---|
| 1. $\sin^{-1}x$ | $\left[0, \frac{\pi}{2}\right]$ |
| 2. $\cos^{-1}x$ | $\left[0, \frac{\pi}{2}\right]$ |
| 3. $\tan^{-1}x$ | $\left(0, \frac{\pi}{2}\right)$ |
| 4. $\cot^{-1}x$ | $\left(0, \frac{\pi}{2}\right]$ |
| 5. $\text{cosec}^{-1}x$ | $\left(0, \frac{\pi}{2}\right]$ |
| 6. $\sec^{-1}x$ | $\left[0, \frac{\pi}{2}\right)$ |

**Case II:** When $x < 0$

| Functions | Principal values |
|---|---|
| 1. $\sin^{-1}x$ | $\left[-\frac{\pi}{2}, 0\right]$ |
| 2. $\text{cosec}^{-1}x$ | $\left[-\frac{\pi}{2}, 0\right)$ |
| 3. $\tan^{-1}x$ | $\left(-\frac{\pi}{2}, 0\right)$ |
| 4. $\cos^{-1}x$ | $\left[\frac{\pi}{2}, \pi\right]$ |
| 5. $\sec^{-1}x$ | $\left(\frac{\pi}{2}, \pi\right]$ |
| 6. $\cot^{-1}x$ | $\left[\frac{\pi}{2}, \pi\right)$ |

Here, we shall discuss, how any inverse trigonometric function can be expressed in terms of any other inverse trigonometric functions.

**Step I:**

(i) $\sin^{-1}(x) = \text{cosec}^{-1}\left(\frac{1}{x}\right), x \in [-1, 1] - \{0\}$

(ii) $\text{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right), |x| \geq 1$

(iii) $\cos^{-1}(x) = \sec^{-1}\left(\frac{1}{x}\right), x \in [-1, 1] - \{0\}$

(iv) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right), |x| \geq 1$

(v) $\tan^{-1}(x) = \begin{cases} \cot^{-1}\left(\frac{1}{x}\right) : x > 0 \\ -\pi + \cot^{-1}\left(\frac{1}{x}\right) : x < 0 \end{cases}$

(vi) $\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) \quad : x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right) : x < 0 \end{cases}$

**Step II:**

(i) $\sin^{-1}x = \begin{cases} \cos^{-1}\left(\sqrt{1-x^2}\right) & : \quad 0 \le x \le 1 \\ -\cos^{-1}(\sqrt{1-x^2}) & : -1 \le x < 0 \end{cases}$

(ii) $\sin^{-1}x = \begin{cases} \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & : \quad 0 \le x \le 1 \\ -\sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & : -1 \le x < 0 \end{cases}$

(iii) $\cos^{-1}x = \begin{cases} \sin^{-1}(\sqrt{1-x^2}) & : \quad 0 \le x \le 1 \\ \pi - \sin^{-1}(\sqrt{1-x^2}) & : -1 \le x < 0 \end{cases}$

(iv) $\cos^{-1}x = \begin{cases} \operatorname{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & : \quad 0 < x \le 1 \\ \pi - \operatorname{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & : -1 \le x < 0 \end{cases}$

(v) $\sin^{-1}x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) : -1 < x < 1$

(vi) $\sin^{-1}x = \begin{cases} \cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) & : 0 < x \le 1 \\ -\pi + \cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) & : -1 \le x < 0 \end{cases}$

## 4.6 Composition of Trigonometric Functions and its Inverse

Let $y = \sin^{-1}x$

$\Rightarrow \quad x = \sin y$

$\Rightarrow \quad x = \sin(\sin^{-1}x)$

$\Rightarrow \quad \sin(\sin^{-1}x) = x$

Therefore, $\sin(\sin^{-1}x)$ provide us a real value lies in $[-1, 1]$

Hence,

(i) $\sin(\sin^{-1}x) = x, |x| \le 1$

(ii) $\cos(\cos^{-1}x) = x, |x| \le 1$

(iii) $\tan(\tan^{-1}x) = x, x \in R$

(iv) $\cot(\cot^{-1}x) = x, x \in R$

(v) $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x, |x| \ge 1$

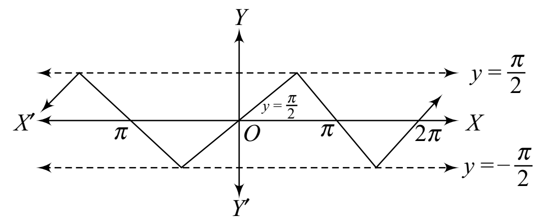
(vi) $\sec(\sec^{-1}x) = x, |x| \ge 1$

## 4.7 Composition of Inverse Trigonometric Functions and Trigonometric Functions

(i) A function $f : R \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ is

defined as $f(x) = \sin^{-1}(\sin x)$

Graph of $f(x) = \sin^{-1}(\sin x)$

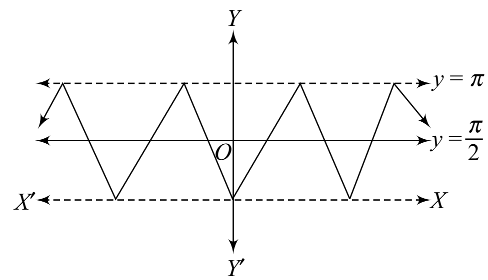



1. $D_f = R$
2. $R_f = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
3. It is an odd function.
4. It is a periodic function with period $2\pi$
5. $\sin^{-1}(\sin x) = \begin{cases} x & : -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ \pi - x & : \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2} \\ x - 2\pi & : \dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2} \\ -\pi - x & : -\dfrac{3\pi}{2} \le x \le -\dfrac{\pi}{2} \end{cases}$

(ii) $\cos^{-1}(\cos x)$ :

A function $f : R \to [0, \pi]$ is defined as $f(x) = \cos^{-1}(\cos x)$

Graph of $f(x) = \cos^{-1}(\cos x)$:



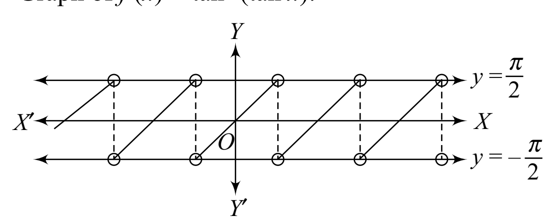

1. $D_f = R$
2. $R_f = [0, \pi]$
3. It is neither an odd nor an even function.
4. It is a periodic function with period $2\pi$
5. $\cos^{-1}(\cos x) = \begin{cases} x & : 0 \le x \le \pi \\ 2\pi - x & : \pi \le x \le 2\pi \\ x - 2\pi & : 2\pi \le x \le 3\pi \\ -x & : -\pi \le x \le 0 \end{cases}$

(iii) $\tan^{-1}(\tan x)$:

A function $f : R - (2n+1)\dfrac{\pi}{2} \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

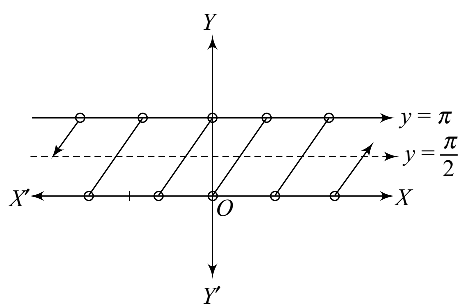
is defined as $f(x) = \tan^{-1}(\tan x)$

Graph of $f(x) = \tan^{-1}(\tan x)$:



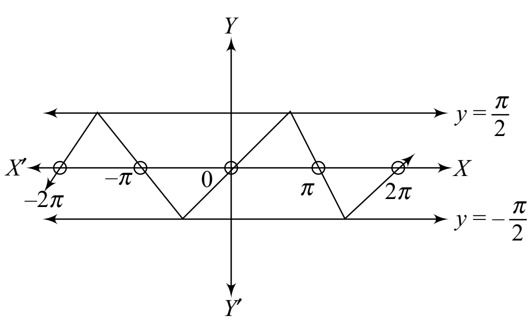
1. $D_f = R - (2n+1)\dfrac{\pi}{2}, n \in I$
2. $R_f = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
3. It is an odd function.
4. It is a periodic function with period $\pi$
5. $\tan^{-1}(\tan x) = \begin{cases} x & : \dfrac{-\pi}{2} < x < \dfrac{\pi}{2} \\ x - \pi & : \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \\ x - 2\pi & : \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \\ x + \pi & : \dfrac{-3\pi}{2} < x < \dfrac{-\pi}{2} \end{cases}$

(iv) $\cot^{-1}(\cot x)$ :
A function $f : R - (n\pi) \to (0, \pi)$ is defined as $f(x) = \cot^{-1}(\cot x)$
Graph of $f(x) = \cot^{-1}(\cot x)$:




1. $D_f = R - n\pi, n \in I$
2. $R_f = (0, \pi)$
3. It is neither an even nor an odd function.
4. It is a periodic function with period $\pi$
5. $\cot^{-1}(\cot x) = \begin{cases} x & : 0 < x < \pi \\ x - \pi & : \pi < x < 2\pi \\ x - 2\pi & : 2\pi < x < 3\pi \\ \pi + x & : -\pi < x < 0 \end{cases}$

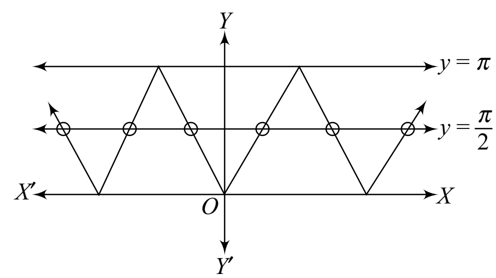
(v) $\text{cosec}^{-1}(\text{cosec}\, x)$: A function
$f : R - (n\pi) \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$ is defined
as $f(x) = \text{cosec}^{-1}(\text{cosec}\, x)$
Graph of $f(x) = \text{cosec}^{-1}(\text{cosec}\, x)$




1. $D_f = R - n\pi, n \in I$
2. $R_f = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$
3. It is an odd function.
4. It is a periodic function with period $2\pi$
5. $\text{cosec}^{-1}(\text{cosec}\, x)$

$$= \begin{cases} x & : \dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2} \\ \pi - x & : \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2} \\ x - 2\pi & : \dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2} \\ -x - \pi & : \dfrac{-3\pi}{2} \le x \le \dfrac{-\pi}{2} \end{cases}$$

(vi) $\sec^{-1}(\sec x)$: A function
$f : R - (2n+1)\dfrac{\pi}{2} \to [0, \pi] - \left\{\dfrac{\pi}{2}\right\}$ is defined as
$f(x) = \sec^{-1}(\sec x)$
Graph of $f(x) = \sec^{-1}(\sec x)$




1. $D_f = R - (2n+1)\dfrac{\pi}{2}, n \in I$
2. $R_f = [0, \pi] - \left\{\dfrac{\pi}{2}\right\}$
3. It is neither an even nor an odd function.
4. It is a periodic function with period $2\pi$.
5. $\sec^{-1}(\sec x) = \begin{cases} x & : 0 \le x \le \pi \\ 2\pi - x & : \pi \le x \le 2\pi \\ x - 2\pi & : 2\pi \le x \le 3\pi \\ -x & : -\pi \le x \le 0 \end{cases}$

## 4.8 Sum of Angles

(i) $\sin^{-1}x + \sin^{-1}y$

$$= \begin{cases} \alpha & : x^2 + y^2 \le 1 \\ \pi - \alpha & : x > 0, y > 0, x^2 + y^2 > 1 \\ \alpha & : xy < 0,\ x^2 + y^2 > 1 \\ -\pi - \alpha & : x < 0,\ y > 0,\ x^2 + y^2 > 1 \end{cases}$$

where $\alpha = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

(ii) $\sin^{-1}x - \sin^{-1}y$

$$= \begin{cases} \alpha : & x^2+y^2 \le 1 \\ \pi-\alpha & : x>0, y<0, x^2+y^2>1 \\ \alpha & : xy>0,\ x^2+y^2>1 \\ -\pi-\alpha : & x<0, y>0, x^2+y^2>1 \end{cases}$$

Where $\alpha = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$

(iii) $\cos^{-1}x + \cos^{-1}y$

$$= \begin{cases} \alpha & : x+y \ge 0 \\ 2\pi-\alpha & : x+y<0 \end{cases}$$

where $\alpha = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$

(iv) $\cos^{-1}x - \cos^{-1}y$

$$= \begin{cases} \alpha & : \quad x \le y \\ -\alpha & : \quad x > y \end{cases}$$

where $\alpha = \cos^{-1}\ (xy + \sqrt{1-x^2}\sqrt{1-y^2})$

(v) $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \alpha : & xy<1 \\ \pi+\alpha & : x>0,\ y>0,\ xy>1 \\ -\pi+\alpha & : x<0, y<0,\ xy>1 \\ \dfrac{\pi}{2} & : x>0,\ y>0,\ xy=1 \\ \dfrac{\pi}{2} & : x<0,\ y<0,\ xy=1 \end{cases}$$

where $\alpha = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

(vi) $\tan^{-1} - \tan^{-1}y$

$$= \begin{cases} \alpha & : \quad xy>-1 \\ \pi+\alpha & : xy<-1, x>0,\ y<0 \\ -\pi+\alpha & : xy<-1, x<0, y>0 \\ \dfrac{\pi}{2} & : xy=-1, x>0,\ y<0 \\ -\dfrac{\pi}{2} & : xy=-1,\ x<0, y>0 \end{cases}$$

where $\alpha = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

## 4.9 Multiple Angles

(i) $\sin^{-1}(2x\sqrt{1-x^2})$

$$= \begin{cases} 2\sin^{-1}x & : -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & : \quad \dfrac{1}{\sqrt{2}} < x \le 1 \\ -\pi - 2\sin^{-1}x & : -1 \le x < -\dfrac{1}{\sqrt{2}} \end{cases}$$

(ii) $\cos^{-1}(2x^2-1) = \begin{cases} 2\cos^{-1}x & : 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & : -1 \le x < 0 \end{cases}$

(iii) $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \begin{cases} \alpha & : -1 < x < \\ -\pi+\alpha & : x>1 \\ \pi+\alpha & : x<-1 \end{cases}$

where $\alpha = 2\tan^{-1}(x)$.

(iv) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \begin{cases} \alpha & : -1 \le x \le 1 \\ \pi-\alpha & : x>1 \\ -\pi-\alpha & : x<-1 \end{cases}$

where $\alpha = 2\tan^{-1}(x)$

(v) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}(x) & : x \ge 0 \\ -2\tan^{-1}(x) & : x \le 0 \end{cases}$

## 4.10 More Multiple Angles

(i) $\sin^{-1}(3x - 4x^3)$

$$= \begin{cases} 3\sin^{-1}x & : -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x & : \dfrac{1}{2} < x \le 1 \\ -\pi - 3\sin^{-1}x & : -1 \le x < -\dfrac{1}{2} \end{cases}$$

(ii) $\cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1}x & : \dfrac{1}{2} \le x \le 1 \\ 2\pi - 3\cos^{-1}x & : -\dfrac{1}{2} \le x < \dfrac{1}{2} \\ -2\pi + 3\cos^{-1}x & : -1 \le x < -\dfrac{1}{2} \end{cases}$$

(iii) $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

$$= \begin{cases} 3\tan^{-1}x & : -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x & : -\infty < x < -\dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & : \dfrac{1}{\sqrt{3}} < x < \infty \end{cases}$$

# EXERCISES

## LEVEL I

### (Problems Based on Fundamentals)

**ABC OF INVERSE FUNCTION**

1. A function $f$: $R \to R$ is defined as $f(x) = 3x + 5$. Find $f^{-1}(x)$.
2. A function $f$: $(0, \infty) \to (2, \infty)$ is defined as $f(x) = x^2 + 2$. Then find $f^{-1}(x)$.
3. A function $f$: $R^+ \to [0, 1)$ is defined as $f(x) = \frac{x^2}{x^2+1}$. Then find $f^{-1}(x)$.
4. A function $f$: $[1, \infty) \to [1, \infty)$ is defined as $f(x) = 2^{x(x-1)}$. Find $f^{-1}(x)$.
5. If a function $f$ is bijective such that $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$, then find $f^{-1}(x)$
6. A function $f$: $R \to R$ is defined as $f(x) = x + \sin x$. Find $f^{-1}(x)$.
7. A function f: $[2, \infty) \to [5, \infty)$ is defined as $f(x) = x^2 - 4x + 9$. Find its inverse.
8. Find all the real solutions to the equation $$x^2 - \frac{1}{4} = \sqrt{x + \frac{1}{4}}$$
9. A function $f$ is defined as $f(x) = 3z + 5$ where $f$: $R \to R$, then find $f^{-1}(x)$
10. A function f is defined as $f(x) = \frac{x}{x-1}$ where $f$: $R - \{1\} \to R - \{1\}$, then find $f^{-1}(x)$
11 A function f is defined as $f(x) = \frac{1}{x^2+1}$ where f: $R^+ \cup \{0\} \to (0, 1]$, find $f^{-1}(x)$
12 A function f is bijective such that $f(x) = \frac{2^x - 2^{-x}}{2^x + 2^{-x}}$, then find $f^{-1}(x)$.
13. A function $f: [-1, 1] \to \left[-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right]$ is defined as $f(x) = \frac{x}{x^2+1}$, then find $f^{-1}(x)$.

**ABC OF INVERSE TRIGONOMETRIC FUNCTIONS**

14. Find the domain of $f(x) = \sin^{-1}(3x + 5)$
15. Find the domain of $f(x) = \sin^{-1}\left(\frac{x}{x+1}\right)$.
16. Find the domain of $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$.
17. Find the domain of $f(x) = \sin^{-1}\left(\frac{|x|-1}{2}\right)$.
18. Find the domain of $f(x) = \sin^{-1}(\log_2 x)$.
19. Find the domain of $f(x) = \sin^{-1}(\log_4 x^2)$.
20. Solve for $x$ and $y$ : $\sin^{-1} x + \sin^{-1} y = \pi$
21. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then find the value of $$x^{2013} + y^{2013} + z^{2013} - \frac{9}{x^{2014} + y^{2014} + z^{2014}}$$
22. Find the range of $f(x) = 2\sin^{-1}(3x+5) + \frac{\pi}{4}$.
23. Solve the inequality: $\sin^{-1} x > \sin^{-1}(3x - 1)$.
24. Find the domain of $f(x) = \cos^{-1}(2x + 4)$.
25. Find the range of $f(x) = 2\cos^{-1}(3x+5) + \frac{\pi}{4}$.
26. Find the range of $f(x) = 3\cos^{-1}(-x^2) - \frac{\pi}{2}$.
27. Solve for $x$: $\cos^{-1} x + \cos^{-1} x^2 = 0$.
28. Solve for $x$: $[\sin^{-1} x] + [\cos^{-1} x] = 0$, where $x$ is a non negative real number and [,] denotes the greatest integer function.
29. Find the domain of $f(x) = \cos^{-1}\left(\frac{x^2}{x^2+1}\right)$.
30. Solve for $x$: $\cos^{-1}(x) > \cos^{-1}(x^2)$.
31. Find the domain of $f(x) = \tan^{-1}(\sqrt{9-x^2})$.
32. Find the range of the function $$f(x) = 2\tan^{-1}(1-x^2) + \frac{\pi}{6}$$
33. Find the range of $f(x) = \cot^{-1}(2x - x^2)$.
34. Solve for $x$: $[\cot^{-1} x] + [\cos^{-1} x] = 0$,
35. Find the number of solutions of $\sin\{x\} = \cos\{x\}$, $\forall x \in [0, 2\pi]$

**Q. Find the domains of each of the following functions:**

36. $f(x) = \sin^{-1}\left(\frac{|x|-2}{3}\right) + \cos^{-1}\left(\frac{1-|x|}{4}\right)$
37. $f(x) = \sin^{-1}(2x^2 - 1)$
38. $f(x) = \sqrt{5\pi \sin^{-1}x - 6(\sin^{-1}x)^2}$
39. $f(x) = \log_2\left(\frac{3\tan^{-1}x + \pi}{\pi - 4\tan^{-1}x}\right)$
40. $f(x) = \cos^{-1}\left(\frac{3}{2+\sin x}\right)$
41. $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$
42. $f(x) = \cos^{-1}\left(\frac{x^2+1}{x^2}\right)$
43. $f(x) = \sin^{-1}(\log_2(x^2 + 3x + 4)$
44. $f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$
45. $f(x) = \sin^{-1}[2 - 3x^2]$
46. $f(x) = \frac{1}{x} + 3^{\sin^{-1}x} + \frac{1}{\sqrt{x-2}}$
47. $f(x) = \sin^{-1}(\log_2 x^2)$
48. $f(x) = e^x + \sin^{-1}\left(\frac{x}{2} - 1\right) + \frac{1}{x}$

49. $f(x)=\sqrt{\sin^{-1}(\log_x 2)}$

50. $f(x)=\sqrt{\sin^{-1}(\log_2 x)}$

**Q. Find the ranges of each of the following functions:**

51. $f(x) = \sin^{-1}(2x - 3)$

52. $f(x)=2\sin^{-1}(2x-1)-\dfrac{\pi}{4}$

53. $f(x) = 2\cos^{-1}(-x^2) - \pi$

54. $f(x)=\dfrac{1}{2}\tan^{-1}(1-x^2)-\dfrac{\pi}{4}$

55. $f(x) = \cot^{-1}(2x - x^2)$

56. $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$

57. $f(x) = \sin^{-1} x + \sec^{-1} x + \tan^{-1} x$

58. $f(x)=3\cot^{-1}x+2\tan^{-1}x+\dfrac{\pi}{4}$

59. $f(x) = \operatorname{cosec}^{-1}[1 + \sin^2 x]$

60. $f(x) = \sin^{-1}(\log_2(x^2 + 3x + 4)$

## CONSTANT PROPERTY

61. Find the range of
$$f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$$

62. Solve for $x$: $4\sin^{-1}(x - 2) + \cos^{-1}(x - 2) = \pi$

63. Solve for $x$:
$$\sin^{-1}(x^2-2x+1)+\cos^{-1}(x^2-x)=\frac{\pi}{2}$$

64. Find the number of real solutions of
$$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}.$$

65. If $\sin^{-1}\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{4}-........\right)$
$$+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-.......\right)=\frac{\pi}{2},\text{ for}$$
$0<|x|<\sqrt{2}$, then find $x$.

66. Solve for $x$: $\sin^{-1} x > \cos^{-1} x$

**Q. Solve for $x$:**

67. $(\sin^{-1} x)^2 - 3\sin^{-1} x + 2 = 0$

68. $\sin^{-1} x + \sin^{-1} 2y = \pi$

69. $\cos^{-1} x + \cos^{-1} x^2 = 2\pi$

70. $\cos^{-1} x + \cos^{-1} x^2 = 0$

71. $4\sin^{-1}(x - 1) + \cos^{-1}(x - 1) = \pi$

72. $\cot^{-1}\left(\dfrac{1}{x^2-1}\right)+\tan^{-1}(x^2-1)=\dfrac{\pi}{2}$

73. $\cot^{-1}\left(\dfrac{x^2-1}{2x}\right)+\tan^{-1}\left(\dfrac{2x}{x^2-1}\right)=\dfrac{2\pi}{3}$

74. $4\sin^{-1}x+\cos^{-1}x=\dfrac{3\pi}{4}$

75. $5\tan^{-1}x+3\cot^{-1}x=\dfrac{7\pi}{4}$

76. $5\tan^{-1} x + 4\cot^{-1} x = 2\pi$

77. $\cot^{-1}x-\cot^{-1}(x+1)=\dfrac{\pi}{2}$

78. $[\sin^{-1} x] + [\cos^{-1} x] = 0$

79. $[\tan^{-1} x] + [\cot^{-1} x] = 0$

80. $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1} x] = 0$

81. $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1} x] = 1$

82. $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\dfrac{5\pi^2}{8}$

## CONVERSION OF INVERSE TRIGONOMETRIC FUNCTIONS

83. Find the value of $\cos\left(\dfrac{1}{2}\cos^{-1}\left(\dfrac{3}{5}\right)\right)$.

84. Find the value of $\sin\left(\dfrac{\pi}{4}+\sin^{-1}\left(\dfrac{1}{2}\right)\right)$.

85. If $m$ is a root of $x^2 + 3x + 1 = 0$, then find the value of $\tan^{-1}(m)+\tan^{-1}\left(\dfrac{1}{m}\right)$.

86. Prove that
$$\cos(\tan^{-1}(\sin(\cot^{-1} x))) = \sqrt{\frac{x^2+1}{x^2+2}}$$

**Q. Solve for $x$:**

87. $6(\sin^{-1} x)^2 - \pi\sin^{-1} x \le 0$

88. $\dfrac{2\tan^{-1}x+\pi}{4\tan^{-1}x-\pi}\le 0$

89. $\sin^{-1} x < \sin^{-1} x^2$

90. $\cos^{-1} x > \cos^{-1} x^2$

91. $\log^2(\tan^{-1} x) > 1$

92. $(\cot^{-1} x)^2 - 5\cot^{-1} x + 6 > 0$

93. $\sin^{-1} x < \cos^{-1} x$

94. $\sin^{-1} x > \sin^{-1}(1 - x)$

95. $\sin^{-1} 2x > \operatorname{cosec}^{-1} x$

96. $\tan^{-1} 3x < \cot^{-1} x$

97. $\cos^{-1} 2x^3 \sin^{-1} x$

98. $x^2 - 2x < \sin^{-1}(\sin 2)$

99. $\sin^{-1}\left(\dfrac{x}{2}\right)<\cos^{-1}(x+1)$

100. $\tan^{-1} 2x > 2\tan^{-1} x$

101. $\tan(\cos^{-1}x)\le\sin\left(\cot^{-1}\left(\dfrac{1}{2}\right)\right)$

## COMPOSITION OF TRIGONOMETRIC FUNCTIONS AND ITS INVERSE

102. Let $f(x) = \sin^{-1} x + \cos^{-1} x$

Then find the value of:

(i) $f\left(\dfrac{1}{m^2+1}\right), m\in R$

(ii) $f\left(\dfrac{m^2}{m^2+1}\right), m\in R$

(iii) $f\left(\dfrac{m}{m^2+1}\right), m\in R$

(iv) $f(m^2 - 2m + 6), m \in R$

(v) $f(m^2 + 1), m \in R$

103. If $\cos^{-1}x+\cos^{-1}y=\frac{2\pi}{3}$, then find the value of $\sin^{-1}x+\sin^{-1}y$.

104. If $m$ is the root of $x^2+3x+1=0$, then find the value of $\tan^{-1}(m)+\tan^{-1}\left(\frac{1}{m}\right)$.

105. Solve for $x$:

$$\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+2}\right)\right)>\sin^{-1}(\sin 3)$$

## COMPOSITION OF INVERSE TRIGONOMETRIC FUNCTIONS AND TRIGONOMETRIC FUNCTIONS

106. Find the values of:
(i) $\sin^{-1}(\sin 3)$ (ii) $\sin^{-1}(\sin 53)$
(iii) $\sin^{-1}(\sin 7)$ (iv) $\sin^{-1}(\sin 10)$
(v) $\sin^{-1}(\sin 20)$

107. Find the values of:
(i) $\cos^{-1}(\cos 2)$ (ii) $\cos^{-1}(\cos 3)$
(iii) $\cos^{-1}(\cos 5)$ (iv) $\cos^{-1}(\cos 7)$
(v) $\cos^{-1}(\cos 10)$

108. Find the values of:
(i) $\tan^{-1}(\tan 3)$ (ii) $\tan^{-1}(\tan 5)$
(iii) $\tan^{-1}(\tan 7)$ (iv) $\tan^{-1}(\tan 10)$
(v) $\tan^{-1}(\tan 15)$

109. Find the value of $\cos^{-1}(\sin(-5))$

110. Find $f'(x)$, where $f(x)=\sin^{-1}(\sin x)$ and $-2\pi\le x\le\pi$

111. Find $f'(x)$, where $f(x)=\cos^{-1}(\cos x)$ and $-\pi\le x\le 2\pi$

112. Solve for $x$: $\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+1}\right)\right)<\pi-3$

113. Find the integral values of $x$ satisfying the inequality, $x^2-3x<\sin^{-1}(\sin 2)$

114. Find the value of

$$\sin^{-1}(\text{si } 50)+\cos^{-1}(\cos 50)+\tan^{-1}(\tan 50)$$

**Q. Find the values of:**

115. $\sin^{-1}(\sin 1)+\sin^{-1}(\sin 2)+\sin^{-1}(\sin 3)$

116. $\sin^{-1}(\sin 10)+\sin^{-1}(\sin 20)$
$+\sin^{-1}(\sin 30)+\sin^{-1}(\sin 40)$

117. $\cos^{-1}(\cos 1)+\cos^{-1}(\cos 2)$
$+\cos^{-1}(\cos 3)+\cos^{-1}(\cos 4)$

118. $\cos^{-1}(\cos 10)+\cos^{-1}(\cos 20)$
$+\cos^{-1}(\cos 30)+\cos^{-1}(\cos 40)$

119. $\sin^{-1}(\sin 10)+\cos^{-1}(\cos 10)$

120. $\sin^{-1}(\sin 50)+\cos^{-1}(\cos 50)$

121. $\sin^{-1}(\sin 100)+\cos^{-1}(\cos 100)$

122. $\cos^{-1}(\sin(-5))+\sin^{-1}(\cos(-5))$

123. Find the number of ordered pairs of $(x, y)$ satisfying the equations $y=|\sin x|$ and $y=\cos^{-1}(\cos x)$, where $x\in[-2\pi, 2\pi]$

124. Let $f(x)=\cos^{-1}(\cos x)-\sin^{-1}(\sin x)$ in $[0,\pi]$. Find the area bounded by $f(x)$ and $x$-axis.

125. $\tan^{-1}(\tan 1)+\tan^{-1}(\tan 2)$
$+\tan^{-1}(\tan 3)+\tan^{-1}(\tan 4)$

126. $\tan^{-1}(\tan 20)+\tan^{-1}(\tan 40)$
$+\tan^{-1}(\tan 60)+\tan^{-1}(\tan 80)$

127. $\sin^{-1}(\sin 15)+\cos^{-1}(\cos 15)+\tan^{-1}(\tan 15)$

128. $\sin^{-1}(\sin 50)+\cos^{-1}(\cos 50)-\tan^{-1}(\tan 50)$

129. $3x^2+8x<2\sin^{-1}(\sin 4)-\cos^{-1}(\cos 4)$

130. $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3$

## SUM OF ANGLES

131. Find the value of $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)$.

132. Find the value of

$$\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)$$

133. Find the value of $\tan^{-1}(9)+\tan^{-1}\left(\frac{5}{4}\right)$.

134. Find the value of

$$\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)-\sin^{-1}\left(\frac{63}{65}\right)$$

135. Prove that $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\frac{\pi}{4}$

136. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, prove that

$$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$$

137. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$
prove that $x^2+y^2+z^2+2xyz=1$

138. If $\cos^{-1}\left(\frac{x}{2}\right)+\cos^{-1}\left(\frac{y}{3}\right)=\theta$, prove that

$$9x^2+12\,xy\cos\theta+4y^2=36\sin^2\theta$$

139. Let $m=\frac{\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)}{\cot^{-1}(1)+\cot^{-1}(2)+\cot^{-1}(3)}$
Then find the value of $(m-1)^{2013}$.

140. Solve for $x$: $\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{3\pi}{4}$

141. Solve for $x$: $\sin^{-1}(x)+\sin^{-1}(2x)=\frac{\pi}{3}$

142. Let $f(x)=\cos^{-1}(x)$

$$+\cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right),\ \text{for } \frac{1}{2}\le x\le 1$$

Then find $f(2013)$

143. $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$

144. $x^2-4x>\sin^{-1}(\sin(\pi^{3/2}])+\cos^{-1}(\cos[\pi^{3/2}])$

145. $\cos(\tan^{-1}x)=x$

146. $\sin(\tan^{-1}x)=\cos(\cot^{-1}(x+1))$

147. $\sec^{-1}\left(\frac{x}{2}\right)-\sec^{-1}x=\sec^{-1}2$

148. $\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(x+\frac{3}{2}\right)\right)\right)\right)+\tan(\sec^{-1}x)=0$

149. Find the smallest +ve integer $x$ so that

$$\tan\left(\tan^{-1}\left(\frac{x}{10}\right)+\tan^{-1}\left(\frac{1}{x+1}\right)\right)=\tan\left(\frac{\pi}{4}\right)$$

150. Find the least integral value of $k$ for which $(k - 2) x^2 + 8x + k + 4 > \sin^{-1} (\sin 12) + \cos^{-1} (\cos 12)$ holds for all $x$ in $R$.

151 If $\alpha = 2 \tan^{-1}\left(\frac{1+x}{1-x}\right)$ and $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that $\alpha + \beta = \pi$.

152. Let $f(x) = \sin^{-1} (\sin x)$, $\forall\, x \in [-\pi, 2\pi]$. Then find $f'(x)$.

153. Let $f(x) = \cos^{-1} (\cos x)$, $\forall\, x \in [-2\pi, \pi]$. Then find $f'(x)$.

154. Let, $f(x) = \tan^{-1}(\tan x)$ , $\forall\, x \in \left[-\frac{3\pi}{2}, \frac{5\pi}{2}\right]$. Then find $f'(x)$.

155. Prove that $\sin^{-1}\left(\frac{1}{5}\right) + \cot^{-1}(3) = \frac{\pi}{4}$.

156. Prove that $2 \tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{12}{5}\right) = \pi$.

**Q. Find the simplest form of:**

157. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

158. $\sin^{-1}(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2})$

159. $\sin^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$

160. $\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), \frac{\pi}{4} < x < \frac{5\pi}{4}$

161. $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

162. $\sin^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$

**MULTIPLE ANGLES**

163. Find the value of $\sin\left(2 \sin^{-1}\left(\frac{1}{4}\right)\right)$

164. Find the value of $\cos\left(2\cos^{-1}\left(\frac{1}{3}\right)\right)$

165. Find the value of $\cos\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)$

166. Find the value of $\sin\left(\frac{1}{2}\cot^{-2}\left(\frac{3}{4}\right)\right)$

167. Find the value of $\tan^{-1}\left(\frac{3\pi}{4}-2\tan^{-1}\left(\frac{3}{4}\right)\right)$

168. Prove that $\sin\left(2\sin^{-1}\left(\frac{1}{2}\right)\right)=\frac{\sqrt{3}}{2}$

169. Prove that $\sin\left(3\sin^{-1}\left(\frac{1}{3}\right)\right)=\frac{23}{27}$

170. Prove that $\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right)=\frac{3}{4}$

171. Prove that $\cos\left(\frac{1}{2}\cos^{-1}\left(-\frac{1}{10}\right)\right)=\frac{3\sqrt{5}}{10}$

172. Prove that $\sin\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{9}\right)\right)=\frac{2}{3}$

173. Prove that $\sin\left(\frac{1}{4}\tan^{-1}(\sqrt{63})\right)=\frac{1}{2\sqrt{2}}$

174. Prove that $\cos\left(\frac{1}{4}\left(\tan^{-1}\left(\frac{24}{7}\right)\right)\right)=\frac{3}{\sqrt{10}}$

175. Prove that $\tan\left(\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right)\right)=\frac{1}{\sqrt{5}}$

176. Prove that $\tan\left(2 \tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right)=-\frac{7}{17}$

177. $\tan\left(\frac{3\pi}{4}-\frac{1}{4}\sin^{-1}\left(-\frac{4}{5}\right)\right)=\frac{1-\sqrt{5}}{2}$

178. Find the integral values of $x$ satisfying the inequation $x^2 - 3x < \sin^{-1} (\sin 2)$

179. Find the value of $x$ satisfying the inequation $3x^2 + 8x < 2 \sin^{-1} (\sin 4) - \cos^{-1} (\cos 4)$

180. For what value of $x$,

$$f(x)=\cos^{-1}x+\cos^{-1}\left\{\frac{x}{2}+\frac{\sqrt{1-3x^2}}{2}\right\}$$

is a constant function.

**MORE MULTIPLE ANGLES**

181. Let $f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)+2 \tan^{-1}(x)$ , $x > 1$

Then find the value of $f(2013)$

182. Let $f(x)=2 \tan^{-1}\left(\frac{1+x}{1-x}\right)+\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

for $0 \le x < 1$. Then find the value of $f\left(\frac{1}{2014}\right)$

183. Let $f(x)=\sin^{-1}\left(\frac{6x}{x^2+9}\right)+2 \tan^{-1}\left(-\frac{x}{3}\right)$

is independent of $x$, then find the value of $x$

184. Find the interval of $x$ for which the function $f(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+2 \tan^{-1}(x)$ is a constant function.

185. Find the interval of $x$ for which the function $f(x) = 3 \cos^{-1}(2x^2 - 1) + 2 \cos^{-1}(4x^3 - 3x)$ is independent of $x$

186. If $\tan^{-1} y : \tan^{-1} x = 4 : 1$, then express $y$ as algebraic function of $x$. Also, prove that $\tan\left(22\frac{1}{2}^\circ\right)$ is a root of $x^4 - 6x^2 + 1 = 0$

**BOARD SPECIAL PROBLEMS**

**Q. Prove that:**

187. $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2\right)$

188. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$

189. $\tan^{-1}\left(\frac{p-q}{1+pq}\right)+\tan^{-1}\left(\frac{q-r}{1+qr}\right)+\tan^{-1}\left(\frac{r-p}{1+pr}\right)=\pi$
where $p > q > 0$ and $pr < -1 < qr$

190. $\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)=0$

191. $\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right)$
$=\left(\frac{x+y}{1-xy}\right), xy<1$

192. $\tan^{-1}\left(\frac{1-x}{1+x}\right)-\tan^{-1}\left(\frac{1-y}{1+y}\right)=\sin^{-1}\left(\frac{y-x}{\sqrt{(1+x^2)(1+y^2)}}\right)$

193. $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=0$

194. $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{\theta}{2}\right)\right)=\cos^{-1}\left(\frac{b+a\cos\theta}{a+b\cos\theta}\right)$

195. $\tan(2\tan^{-1}a)=2\tan(\tan^{-1}a+\tan^{-1}a^3)$

196. $\cos^{-1}x+\cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right)=\frac{\pi}{3}, \frac{1}{2}<x<1$

197. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$,
then prove that
$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$

198. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$,
then prove that
$x^2+y^2+z^2+2xyz=1$

199. If $\cos^{-1}\left(\frac{x}{2}\right)+\cos^{-1}\left(\frac{y}{3}\right)=\theta$, then prove that
$9x^2-12xy\cos\theta+4y^2=36\sin^2\theta$.

200. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then prove that
$x^2+y^2+z^2-2xyz=1$.

201. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then prove that
$xy+yz+zx=3$.

202. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then find the value of
$x^{2012}+y^{2012}+z^{2012}-\frac{9}{x^{2013}+y^{2013}+z^{2013}}$

203. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then prove that, $xy+yz+zx=3$.

204. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then find the value of
$\left(\frac{x^{2013}+y^{2013}+z^{2013}+6}{x^{2014}+y^{2014}+z^{2014}}\right)$

205. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then prove that $xy+yz+zx=1$.

206. If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$, then prove that $x+y+xy=1$.

207. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then prove that $x+y+z=xyz$.

208. If $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)=\alpha$, then prove that $x^2=\sin 2\alpha$

209. Let $m=\tan^2(\sec^{-1}2)+\cot^2(\operatorname{cosec}^{-1}3)$. Then find the value of $(m^2+m+10)$.

210. If $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)=\frac{\pi}{4}$, then find the value of $\tan\theta$.

211. Let $m=\frac{(\tan^{-1}1+\tan^{-1}2+\tan^{-1}3)}{(\cot^{-1}1+\cot^{-1}2+\cot^{-1}3)}$, then prove that $(m+2)^{m+1}=64$.

**Q. Solve for $x$:**

212. $\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{3\pi}{4}$

213. $\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7)$

214. $\sin^{-1}(2x)+\sin^{-1}(x)=\frac{\pi}{3}$

215. $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cos^{-1}x=\frac{\pi}{4}$

216. $\sin^{-1}(x)+\sin^{-1}(3x)=\frac{\pi}{3}$

217. $\tan^{-1}\left(\frac{1}{1+2x}\right)+\tan^{-1}\left(\frac{1}{1+4x}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$

218. $2\tan^{-1}(2x+1)=\cos^{-1}x$

219. $\cos^{-1}x-\sin^{-1}x=\cos^{-1}(x\sqrt{3})$

220. If $\tan^{-1}y:\tan^{-1}x=4:1$, express $y$ as an algebraic function of $x$. Hence, prove that $\tan\left(\frac{\pi}{8}\right)$ is a root of $x^4+1=6x^2$

## LEVEL II

### (Mixed Problems)

1. The set of values of $k$ for which $x^2-kx+\sin^{-1}(\sin 4)>0$ for all real $x$ is
   (a) $\{0\}$ (b) $(-2, 2)$
   (c) $R$ (d) None of these

2. If $x < 0$ then value of

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) =$$

(a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$
(c) 0 (d) None of these

3. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then $\cos^{-1} x + \cos^{-1} y$ is

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\pi$

4. Let $f(x) = \sin^{-1} x + \cos^{-1} x$. Then $\frac{\pi}{2}$ is equal to

(a) $f\left(\frac{1}{2}\right)$ (b) $f(k^2 - 2k + 3), k \in R$
(c) $f\left(\frac{1}{1+k^2}\right), k \in R$ (d) $f(-2)$

5. Which one of the following is correct?
(a) $\tan 1 > \tan^{-1} 1$ (b) $\tan 1 < \tan^{-1} 1$
(c) $\tan 1 = \tan^{-1} 1$ (d) None

6. If $a \sin^{-1} x - b \cos^{-1} x = c$, then the value of $a \sin^{-1} x + b \cos^{-1} x$ is

(a) 0 (b) $\frac{\pi ab + c(b-a)}{a+b}$
(c) $\frac{\pi ab - c(b-a)}{a+b}$ (d) $\frac{\pi}{2}$

7. The number of solutions of the equation $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$ is
(a) 0 (b) 1
(c) 2 (d) More than two

8. The smallest and the largest values of $\tan^{-1}\left(\frac{1-x}{1+x}\right), 0 \le x \le 1$ are

(a) $0, \pi$ (b) $0, \frac{\pi}{4}$ (c) $-\frac{\pi}{4}, \frac{\pi}{4}$ (d) $\frac{\pi}{4}, \frac{\pi}{2}$

9. The equation $\sin^{-1} x - \cos^{-1} x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ has
(a) No solution
(b) Unique solution
(c) Infinite number of solution
(d) None

10. If $-\pi \le x \le 2\pi$, then $\cos^{-1}(\cos x)$ is
(a) $x$ (b) $\pi - x$ (c) $2\pi + x$ (d) $2\pi - x$

11. If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then $x$ is equal to

(a) 0 (b) $\frac{1}{\sqrt{5}}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{3}}{2}$

12. If $\cos[\tan^{-1}\{\sin(\cot^{-1}\sqrt{3})\}] = y$, then the value of $y$ is

(a) $y = \frac{4}{5}$ (b) $y = \frac{2}{\sqrt{5}}$
(c) $y = -\frac{2}{\sqrt{5}}$ (d) $y = \frac{\sqrt{3}}{2}$

13. If $x = \frac{1}{5}$, then the value of $\cos(\cos^{-1} x + 2\sin^{-1} x)$ is

(a) $\sqrt{\frac{24}{25}}$ (b) $-\sqrt{\frac{24}{25}}$ (c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

14. $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$ (d) None of these

15. $\tan^{-1} a + \tan^{-1} b$, where $a > 0, b > 0, ab > 1$ is equal to

(a) $\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (b) $\tan^{-1}\left(\frac{a+b}{1-ab}\right) - \pi$
(c) $\pi + \tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (d) $\pi - \tan^{-1}\left(\frac{a+b}{1-ab}\right)$

16. A solution to the equation

$$\tan^{-1}(1+x) + \tan^{-1}(1-x) = \frac{\pi}{2} \text{ is}$$

(a) $x = 1$ (b) $x = -1$ (c) $x = 0$ (d) $x = \pi$

17. All possible values of $p$ and $q$ for which

$$\cos^{-1}(\sqrt{p}) + \cos^{-1}(\sqrt{1-p}) + \cos^{-1}(\sqrt{1-q}) = \frac{3\pi}{4}$$

holds, is
(a) $p = 1, q = 1/2$ (b) $q > 1, p = 1/2$
(c) $0 < p < 1, q = 1/2$ (d) None

18. $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1} x\right)$, $x \ne 0$, is equal to

(a) $x$ (b) $2x$ (c) $\frac{2}{x}$ (d) $\frac{x}{2}$

19. The value of $\cot^{-1}(3) + \operatorname{cosec}^{-1}(\sqrt{5})$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

20. If $\sum_{i=1}^{2n} \sin^{-1} x_i = n\pi$, then $\sum_{i=1}^{2n} x_i$ is

(a) $n$ (b) $2n$
(c) $\frac{n(n+1)}{2}$ (d) $\frac{n(n-1)}{2}$

21. If $u = \cot^{-1}(\sqrt{\tan\alpha}) - \tan^{-1}(\sqrt{\tan\alpha})$, then $\tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$ is equal to

(a) $\sqrt{\tan\alpha}$ (b) $\sqrt{\cot\alpha}$
(c) $\tan\alpha$ (d) $\cot\alpha$

22. The value of $\tan^{-1}\left(\frac{a}{b+c}\right) + \tan^{-1}\left(\frac{b}{a+c}\right)$, if $\angle C = 90$, in triangle $ABC$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\pi$

23. If $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$, $n \in N$, then the maximum value of '$n$' is
(a) 1 (b) 5
(c) 9 (d) None of these

24. $\sin^{-1} x > \cos^{-1} x$ holds for
(a) all values of $x$ (b) $x \in \left(0, \frac{1}{\sqrt{2}}\right)$
(c) $\left(\frac{1}{\sqrt{2}}, 1\right)$ (d) $x = 0.75$

25. The value of $\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right)$ is equal to
(a) $\frac{3}{4}$ (b) $-\frac{3}{4}$ (c) $\frac{1}{16}$ (d) 4

26. The values of $x$ satisfying
$\tan(\sec^{-1}x) = \sin\left(\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)$ is
(a) $\pm\frac{\sqrt{5}}{3}$ (b) $\pm\frac{3}{\sqrt{5}}$ (c) $\pm\frac{\sqrt{3}}{5}$ (d) $\pm\frac{3}{5}$

27. If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then the value of $x$ is
(a) 0 (b) –1 (c) –2 (d) –3

28. The number of real solutions of $\cos^{-1} x + \cos^{-1} 2x = -\pi$ are
(a) 0 (b) 1
(c) 2 (d) infinitely many

29. Let $a$, $b$, $c$ be positive real numbers and
$$\theta = \tan^{-1}\left(\sqrt{\frac{a(a+b+c)}{bc}}\right) + \tan^{-1}\left(\sqrt{\frac{b(a+b+c)}{ac}}\right) + \tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ba}}\right),$$
then the value of $\tan\theta$ is
(a) 0 (b) 1 (c) –1 (d) None

30. The set of values of $x$ satisfying the inequation $\tan^2(\sin^{-1} x) > 1$ is
(a) $[-1, 1]$ (b) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$
(c) $(-1, 1) - \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (d) $[-1, 1] - \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

31. The value of $a$ for which
$ax^2 + \sin^{-1}(x^2 - 2x + 2) + \cos^{-1}(x^2 - 2x + 2) = 0$
has a real solution, is
(a) $\frac{\pi}{2}$ (b) $\frac{-\pi}{2}$ (c) $\frac{2}{\pi}$ (d) $\frac{-2}{\pi}$

32. The value of
$$\sin^{-1}\left[\cot\left(\sin^{-1}\sqrt{\left(\frac{2-\sqrt{3}}{4}\right)} + \cos^{-1}\left(\frac{\sqrt{12}}{4}\right) + \sec^{-1}\sqrt{2}\right)\right]$$ is
(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

33. The number of positive integral solutions of $\tan^{-1}x + \cot^{-1}\left(\frac{1}{y}\right) = \sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ is
(a) 0 (b) 1 (c) 2 (d) 3

34. The value of
$\cos\left[\frac{1}{2}\cos\left\{\cos\left(\sin^{-1}\left(\frac{\sqrt{63}}{8}\right)\right)\right\}\right]$ is
(a) $\frac{3}{16}$ (b) $\frac{3}{8}$ (c) $\frac{3}{4}$ (d) $\frac{3}{2}$

35. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, then the value of $\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy}$ is
(a) 0 (b) 1 (c) $\frac{1}{xyz}$ (d) $xyz$

36. If $x < 0$, then $\tan^{-1}\left(\frac{1}{x}\right)$ is
(a) $\cot^{-1}(x)$ (b) $-\cot^{-1}(x)$
(c) $-\pi + \cot^{-1}(x)$ (d) None

37. The number of triplets satisfying
$\sin^{-1} x + \cos^{-1} y + \sin^{-1} z = 2\pi$, is
(a) 0 (b) 2 (c) 1 (d) infinite

38. If $x^2 + y^2 + z^2 = r^2$, then
$\tan^{-1}\left(\frac{xy}{zr}\right) + \tan^{-1}\left(\frac{zy}{xr}\right) + \tan^{-1}\left(\frac{zx}{yr}\right)$ is equal to ....
(a) $\pi$ (b) $\frac{\pi}{2}$ (c) 0 (d) None

39. If $\tan^{-1} x + \tan^{-1} 2x + \tan^{-1} 3x = \pi$, then the value of $x$ is
(a) 0 (b) –1 (c) 1 (d) $\phi$

40. The number of solutions of the equation $1 + x^2 + 2x \sin(\cos^{-1} y) = 0$ is
(a) 1 (b) 2 (c) 3 (d) 4

41. If $\alpha$ is the only real root of the equation $x^3 + bx^2 + cx + 1 = 0$, then the value of $\tan^{-1}\alpha + \tan^{-1}\left(\frac{1}{\alpha}\right)$ is equal to
(a) $\frac{\pi}{2}$ (b) $\frac{-\pi}{2}$ (c) 0 (d) None

42. If $\alpha$, $\beta$, $\gamma$ are the roots of $x^3 + px^2 + 2x + p = 0$, the the general value of $\tan^{-1} \alpha + \tan^{-1} \beta \tan^{-1} \gamma$ is
(a) $n\pi$ (b) $\frac{n\pi}{2}$
(c) $\frac{(2n+1)\pi}{2}$ (d) depend on $p$

43. If $[\sin^{-1} (\cos^{-1})(\cos^{-1}(\sin^{-1}(\tan^{-1} x)))] = 1$, where [,] = GIF, the value of $x$ lies in
(a) [tan sin cos 1, tan sin cos sin 1]
(b) (tan sin cos 1, tan sin cos sin 1)
(c) [–1, 1]
(d) [sin cos tan 1, sin cos sin tan 1]

## Level IIA
### (Problems For JEE Main)

1. Find the principal value of $\sin^{-1}(\sin 10)$
2. Find the principal value of $\cos^{-1}(\cos 5)$

3. Find the value of
   $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$
4. Find $x$ if $\sin^{-1} x > \cos^{-1} x$
5. Find $x$ if $\sin^{-1} x < \cos^{-1} x$
6. Find $x$ if $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$
7. Find $x$ if $3\sin^{-1} x = \pi + \sin^{-1}(3x - 4x^3)$
8. Find $x$ if $2\tan^{-1}x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$
9. Find the value of $\cos\left(\frac{\pi}{6} + \cos^{-1}\left(-\frac{1}{2}\right)\right)$
10. Find the value of $\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2}-1)))$
11. Find the value of $\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)$.
12. Find the value of $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right)$.
13. Find the value of $\sum_{r=1}^{n} \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$
14. Find the value of
   $\tan^{-1}\left(\frac{a_1x-y}{a_1y+x}\right) + \tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)$
   $+ \tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right) + \ldots + \tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)$
   $+ \tan^{-1}\left(\frac{1}{a_n}\right)$, where $x, y, a_1, a_2, \ldots a_n \in R^+$
15. Find $x$ if $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{\pi^2}{8}$
16. Find the maximum value of $f(x)$, if
   $f(x) = (\sec^{-1} x)^2 + (\operatorname{cosec}^{-1} x)^2$
17. Find the minimum value of $f(x)$, if
   $f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$
18. Find $x$ if $[\cot^{-1} x] + [\cos^{-1} x] = 0$
19. Find $x$ if $[\sin^{-1} x] + [\cos^{-1} x] = 0$.
20. Find $x$ if $[\tan^{-1} x] + [\cot^{-1} x] = 2$
21. Find $x$ if $[\sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1} x)))] = 1$
22. Find the range of
   $f(x) = \sin^{-1} x + \tan^{-1} x + \cot^{-1} x$
23. Find the range of
   $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$.
24. Find the range of
   $f(x) = \sin^{-1} x + \sec^{-1} x + \tan^{-1} x$
25. If $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$, then find $x$
26 If $\cos^{-1}x = \cot^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$,
   then find $x$.
27. If $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$, where $n \in N$, then find the maximum value of $n$.
28. If $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$, then find $x$.
29. If $x = \sin^{-1}(b^6 + 1) + \cos^{-1}(b^4 + 1) + \tan^{-1}(a^2 + 1)$ then find the value of
   $\sin\left(x + \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)$
30. Find the number of integral values of $k$ for which the equation $\sin^{-1} x + \tan^{-1} x = 2k + 1$ has a solution.
31. If $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$, then find the value of $\left(\frac{1+x^4+y^4}{x^2-x^2y^2+y^2}\right)$
32. If $\cos^{-1} x + \cos^{-1}(2x) + \cos^{-1}(3x)$ and $x$ satisfies the equation $ax^3 + bx^2 + cx = 1$ then find the value of
   $a^2 + b^2 + c^2 + 10$.
33. If $f(x) = \sin^{-1} x + \tan^{-1} x + x^2 + 4x + 5$ such that $R_f = [a, b]$, find the value of $a + b + 5$.
34. If $\cot^{-1}(\sqrt{\cos\alpha}) + \tan^{-1}(\sqrt{\cos\alpha}) = x$, then $\sin x$ is
   (a) 1 (b) $\cot^2(\alpha/2)$
   (c) $\tan\alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$
   **[JEE Main, 2002]**
35. The domain of $\sin^{-1}\left(\log_3\left(\frac{x}{3}\right)\right)$ is
   (a) [1, 9] (b) [–1, 9]
   (c) [–9, 1] (d) [–9, –1]
   **[JEE Main, 2002]**
36. The trigonometric equation $\sin^{-1} x = 2\sin^{-1} a$ has a solution for
   (a) all real values (b) $|a| < \frac{1}{2}$
   (c) $|a| \le \frac{1}{\sqrt{2}}$ (d) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$
   **[JEE Main, 2003]**
37. The domain of the function
   $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is
   (a) [1, 2] (b) [2, 3) (c) [2, 3] (d) [1, 2)
   **[JEE Main, 2004]**
38. Let $f: (-1, 1) \to B$ be a function defined as $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then $f$ is both one one and onto, when $B$ lies in
   (a) $\left[0, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{2}\right)$
   (c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
   **[JEE Main, 2005]**

39. If $\cos^{-1}x - \cos^{-1}\left(\frac{y}{2}\right) = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is
(a) 4 (b) $2\sin\alpha$
(c) $-4\sin^2\alpha$ (d) $4\sin^2\alpha$
**[JEE Main, 2005]**

40. If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then $x$ is
(a) 4 (b) 5 (c) 1 (d) 3
**[JEE Main, 2007]**

41. Find the value of
$\cot\left(\operatorname{cosec}^{-1}\left(\frac{5}{3}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$ is
(a) $\frac{5}{17}$ (b) $\frac{6}{17}$ (c) $\frac{3}{17}$ (d) $\frac{4}{17}$
**[JEE Main, 2008]**

## Level III

## (Problems for JEE Advanced)

1. Find the domain of
$$f(x) = \sin^{-1}\left(\frac{|x|-2}{3}\right) + \cos^{-1}\left(\frac{1-|x|}{4}\right)$$
2. Find the domain of
$$f(x) = \sqrt{5\pi\sin^{-1}x - 6(\sin^{-1}x)^2}$$
3. Find the domain of
$$f(x) = \sin^{-1}(\log_2(x^2 + 3x + 4)).$$
4. Solve for $x$: $\cos^{-1}x + \cos^{-1}x^2 = 2\pi$
5. Solve for $x$:
$$\cot^{-1}\left(\frac{1}{x^2-1}\right) + \tan^{-1}(x^2-1) = \frac{\pi}{2}$$
6. Solve for $x$:
$$\cot^{-1}\left(\frac{x^2-1}{2x}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$
7. Solve for $x$:
$$\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right) < \pi - 3$$
8. Solve for $x$:
$$x^2 - 4x > \sin^{-1}(\sin[\pi^{3/2}]) + \cos^{-1}(\cos[\pi^{3/2}])$$
9. Solve for $x$:
$$\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(x + \frac{3}{2}\right)\right)\right)\right) + \tan(\sec^{-1}x) = 0$$
10. Solve for $x$:
$$\tan\left(\tan^{-1}\left(\frac{x}{10}\right) + \tan^{-1}\left(\frac{1}{x+1}\right)\right) = \tan\left(\frac{\pi}{4}\right)$$
11. If $\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$ and $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that $\alpha + \beta = \pi$
12. Find the range of $f(x) = 2\sin^{-1}(2x - 3)$
13. Find the range of
$$f(x) = 2\sin^{-1}(2x-1) - \frac{\pi}{4}$$
14. Find the range of
$$f(x) = 2\cos^{-1}(-x)^2 - \pi$$
15. Find the range of
$$f(x) = \frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4}$$
16. Find the range of $f(x) = \cot^{-1}(2x - x^2)$.
17. Find the range of
$$f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$$
18. Find the range of
$$f(x) = \sin^{-1}x + \sec^{-1}x + \tan^{-1}x$$
19. Find the range of
$$f(x) = 3\cot^{-1}x + 2\tan^{-1}x + \frac{\pi}{4}$$
20. Prove that $\sin(\cot^{-1}(\tan(\cos^{1}x))) = x$, $\forall x \in (0, 1]$
21. Prove that $\sin(\operatorname{cosec}^{-1}(\cot(\tan^{-1}x))) = x$, $\forall x \in (0, 1]$
22. Find the value of $\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) + \tan^{-1}(\tan(-6)) + \cot^{-1}(\cot(-10))$
23. If $U = \cot^{-1}(\sqrt{\cos 2\theta}) - \tan^{-1}(\sqrt{\cos 2\theta})$, then prove that $\sin U = \tan^2\theta$
24. Prove that
$$\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}$$
25. Prove that
$$\cos^{-1}\left(\frac{\cos x + \cos y}{1 + \cos x\cos y}\right) = 2\tan^{-1}\left(\tan\left(\frac{x}{2}\right)\tan\left(\frac{y}{2}\right)\right)$$
26. Prove that
$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{x}{2}\right)\right) = \cos^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right)$$
27. If $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are in AP then prove that $(x + z)y^2 + 2y(1 - xz)$, where $y \in (0, 1)$, $xz < 1$, $x > 0$ and $z > 0$.
28. Prove that
$$\sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right) = \frac{13\pi}{7}$$
29. Solve for $x$ and $y$:
$$\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}, \cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}.$$
30. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right)$, then prove that $x^2 = \sin(2y)$.

31. Prove that

$$\frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)+\frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\alpha}{\beta}\right)\right)=(\alpha+\beta)(\alpha^2+\beta^2)$$

32. Find the minimum value of $n$, if

$$\cot^{-1}\left(\frac{n^2-10n+21.6}{\pi}\right)>\frac{\pi}{6}, n\in N$$

33. Prove that

$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right)+\cos^{-1}\left(\frac{\sqrt{12}}{4}\right)+\sec^{-1}(\sqrt{2})\right\}\right\}=0$$

34. Solve for $x$:

$$[\sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1}x)))]=1$$

where [,] = GIF

35. Find the interval for which

$$2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right) \text{ is independent of } x.$$

36. If $x=\operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ and $y=\sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1}a)))))$, where $a\in[0,1]$, then find the relation between $x$ and $y$.

37. Find the sum of the infinite series.

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\ldots$$

38. Find the sum of

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right)+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right)+\ldots\ldots+\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}\times\sqrt{n+1}}\right)+\ldots\text{ to }\infty$$

39. Find the sum of infinite series:

$$\cot^{-1}(2.1^2)+\cot^{-1}(2.2^2)+\cot^{-1}(2.3^2)+\ldots$$

40. If $\cos^{-1}\left(\frac{x}{2}\right)+\cos^{-1}\left(\frac{y}{3}\right)=\theta$, then prove that

$$9x^2-2xy\cos\theta+4y^2=36\sin^2\theta$$

**[Roorkee, 1984]**

41. Evaluate: $\tan\left(\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right)$ **[Roorkee, 1986]**

***Note:*** No questions asked between 1987 and 1991.

42. Solve for $x$: $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}]$

**[Roorkee, 1992]**

43. Find all positive integral solutions of

$$\tan^{-1}x+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$$

**[Roorkee, 1993]**

44. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then find the value of $x^2+y^2+z^2+2xyz$.

**[Roorkee, 1994]**

45. Convert the trigonometric function $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}]$ into an algebraic function $f(x)$. Then from the algebraic function $f(x)$, find all values of $x$ for which $f(x)$ is zero.

Also, express the values of $x$ in the form of $a\pm\sqrt{b}$, where $a$ and $b$ are rational numbers.

**[Roorkee, 1995]**

***Note*** No questions asked in 1996.

46. If $\theta=\tan^{-1}(2\tan^2\theta)-\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$, then find the general value of $\theta$

**[Roorkee, 1997]**

***Note*** No questions asked in 1998.

47. Using the principal values, express the following expression as a single angle

$$3\tan^{-1}\left(\frac{1}{2}\right)+2\tan^{-1}\left(\frac{1}{5}\right)+\sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)$$

**[Roorkee, 1999]**

48. Solve for $x$:

$$\sin^{-1}\left(\frac{ax}{c}\right)+\sin^{-1}\left(\frac{bx}{c}\right)=\sin^{-1}x$$

where $a^2+b^2=c^2$, $c\neq 0$ **[Roorkee, 2000]**

49. Solve for $x$:

$$\cos^{-1}(x\sqrt{6})+\cos^{-1}(3\sqrt{3}\,x^2)=\frac{\pi}{2}$$

**[Roorkee, 2001]**

50. Let $x_1, x_2, x_3, x_4$ be four non zero numbers satisfying the equation

$$\tan^{-1}\left(\frac{a}{x}\right)+\tan^{-1}\left(\frac{b}{x}\right)+\tan^{-1}\left(\frac{c}{x}\right)+\tan^{-1}\left(\frac{d}{x}\right)=\frac{\pi}{2}$$

then prove that

(i) $\sum_{i=1}^{4}x_i=0$ (ii) $\sum_{i=1}^{4}\left(\frac{1}{x_i}\right)=0$

(iii) $\prod_{i=1}^{4}(x_i)=abcd$ (iv) $\Pi(x_1+x_2+x_3)=abcd$

51. Let $\cos^{-1}(x)+\cos^{-1}(2x)+\cos^{-1}(3x)=\pi$

If $x$ satisfies the cubic equation

$$ax^3+bx^2+cx-1=0,$$

then find the value of $(a+b+c+2)$.

52. If $x=\sin(2\tan^{-1}2)$, $y=\sin\left(\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right)$ then prove that $y^2=1-x$

## Level IV

### (Tougher Problems for JEE Advanced)

1. Prove that
$\sin^{-1}(\cos(\sin^{-1} x)) + \cos^{-1}(\sin(\cos^{-1} x))$
2. Prove that
$$\tan^{-1}\{\text{cosec}(\tan^{-1}x) - \tan(\cot^{-1}x)\} = \frac{1}{2}\tan^{-1}x$$
where $x \neq 0$
3. Prove that
$\tan(\tan^{-1} x + \tan^{-1} y + \tan^{-1} z)$
$= \cot(\cot^{-1} x + \cot^{-1} y + \cot^{-1} z)$.
4. Prove that $\sin(\cot^{-1}(\tan(\cos^{-1} x))) = \forall x \in (0, 1]$
5. Prove that $\sin(\text{cosec}^{-1}(\cot(\tan^{-1} x))) = x \ \forall x \in (0, 1]$
6. Find the value of
$\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) + \tan^{-1}(\tan(-6)) + \cot^{-1}(\cot(1-10))$
7. Find the simplest value of
$$\cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right), \forall x \in \left(\frac{1}{2}, 1\right)$$
8. Find the value of
$$\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{\sqrt{5-2\sqrt{6}}}{1+\sqrt{6}}\right)$$
9. Let $m = \sin^{-1}(a^6 + 1) + \cos^{-1}(a^4 + 1) - \tan^{-1}(a^2 + 1)$
Then find the image of the line $x + y = m$ about the $y$-axis.
10. If $(\sin^{-1}x)^3 + (\sin^{-1}y)^3 + (\sin^{-1}z)^3 = \dfrac{(3\pi)^3}{8}$
then find the value of $(3x + 4y - 5z + 2)$
11. Let $S = \sum_{r=1}^{n} \cot^{-1}\left(2^{r+1} + \dfrac{1}{2^r}\right)$
Then find $\lim_{n\to\infty}(S)$
12. Find the value of
$$\lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{4}{4r^2+3}\right)\right)\right)$$
13. Find the number of solution of the equation
$$2\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi x^3$$
14. If $\cos^{-1}\left(\dfrac{x}{a}\right) + \cos^{-1}\left(\dfrac{y}{b}\right) = \alpha$, then prove that,
$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$$
15. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$,
then prove that,
$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$
16. Find the greatest and least value of the function
$f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$.
17. Solve for $x$: $\sin^{-1}x + \sin^{-1}2x = \dfrac{\pi}{2}$
18. Solve for $x$:
$$\tan^{-1}\left(\frac{1}{1+2x}\right) + \tan^{-1}\left(\frac{1}{1+4x}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$
19 Solve for $x$:
$$\tan^{-1}(x-1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}(3x)$$
20. Solve for $x$: $\sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right) + \cos^{-1}x = \dfrac{\pi}{4}$
21. Solve for $x$:
$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$
22. Solve for $x$:
$$2\tan^{-1}x = \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) - \cos^{-1}\left(\frac{1-b^2}{1+b^2}\right),$$
$a > 0, b > 0$
23. Solve for $x$:
$$\cot^{-1} x + \cot^{-1}(n^2 - x + 1) = \cot^{-1}(n - 1)$$
24. Solve for $x$:
$$\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$
25. Solve for $x$:
$$\sec^{-1}\left(\frac{x}{a}\right) - \sec^{-1}\left(\frac{x}{b}\right) = \sec^{-1}b - \sec^{-1}a$$
26. Find the sum of
$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{8n}{n^4 - 2n^2 + 5}\right)$$
27. Find the number of real solutions of the equation
$$\sin^{-1}(e^x) + \cos^{-1}(x^2) = \frac{\pi}{2}$$
28. Find the number of real roots of
$$\sqrt{\sin(x)} = \cos^{-1}(\cos x) \text{ in } (0, 2\pi)$$
29. If $\tan^{-1}\left(\dfrac{1}{3}\right) + \tan^{-1}\left(\dfrac{1}{4}\right) + \tan^{-1}\left(\dfrac{1}{5}\right) + \tan^{-1}\left(\dfrac{1}{n}\right) = \dfrac{\pi}{4}$
where $n \in N$, then find $n$
30. If $\alpha$ is the real root of $x^3 + bx^2 + cx + 1 = 0$ where $b < c$, then find the value of
$$\tan^{-1}(\alpha) + \tan^{-1}\left(\frac{1}{\alpha}\right)$$
31. If the equation $x^3 + bx^2 + cx + 1 = 0$ has only one root $\alpha$, then find the value of
$2\tan^{-1}(\text{cosec}\,\alpha) + \tan^{-1}(2\sin\alpha\sec^2\alpha)$

**Q. Solve the following inequalities:**

32. $\sin^{-1} x > \cos^{-1} x$
33. $\cos^{-1} x > \sin^{-1} x$

34. $(\cos^{-1} x)^2 - 5(\cot^{-1} x) + 6 > 0$
35. $\tan^2(\sin^{-1} x) > 1$
36. $4(\tan^{-1} x)^2 - 8(\tan^{-1} x) + 3 < 0$
37. $4 \cot^{-1} x - (\cot^{-1} x)^2 - 3 \geq 0$
38. $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 2$
39. Find the maximum value of
$$f(x) = \{\sin^{-1}(\sin x)\}^2 - \sin^{-1}(\sin x)$$
40. Find the minimum value of
$$f(x) = 8^{\sin^{-1}x} + 8^{\cos^{-1}x}$$
41. Find the set of values of $k$ for which $x^2 - kx + \sin^{-1}(\sin 4) > 0$, for all real $x$
42. If $A = 2\tan^{-1}(2\sqrt{2}-1)$ and
$$= 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right),$$
then prove that $A > B$
43. Prove that
$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right) + \cos^{-1}\left(\frac{\sqrt{12}}{4}\right) + \sec^{-1}(\sqrt{2})\right\}\right\}$$
$= 0$
44. Find the domain of the function
$$f(x) = \sin^{-1}(\cos^{-1} x + \tan^{-1} x + \cot^{-1} x)$$
45. If $\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sin^{-1}\left(\sqrt{1-\frac{y}{4}}\right) + \tan^{-1} y = \frac{2\pi}{3}$
then find the maximum value of $(x^2 + y^2 + 1)$
46. Find the number of integral ordered pairs $(x, y)$ satisfying the equation
$$\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{1}{10}\right)$$
47. Let $\left[\cot\left(\sum_{k=1}^{10}\cot^{-1}(k^2+k+1)\right)\right] = \frac{a}{b}$
where $a$ and $b$ are co-prime, then find the value of $(a + b + 10)$.
48. If $p > q > 0$, $pr < -1 < qr$, then prove that
$$\tan^{-1}\left(\frac{p-q}{1+pq}\right) + \tan^{-1}\left(\frac{q-r}{1+qr}\right) + \tan^{-1}\left(\frac{r-p}{1+rp}\right) = \pi$$
49. Consider the equation
$$(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = a\pi^3$$
find the values of '$a$' so that the given equation has a solution.
50. If the range of the function $f(x) = \cot^{-1}\left(\frac{x^2}{x^2+1}\right)$ is $(a, b)$, find the value of $\left(\frac{b}{a}+2\right)$
51. If $\tan^{-1} y = 4\tan^{-1} x$, $\left(|x| < \tan\left(\frac{\pi}{8}\right)\right)$, find $y$ as an algebraic function of $x$ and hence prove that $\tan\left(\frac{\pi}{8}\right)$ is a root of the equation $x^4 - 6x^2 + 1 = 0$
52. Prove that
$$\tan^{-1}\left(\sqrt{\frac{a(a+b+c)}{bc}}\right) + \tan^{-1}\left(\sqrt{\frac{b(a+b+c)}{ac}}\right)$$
$$+ \tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right) = \pi, \text{ where } a, b, c > 0$$
53. Solve
$$\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$$
54. Simplify
$$\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x-\sin\theta}\right)$$
55. Solve
$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \sin^{-1}\left(\frac{2x}{x^2+1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$
56. Prove that
$$\tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{xz}{yr}\right) + \tan^{-1}\left(\frac{xy}{zr}\right) = \frac{\pi}{2}.$$
where $x^2 + y^2 + z^2 = r^2$.
57. If $\sum_{r=1}^{10}\tan^{-1}\left(\frac{3}{9r^2+3r-1}\right) = \cot^{-1}\left(\frac{m}{n}\right)$
where $m$ and $n$ are co-prime, find the value of $(2m + n + 4)$
58. If the sum $\sum_{b=1}^{10}\sum_{a=1}^{10}\tan^{-1}\left(\frac{a}{b}\right) = m\pi$, then find the value of $(m + 4)$
59. Let $f(x) = \frac{1}{\pi}(\sin^{-1}x + \cos^{-1}x + \tan^{-1}x) + \frac{(x+1)}{x^2+2x+10}$
such that the maximum value of $f(x)$ is $m$, then find the value of $(104m - 90)$.
60. Let $m$ be the number of solutions of
$\sin(2x) + \cos(2x) + \cos x + 1 = 0$ in
$0 < x < \frac{\pi}{2}$ and
$$n = \sin\left[\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right) + \cos^{-1}\left(\cos\left(\frac{7\pi}{3}\right)\right)\right]$$
then find the value of $(m^2 + n^2 + m + n + 4)$
61. Let $f(n) = \sum_{k=-n}^{n}\left(\cot^{-1}\left(\frac{1}{k}\right) - \tan^{-1}(k)\right)$
such that $\sum_{n=2}^{10}(f(n) + f(n-1)) = a\pi$
then find the value of $(a = 1)$

## Integer Type Questions

1. If the solution set of

$$\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right) < \pi - 3 \text{ is}$$

$(a, b)$, where $a, b \in I$, then find $(b - a + 5)$

2. If $a \sin^{-1} x - b \cos^{-1} x = c$, such that the value of $a \sin^{-1} x + b \cos^{-1} x$ is $\frac{m\pi ab + c(a-b)}{a+b}, m \in N$, then find the value of $(m^2 + m + 2)$

3 If $m$ is a root of $x^2 + 3x + 1 = 0$, such that the value of $\tan^{-1}(m) + \tan^{-1}\left(\frac{1}{m}\right)$ is $\frac{k\pi}{2}, k \in I$, then find the value of $(k + 4)$

4. Find the number of real solutions of

$$\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$$

5. Let $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$
If $x$ satisfies the cubic equation
$ax^3 + bx^2 + cx + d = 0$, then find the value of $(b + c) - (a + d)$

6. Consider $\alpha, \beta, \gamma$ are the roots of $x^3 - x^2 - 3x + 4 = 0$ such that $\text{an}^{-1} \alpha + \tan^{-1} \beta + \tan^{-1} \gamma = \theta$
If the positive value of $\tan(\theta)$ is $\frac{p}{q}$, where $p$ and $q$ are natural numbers, then find the value of $(p + q)$

7. If $M$ is the number of real solution of $\cos^{-1} x + \cos^{-1}(2x) + \pi = 0$ and $N$ is the number of values of $x$ satisfying the equation $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$, then find the value of $M + N + 4$

8. Find the value of

$$4\cos\left[\cos^{-1}\left(\frac{1}{4}(\sqrt{6}-\sqrt{2})\right) - \cos^{-1}\left(\frac{1}{4}(\sqrt{6}+\sqrt{2})\right)\right]$$

9. Find the value of

$$5\cot\left(\sum_{k=1}^{5}\cot^{-1}(k^2+k+1)\right)$$

10. Let $3\sin^{-1}(\log_2 x) + \cos^{-1}(\log_2 y) = \frac{\pi}{2}$

and $\sin^{-1}(\log_2 x) + 2\cos^{-1}(\log_2 y) = \frac{11\pi}{6}$

then find the value of $\left(\frac{1}{x^2} + \frac{1}{y^2} + 2\right)$

11. If $\alpha$ and $\beta$ are the roots of $x^2 + 5x - 44 = 0$, then find the value of $\cot(\cot^{-1} \alpha + \cot^{-1} \beta)$

12. If $x$ and $y$ are positive integers satisfying $\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{1}{7}\right)$, then find the number of ordered pairs of $(x, y)$

## Comprehensive Link Passages

In these questions, a passage (paragraph) has been given followed by questions based on each of the passage. You have to answer the questions based on the passage given.

**Passage 1**

| Function | Domain | Co-domain |
|---|---|---|
| $\sin^{-1}x$ | $[-1, 1]$ | $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ |
| $\tan^{-1}x$ | $R$ | $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ |
| $\cos^{-1}x$ | $[-1, 1]$ | $[\pi, 2\pi]$ |
| $\cot^{-1}x$ | $R$ | $[\pi, 2\pi]$ |

1. $\sin^{-1}(-x)$ is
   (a) $-\sin^{-1} x$
   (b) $p + \sin^{-1} x$
   (c) $2\pi - \sin^{-1} x$
   (d) $2\pi - \cos^{-1}\sqrt{1-x^2}, x > 0$

2. If $f(x) = 3\sin^{-1} x - 2\cos^{-1} x$, then $f(x)$ is
   (a) even function
   (b) odd function
   (c) neither even nor odd
   (d) even as well as odd function.

3. The minimum value of $(\sin^{-1} x)^3 - (\cos^{-1} x)^3$ is
   (a) $-\frac{63\pi^3}{8}$ (b) $\frac{63\pi^3}{8}$
   (c) $\frac{125\pi^3}{32}$ (d) $-\frac{125\pi^3}{32}$

4. The value of $\sin^{-1} x + \cos^{-1} x$ is
   (a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{2}$ (d) $\frac{7\pi}{2}$

5. If the co-domain of $\sin^{-1} x$ is $\left[-\frac{5\pi}{2}, -\frac{3\pi}{2}\right]$ such that $\sin^{-1}x + \cos^{-1}x = \frac{5\pi}{2}$, then the co-domain of $\cos^{-1} x$ is
   (a) $[4\pi, 5\pi]$ (b) $[3\pi, 4\pi]$
   (c) $[6\pi, 7\pi]$ (d) $[5\pi, 6\pi]$

**Passage II**

We know that corresponding to every bijection function

$f: A \to B$, there exist a bijection.
$g: B \to A$ defined by $g(y) = x$ if and only if $f(x) = y$

The function $g: B \to A$ is called the inverse of function $f: A \to B$ and is denoted by $f^{-1}$.

Thus, we have $f(x) = y \Rightarrow f^{-1}(y) = x$

We know that trigonometric functions are periodic functions and hence, in general all trigonometric functions are not bijectives.

Consequently, their inverse do not exist.

However, if we restrict their domains and co-domains, they we can make the bijectives and also we can find their inverse.

Now, answer the following questions.

1. $\sin^{-1}(\sin\theta) = \theta$, for all $\theta$ belonging to
   (a) $[0, \pi]$ (b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
   (c) $\left[-\frac{\pi}{2}, 0\right]$ (d) None of these
2. $\cos^{-1}(\cos\theta) = \theta$, for all $\theta$ belonging to
   (a) $[0, \pi]$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
   (c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) None of these
3. $\tan^{-1}(\tan\theta) = \theta$, for all $\theta$ belonging to
   (a) $[0, \pi]$ (b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
   (c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) None of these
4. $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, for all $\theta$ belonging to
   (a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$
   (c) $[0, \pi]$ (d) $(0, \pi)$
5. $\sec^{-1}(\sec\theta) = \theta$, for all $\theta$ belonging to
   (a) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (b) $(0, \pi) - \left\{\frac{\pi}{2}\right\}$
   (c) $(0, \pi)$ (d) None of these
6. $\sin^{-1}(\sin x) = x$, for all $x$ belonging to
   (a) $R$ (b) $\varphi$
   (c) $[-1,1]$ (d) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
7. The value of $\sin^{-1}(\sin 2) + \cos^{-1}(\cos 2)$ is
   (a) 0 (b) $\frac{\pi}{2}$
   (c) $-\frac{\pi}{2}$ (d) None of these

**Passage III**

Let $f(x) = \sin\{\cot^{-1}(x + 1)\} - \cos(\tan^{-1} x)$ and $a = \cos(\tan^{-1}(\sin(\cot^{-1} x)))$ and $b = \cos(2\cos^{-1} x + \sin^{-1} x)$

1. The value of $x$ for which $f(x) = 0$ is
   (a) $-1/2$ (b) 0 (c) 1/2 (d) 1
2. If $f(x) = 0$, then $a^2$ is equal to
   (a) 1/2 (b) 2/3 (c) 5/9 (d) 9/5
3. If $a^2 = \frac{26}{51}$, then $b^2$ is equal to
   (a) 1/25 (b) 24/25 (c) 25/26 (d) 50/51

**Passage IV**

Every bijective (one-one onto function)

$f: A \to B$ there exists a bijection

$g: B \to A$ is defined by $g(y) = x$

if and only if $f(x) = y$.

The function $g : B \to A$ is called the inverse of function $f : A \to B$ and is denoted by $f^{-1}$.

If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of that function.

1. The value of $\cos\{\tan^{-1}(\tan 2)\}$ is
   (a) $1/\sqrt{5}$ (b) $-1/\sqrt{5}$ (c) $\cos 2$ (d) $-\cos 2$
2. If $x$ takes negative permissible value then $\sin^{-1} x$ is
   (a) $\cos^{-1}(\sqrt{1-x^2})$ (b) $-\cos^{-1}(\sqrt{1-x^2})$
   (c) $\cos^{-1}(\sqrt{x^2-1})$ (d) $\pi - \cos^{-1}(\sqrt{1-x^2})$.
3. If $x + \frac{1}{x} = 2$, then the value of $\sin^{-1} x$ is
   (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\pi$ (d) $\frac{3\pi}{2}$

**Passage V**

$$\text{Let } \cos^{-1}x + (\sin^{-1}y)^2 = \frac{a\pi^2}{4} \quad \ldots\text{(i)}$$

$$\text{and } \cos^{-1}x \cdot (\sin^{-1}y)^2 = \frac{\pi^2}{16} \quad \ldots\text{(ii)}$$

where $-1 \le x, y \le 1$. Then

1. The set of values of '$a$' for which the equation (i) holds good is
   (a) $\left(0, 2+\frac{4}{\pi}\right)$ (b) $\left[0, 1+\frac{4}{\pi}\right)$
   (c) $R$ (d) $\left[0, -1+\frac{4}{\pi}\right)$
2. The set of values of '$a$' for which equations (i) and (ii) posses solutions
   (a) $(-\infty, 2] \cup [2, \infty)$ (b) $(-2, 2)$
   (c) $\left[2, 1+\frac{4}{\pi}\right]$ (d) $R$
3. The values of $x$ and $y$, the system of equations (i) and (ii) posses solutions for integral values of '$a$'
   (a) $\left\{\cos\left(\frac{\pi^2}{4}\right), 1\right\}$ (b) $\left\{\cos\left(\frac{\pi^2}{4}\right), -1\right\}$
   (c) $\left\{\cos\left(\frac{\pi^2}{4}\right), \pm 1\right\}$ (d) $\{(x, y): x \in R, y \in R\}$

## Matrix Match
## (For JEE-Advanced Examination Only)

Given below are matching type questions, with two columns (each having some items) each.

Each item of column I has to be matched with the items of column II, by encircling the correct match(es).

***Note:*** An item of Column I can be matched with more than one items of Column II. All the items of Column II have to be matched.

1. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | The principal value of $\sin^{-1}(\sin 20)$ is | (P) | $(20 - 6\pi)$ |
| (B) | The principal value of $\sin^{-1}(\sin 10)$ is | (Q) | $(3\pi - 10)$ |
| (C) | The principal value of $\cos^{-1}(\cos 10)$ is | (R) | $(4\pi - 10)$ |
| (D) | The principal value of $\cos^{-1}(\cos 20)$ is | (S) | $(5\pi - 20)$ |

2. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | The range of $f(x) = 3\sin^{-1}x + 2\cos^{-1}x$ is | (P) | $(0, \pi)$ |
| (B) | The range of $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ is | (Q) | $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ |
| (C) | The range of $f(x) = \sqrt{\sin^{-1}x + \pi}$ is | (R) | $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ |
| (D) | The range of $f(x) = 2\tan^{-1}x + \sin^{-1}x + \sec^{-1}\left(\frac{1}{x}\right)$ is | (S) | $[0, \pi]$ |

3. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | $\sin(\sin^{-1}x) = \sin^{-1}(\sin x)$, if | (P) | $-1 \le x \le 1$ |
| (B) | $\cos(\cos^{-1}x) = \cos^{-1}(\cos x)$, if | (Q) | $0 \le x \le 1$ |
| (C) | $\tan(\tan^{-1}x) = \tan^{-1}(\tan x)$, if | (R) | $0 < x < \pi$ |
| (D) | $\cot(\cot^{-1}x) = \cot^{-1}(\cot x)$, if | (R) | $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ |

4. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| The value of | | | |
| (A) | $\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$ | (P) | $\frac{7\pi}{6}$ |
| (B) | $\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{2}\right)$ | (Q) | $\frac{5\pi}{6}$ |
| (C) | $\tan^{-1}(\sqrt{3}) + \cot^{-1}(-\sqrt{3})$ | (R) | $\frac{\pi}{6}$ |
| (D) | $\sin^{-1}\left(\frac{1}{2013}\right) + \cos^{-1}\left(\frac{1}{2013}\right)$ | (S) | $\frac{\pi}{2}$ |

5. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | The value of $\tan^{-1} + \tan^{-1}2 + \tan^{-1}3$ is | (P) | $\frac{3\pi}{4}$ |
| (B) | The value of $\tan^{-1}1 + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$ is | (Q) | $\frac{\pi}{2}$ |
| (C) | The value of $\tan^{-1}(9) + \tan^{-1}\left(\frac{5}{4}\right)$ is | (R) | $\pi$ |
| (D) | The value of $2\tan^{-1}x - \tan^{-1}\left(\frac{2x}{1-x^2}\right), x > 1$ is | (S) | $-\frac{\pi}{2}$ |

6. Match the following columns:

| Column I | | Column II | |
|---|---|---|---|
| (A) | $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$ is maximum at | (P) | $x = \frac{1}{\sqrt{2}}$ |
| (B) | $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$ is minimum at | (Q) | $x = 1$ |
| (C) | $(\sin^{-1}x) - (\cos^{-1}x)$ is minimum at | (R) | $x = -1$ |
| (D) | $(\tan^{-1}x)^2 + (\cot^{-1}x)^2$ is minimum at | (S) | $x = 0$ |

## Assertion and Reason

**Codes:**

(A) Both A and R are individually true and R is the correct explanation of A

(B) Both A and R are individually true and R is not the correct explanation of A.

(C) A is true but R is false.

(D) A is false but R is true.

1. *Assertion (A):* If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then the value of

$$(x^{2013} + y^{2013} + z^{2013}) - \frac{9}{(x^{2014} + y^{2014} + z^{2014})}$$ is zero.

*Reason (R):* Maximum value of $\sin^{-1}x$ is $\frac{\pi}{2}$

(a) A (b) B (c) C (d) D

2. *Assertion (A):* The value of $2\tan^{-1}x - \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ is $\pi$

*Reason (R):* $x > 1$

(a) A (b) B (c) C (d) D

3. *Assertion (A):* The value of

$$\tan^{-1}(p) + \tan^{-1}\left(\frac{1}{p}\right) \text{ is } -\frac{\pi}{2}$$

*Reason (R):* $P$ is the root of $x^2 + 2013x + 2014 = 0$.

(a) A (b) B (c) C (d) D

4. *Assertion (A):*

If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, then the value of $\cos^{-1} x + \cos^{-1} y$ is $\frac{\pi}{3}$

*Reason (R):* $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, when $x \in [-1, 1]$

(a) A (b) B (c) C (d) D

5. *Assertion (A):*

The value of $\cos^{-1}(\cos 10)$ is $(2\pi - 5)$

*Reason (R):*

The range of $\cos^{-1} x$ is $[0, \pi]$

(a) A (b) B (c) C (d) D

6. *Assertion (A):*

If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then $x^2 + y^2 + z^2 + 2xyz = 1$

*Reason(R):* For $-1 \le x, y, z \le 1$

(a) A (b) B (c) C (d) D

7. *Assertion (A):*

If $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$, then $x$ is $\frac{1}{6}$

*Reason (R):* For $0 < 2x, 3x < 1$

(a) A (b) B (c) C (d) D

8. *Assertion (A):* $\cos\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$

*Reason (R):* for $x \ge 0$

(a) A (b) B (c) C (d) D

9. *Assertion (A):* $\sin^{-1}(3x - 4x^3) = \pi - 3\sin^{-1} x$

*Reason (R):* for $\frac{1}{2} < x \le 1$

(a) A (b) B (c) C (d) D

10. *Assertion (A):* $\cos^{-1}(4x^3 - 3x) = 2\pi - 3\cos^{-1}(x)$

*Reason (R):* For $-\frac{1}{2} \le x < \frac{1}{2}$

(a) A (b) B (c) C (d) D

11. *Assertion (A):* $\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right), x > 0$

*Reason (R):* $\cot^{-1}(x) = \pi + \tan^{-1}\left(\frac{1}{x}\right), x < 0$

(a) A (b) B (c) C (d) D

12. *Assertion (A):*

If $\alpha$. $\beta$ are the roots of $x^2 - 3x + 2 = 0$, then $\sin^{-1}\alpha$ exists but not $\sin^{-1}\beta$, where $\alpha > \beta$

*Reason (R):* Domain of $\sin^{-1} x$ is $[-1, 1]$

(a) A (b) B (c) C (d) D

## Questions Asked In Previous Years' JEE-Advanced Examinations

1. Let $a$, $b$, $c$ be positive real numbers such that

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

Then $\tan\theta$ is equal to? **[IIT-JEE, 1981]**

2. The numerical value of

$$\tan^{-1}\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\} \text{ is ?}$$

**[IIT-JEE, 1981]**

3. Find the value of $\cos(2\cos^{-1} x + \sin^{-1} x)$ at $x = 1/5$, where $0 \le \cos^{-1} x \le \pi$ and $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

**[IIT-JEE, 1981]**

4. The value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$ is

(a) 6/17 (b) 17/6 (c) –17/6 (d) –6/17

**[IIT-JEE, 1983]**

5. No questions asked between 1984 and 1985.

6. The principal value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is

(a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{3}$

**[IIT-JEE, 1986]**

7. No questions asked between 1987 and 1988.

8. The greater of the two angles $A = 2\tan^{-1}(2\sqrt{2} - 1)$ and $B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$ is.......

**[IIT-JEE, 1989]**

9. No questions asked between 1990 and 1998.

10. The number of real solutions of

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2} \text{ is}$$

(a) 0 (b) 1 (c) 2 (d) $\infty$

**[IIT-JEE, 1999]**

11. No questions asked in 2000.

12. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - .......\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ....\right) = \frac{\pi}{2}$,

for $0 < |x| < \sqrt{2}$, then $x$ is

(a) 1/2 (b) 1 (c) –1/2 (d) –1

**[IIT-JEE, 2001]**

13. Prove that $\cos(\tan^{-1}(\sin(\cot^{-1} x))) = \sqrt{\frac{x^2+1}{x^2+2}}$

**[IIT-JEE, 2002]**

14. The domain of $f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ is
(a) $\left(-\frac{1}{2},\frac{1}{2}\right]$ (b) $\left[-\frac{1}{4},\frac{3}{4}\right)$
(c) $\left[-\frac{1}{4},\frac{1}{4}\right]$ (d) $\left[-\frac{1}{4},\frac{1}{2}\right]$
**[IIT-JEE, 2003]**

15. If $\sin(\cot^{-1}(x+1))=\cos(\tan^{-1}x)$, then the value of $x$ is
(a) $-1/2$ (b) $1/2$ (c) $0$ (d) $9/4$
**[IIT-JEE, 2004]**

16. No questions asked in between 2005 to 2006.

17. Match the following columns:
Let $(x, y)$ be such that
$$\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\frac{\pi}{2}$$

| | Column I | | Column II |
|---|---|---|---|
| (A) | If $a=1$ and $b=0$, then $(x, y)$ | (P) | lies on the circle $x^2+y^2=1$ |
| (B) | If a = 1 and b = 1, then $(x, y)$ | (Q) | lies on $(x^2-1)(y^2-1)=0$ |
| (C) | If $a=1$ and $b=2$, then $(x, y)$ | (R) | lies on the line $y=x$ |
| (D) | If $a=2$ and $b=2$, then $(x, y)$ | (S) | lies on $(4x^2-1)(y^2-1)=0$ |

**[IIT-JEE, 2007]**

18. If $0<x<1$, then
$\sqrt{1+x^2}\times[\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2-1]^{1/2}$ equals
(a) $\frac{x}{\sqrt{1+x^2}}$ (b) $x$
(c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$
**[IIT-JEE, 2008]**

19. No questions asked between 2009 to 2010.

20. Let $f(\theta)=\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$
where $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is.......
**[IIT-JEE, 2011]**

21 No questions asked in 2012.

22 The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is
(a) 23/25 (b) 25/23 (c) 23/24 (d) 24/23
**[IIT-JEE, 2013]**

23. The value of
$$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)^2+y^4\right)^{1/2}$$
is......... **[IIT-JEE, 2013]**

24 If $\cot(\sin^{-1}\sqrt{1-x^2})=\sin(\tan^{-1}(x\sqrt{6}))$, $x\neq 0$ then the value of $x$ is.......
**[IIT-JEE, 2013]**

25. The number of positive solutions satisfying the equation
$$\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$
is.... **[IIT-JEE, 2014]**

26. No questions asked in 2015.

## Answers

### Level II

| | | | | |
|---|---|---|---|---|
| 1. (d) | 2. (b) | 3. (b) | 4. (b) | 5. (d) |
| 6. (b) | 7. (b) | 8. (d) | 9. (a) | 10. (c) |
| 11. (c) | 12. (c) | 13. (a, c) | 14. (a) | 15. (d) |
| 16. (b) | 17. (b) | 18. (c) | 19. (c) | 20. (b) |
| 21. (a) | 22. (a) | 23. (b) | 24. (c, d) | 25. (a) |
| 26. (b) | 27. (b) | 28. (a) | 29. (a) | 30. (c) |
| 31. (b) | 32. (a) | 33. (a) | 34. (c) | 35. (b) |
| 36. (c) | 37. (c) | 38. (b) | 39. (c) | 40. (a) |
| 41. (b) | 42. (a) | 43. (a) | | |

### Level III

6. $(8\pi-21)$
7. $\frac{\pi}{3}$
16. Maximum Value $\frac{7\pi^3}{8}$, when $x=-1$
and minimum Value $=\frac{\pi^3}{32}$, when $x=\frac{1}{\sqrt{2}}$
17 $x=\frac{1}{2}\sqrt{\frac{3}{7}}$
18. $x=3$
19. $x=0, \frac{1}{2}, \frac{-1}{2}$
20. $x=\frac{3}{\sqrt{10}}$
21. $x=2-\sqrt{3}, \sqrt{3}$
22. $x=\frac{a-b}{1+ab}$
23. $x=n^2-n+1, n$
24. $x=\frac{4}{3}$
25. $x=ab$

26. $x = \frac{1}{2}, y = 1$
27. $x = 1, y = 2$ ; $x = 2, y = 7$
29. $-\pi$
32. $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$
33. $x \in \left(-1, \frac{1}{\sqrt{2}}\right]$
34. $x \in (-\infty, \cot 3) \cup (\cot 2, \infty)$
35. $x \in \left(\frac{1}{\sqrt{2}}, 1\right) \cup \left(-1, -\frac{1}{\sqrt{2}}\right)$
36. $\tan\left(\frac{1}{2}\right) < x < \tan\left(\frac{3}{2}\right)$
37. $\cot(3) \le x \le \cot(1)$
38. $x \in (-1, 1)$
39. $n = 5$
40. $n = 8$
41. $k = \phi$
44. $\tan(\sin(\cos(\sin 1))) \le x < \tan(\sin(1))$.
45. $[1, \infty)$
49. $a \in \left[\frac{1}{32}, \frac{7}{8}\right]$
50. $x = y = \sqrt{a^2 + 1}$
53. $\theta \in n\pi + \tan^{-1}(-2), n \in Z$
54. $\theta$
55. $x = \frac{1}{\sqrt{3}}$,
56. $\frac{\pi}{2}$
57. 36
58. 29
59. 4
60. 6
61. 100

**INTEGER TYPE QUESTIONS**

1. 7
2. 4
3. 3
4. $x = 1$
5. 3
6. 9
7. 5, where $M = 0$, $N = 1$
8. 2
9. 7
10. 8
11, 9
12. 6

**COMPREHENSIVE LINK PASSAGES**

| | | | | |
|---|---|---|---|---|
| Passage I: | 1. (c) | 2. (b) | 3. (a) | 4. (c) |
| | 5. (a) | | | |
| Passage II: | 1. (b) | 2. (a) | 3. (c) | 4. (a) |
| | 5. (a) | 6. (c) | 7. (a) | |
| Passage III: | 1. (a) | 2. (c) | 3. (b) | |
| Passage IV: | 1. (d) | 2. (b) | 3. (b) | |
| Passage V: | 1. (b) | 2. (c) | 3. (c) | |

**MATRIX MATCH**

1.(A) $\to$ (P); (B) $\to$ (Q); (C) $\to$ (R); (D) $\to$ (P)
2.(A) $\to$ (S); (B) $\to$ (P); (C) $\to$ (Q); (D) $\to$ (R)
3.(A) $\to$ (P); (B) $\to$ (Q); (C) $\to$ (S); (D) $\to$ (R)
4.(A) $\to$ (Q); (B) $\to$ (R); (C) $\to$ (P); (D) $\to$ (S)
5.(A) $\to$ (R); (B) $\to$ (Q); (C) $\to$ (P); (D) $\to$ (R)
6.(A) $\to$ (R); (B) $\to$ (P); (C) $\to$ (Q); (D) $\to$ (Q)

**ASSERTION AND REASON**

| | | | | |
|---|---|---|---|---|
| 1. (a) | 2. (a) | 3. (a) | 4. (a) | 5. (a) |
| 6. (a) | 7. (a) | 8. (a) | 9. (a) | 10. (a) |
| 11. (b) | 12. (a) | | | |

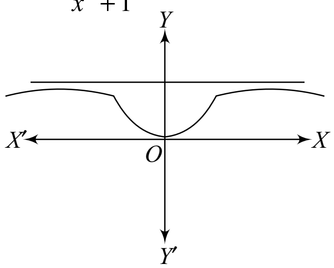
# HINTS AND SOLUTIONS

## LEVEL I

1. Given, $f(x) = 3x + 5$
   $\Rightarrow$ $f'(x) = 3 > 0$
   $\Rightarrow$ $f$ is strictly increasing function.
   $\Rightarrow$ $f$ is one one function
   Also, $R_f = R$ = Co-domain
   $\Rightarrow$ $f$ is onto function.
   Thus, $f$ is a bijective function.
   Hence, $f^{-1}$ is exists.
   Let $y = 3x + 5$
   $\Rightarrow$ $x = \frac{y-5}{3}$
   Thus, $f^{-1}(x) = \frac{x-5}{3}$.
2. Given, $f(x) = x^2 + 2$
   $\Rightarrow$ $f'(x) = 2x > 0$ for every $x > 0$
   $\Rightarrow$ $f$ is strictly increasing function.
   $\Rightarrow$ $f$ is one one function.
   Also, $R_f = (2, \infty)$ = Co-domain
   $\Rightarrow$ $f$ is ont function.
   Thus, $f$ is a bijective function.
   Therefore, the inverse of the given function exists.
   Let $y = x^2 + 2$
   $\Rightarrow$ $x^2 = y - 2$
   $\Rightarrow$ $x = \sqrt{y-2}$

Hence, $f^{-1}(x) = \sqrt{x-2}$

3. Given, $f(x) = \dfrac{x^2}{x^2+1}$




$\Rightarrow \quad f(x) = 1 - \dfrac{1}{x^2+1}$

$\Rightarrow \quad f'(x) = \dfrac{2x}{(x^2+1)^2} > 0, \forall\, x \in R^+$

$\Rightarrow$ $f$ is strictly increasing function.

$\Rightarrow$ $f$ is a one one function.

Also, let $y = \dfrac{x^2}{x^2+1}$

$\Rightarrow$ $y \cdot x^2 + y = x^2$

$\Rightarrow$ $x^2(y-1) = -y$

$\Rightarrow \quad x^2 = -\dfrac{y}{(y-1)} = \dfrac{y}{(1-y)}$

$\Rightarrow \quad x = \sqrt{\dfrac{y}{(1-y)}}$

$\Rightarrow$ $R_f = (0, 1) =$ Co-domain

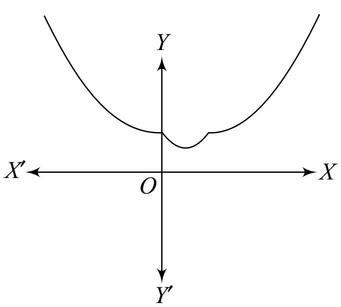
$\Rightarrow$ $f$ is onto function.

Thus, $f$ is a bijective function.

$\Rightarrow$ $f^{-1}(x)$ is exists.

Hence, $f^{-1}(x) = \sqrt{\dfrac{x}{1-x}}$.

4. Given, $f(x) = 2^{x(x-1)}$.




$\Rightarrow$ $f'(x) = 2^{x(x-1)} \times (2x-1) \times \log, 2 > 0$ for all $x$ in $[1, \infty)$

$\Rightarrow$ $f$ is strictly increasing function.

$\Rightarrow$ $f$ is a one one function.

Also, $R_t = [1, \infty)$

$\Rightarrow$ $R_f = [1, \infty)$= Co-domain

$\Rightarrow$ $f$ is onto function.

Thus, $f$ is a bijective function.

So its inverse is exists.

Let $\quad y = 2^{x(x+1)}$

$\Rightarrow \quad y = 2^{x^2-x}$

$\Rightarrow \quad x^2 - x = \log_2(y)$

$\Rightarrow \quad x^2 - x - \log_2(y) = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{1 + 4\log_2(y)}}{2}$

$\Rightarrow \quad x = \dfrac{1 + \sqrt{1 + 4\log_2(y)}}{2}$

Thus, $f^{-1}(x) = \dfrac{1 + \sqrt{1 + 4\log_2(x)}}{2}$

5. Since $f$ is a bijective function, so its inverse exists.

Let $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}} = \dfrac{10^{2x}-1}{10^{2x}+1}$.

$\Rightarrow \quad y \times 10^{2x} + y = 10^{2x} - 1$

$\Rightarrow \quad 10^{2x}(y-1) = -y - 1$

$\Rightarrow \quad 10^{2x} = -\dfrac{y+1}{y-1} = \dfrac{y+1}{1-y}$

$\Rightarrow \quad 2x = \log_{10}\left(\dfrac{y+1}{1-y}\right)$

$\Rightarrow \quad x = \dfrac{1}{2}\log_{10}\left(\dfrac{1+y}{1-y}\right)$

Thus, $f^{-1}(x) = \dfrac{1}{2}\log_{10}\left(\dfrac{1+x}{1-x}\right)$.

6. Given, $f(x) = x + \sin x$

$\Rightarrow$ $f'(x) = 1 + \cos x \geq 0$ for all $x$ in $R$.

$\Rightarrow$ $f$ is strictly increasing function

$\Rightarrow$ $f$ is a one one function.

Also, the range of a function is $R$.

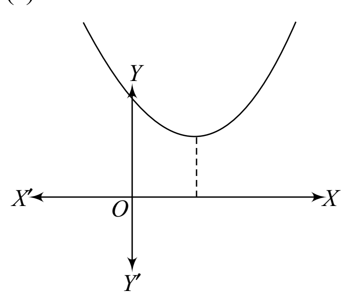
$\Rightarrow$ $f$ is a onto function

Thus, $f$ is a bijective function.

Hence, $f^{-1}$ exists.

Therefore, $f^{-1}(x) = x - \sin x$

7. Given, $f(x) = x^2 - 4x + 9$




$\Rightarrow$ $f'(x) = 2x - 4 \geq 0$ for all $x$ in $D_f$

$\Rightarrow$ $f$ is strictly increasing function.

$\Rightarrow$ $f$ is a one one function.

Also, $R_f = [5, \infty) =$ Co-domain

$\Rightarrow$ $f$ is onto function.

Therefore, $f$ is a bijective function.
Hence, its inverse is exists.
Let $y = x^2 - 4x + 3$

$\Rightarrow \quad x^2 - 4x + (5 - y) = 9$

$\Rightarrow \quad x = \dfrac{4 \pm \sqrt{16 - 4(5 - y)}}{2}$

$\Rightarrow \quad x = \dfrac{4 \pm \sqrt{4y + 16 - 20}}{2}$

$\Rightarrow \quad x = \dfrac{4 \pm \sqrt{4(y-1)}}{2} = 2 \pm \sqrt{(y-1)}$

$\Rightarrow \quad x = 2 + \sqrt{(y-1)}$, since $x \geq 2$

$\Rightarrow \quad f^{-1}(x) = 2 + \sqrt{(x-1)}$

8. Consider the function

$f : [0, \infty) \to \left[-\dfrac{1}{4}, \infty\right)$, where

$f(x) = x^2 - \dfrac{1}{4}$

Clearly, $f$ is a one one and onto function.
So its inverse is exists.

Let its inverse be $f^{-1} : \left[-\dfrac{1}{4}, \infty\right) \to [0, \infty)$.

$\Rightarrow \quad f^{-1}(x) = \sqrt{x + \dfrac{1}{4}}$.

Consequently, we can say that, the two sides of the given equation are inverse to each other.
Thus, the intersection point is the solution of the given equation. $f(x) = x$

$\Rightarrow \quad x^2 - \dfrac{1}{4} = x$

$\Rightarrow \quad x^2 - x = \dfrac{1}{4}$

$\Rightarrow \quad \left(x - \dfrac{1}{2}\right)^2 = \dfrac{1}{2}$

$\Rightarrow \quad \left(x - \dfrac{1}{2}\right) = \pm \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad x = \dfrac{1}{2} \pm \dfrac{1}{\sqrt{2}}$

Hence, the solutions are

$$\left\{\frac{1}{2} + \frac{1}{\sqrt{2}}, \frac{1}{2} - \frac{1}{\sqrt{2}}\right\}$$

9. Clearly, $f$ is bijective
So, its inverse exists
Let $\quad y = 3x + 5$

$\Rightarrow \quad x = \dfrac{y - 5}{3}$

Thus, $f^{-1}(x) = \dfrac{x - 5}{3}$

10. Clearly, $f$ is bijective
So, its inverse exists

Let $\quad y = \dfrac{x}{x - 1}$

$\Rightarrow \quad xy - y = x$

$\Rightarrow \quad x(y - 1) = y$

$\Rightarrow \quad x = \dfrac{y}{(y - 1)}$

Thus, $f^{-1}(x) = \dfrac{x}{(x - 1)}$

11. Clearly, $f$ is bijective
So its inverse exists
Let $\quad y = x^2 + 1$

$\Rightarrow \quad x = \sqrt{y - 1}$

Thus, $f^{-1}(x) = \sqrt{x - 1}$

12. Since $f$ is bijective, so its inverse exists

Let $\quad y = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$

$\Rightarrow \quad \dfrac{y}{1} = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$

$\Rightarrow \quad \dfrac{y+1}{y-1} = \dfrac{2^x - 2^{-x} + 2^x + 2^{-x}}{2^x - 2^{-x} - 2^x - 2^{-x}}$

$\Rightarrow \quad \dfrac{y+1}{y-1} = -\dfrac{2.2^x}{2.2^{-x}}$

$\Rightarrow \quad 2^{2x} = \dfrac{y+1}{1-y}$

$\Rightarrow \quad 2x = \log_2\left(\dfrac{y+1}{1-\mathrm{y}}\right)$

$\Rightarrow \quad x = \dfrac{1}{2}\log_2\left(\dfrac{y+1}{1-\mathrm{y}}\right)$

Thus, $f^{-1}(x) = \dfrac{1}{2}\log_2\left(\dfrac{x+1}{1-x}\right)$

13. Clearly, $f$ is bijective.
So, its inverse exists

Let $\quad y = \dfrac{x}{x^2 + 1}$

$\Rightarrow \quad yx^2 - x + y = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{1 - 4y^2}}{2y}$

Thus, $f^{-1}(x) = \dfrac{1 + \sqrt{1 - 4x^2}}{2x}$

14. We have

$\quad -1 \leq 3x + 5 \leq 1$

$\Rightarrow \quad -6 \leq 3x \leq -1$

$\Rightarrow \quad -2 \leq x \leq -\dfrac{4}{3}$

$\Rightarrow \quad D_f = x \in \left[-2, -\frac{4}{3}\right]$

15. We have

$$-1 \le \frac{x}{x+1} \le 1$$

**Case I:** When $\frac{x}{x+1} \le 1$

$\Rightarrow \quad \frac{x}{x+1} - 1 \le 0$

$\Rightarrow \quad \frac{-1}{x+1} \le 0$

$\Rightarrow \quad \frac{1}{x+1} \ge 0$

$\Rightarrow \quad x > -1$

**Case II:** When $\frac{x}{x+1} \ge -1$

$\Rightarrow \quad \frac{x}{x+1} + 1 \ge 0$

$\Rightarrow \quad \frac{2x+1}{x+1} \ge 0$

$\Rightarrow \quad x \in (-\infty, -1) \cup \left[-\frac{1}{2}, \infty\right)$

Hence, $D_f = \left[-\frac{1}{2}, \infty\right)$.

16. We have

$$-1 \le \frac{x^2+1}{2x} \le 1$$

$\Rightarrow \quad \left|\frac{x^2+1}{2x}\right| \le 1$

$\Rightarrow \quad \frac{|x^2+1|}{|2x|} \le 1$

$\Rightarrow \quad \frac{|x^2+1|}{2|x|} \le 1$

$\Rightarrow \quad x^2 + 1 \le 2|x|$

$\Rightarrow \quad |x|^2 - 2|x| + 1 \le 0$

$\Rightarrow \quad (|x| - 1)^2 \le 0$

$\Rightarrow \quad (|x| - 1)^2 = 0$

$\Rightarrow \quad (|x| - 1) = 0$

$\Rightarrow \quad |x| = 1$

$\Rightarrow \quad x = \pm 1$

Hence, $D_f = \{-1, 1\}$

17. We have $-1 \le \frac{|x|-1}{2} \le 1$

$\Rightarrow \quad -2 \le |x| - 1 \le 2$

$\Rightarrow \quad -1 \le |x| \le 3$

$\Rightarrow \quad |x| \le 3$ ($\because |x| \ge -1$ is rejected)

$\Rightarrow \quad -3 \le x \le 3$

Hence, $D_f$ $[-3, 3]$

18. We have $-1 \le (\log_2 x) \le 1$

$\Rightarrow \quad 2^{-1} \le x \le 2^1$

$\Rightarrow \quad \frac{1}{2} \le x \le 2$

Hence, $D_f = \left[\frac{1}{2}, 2\right]$

19. We have

$$-1 \le \log_3 x^2 \le 1$$

$\Rightarrow \quad 4^{-1} \le x^2 \le 4^1$

$\Rightarrow \quad \frac{1}{4} \le x^2 \le 4$

$\Rightarrow \quad \frac{1}{2} \le |x| \le 2$

$\Rightarrow \quad |x| \le 2$ and $|x| \ge \frac{1}{2}$

$\Rightarrow \quad -2 \le x \le 2$ and $x \ge \frac{1}{2}$ and $x \le -\frac{1}{2}$

$\Rightarrow \quad x \in \left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right]$

Hence, $D_f = \left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right]$

20. Given, $\sin^{-1} x + \sin^{-1} y = \pi$

It is possible only when each term of the given equation provides the maximum value.

Thus, $\sin^{-1} x = \frac{\pi}{2}$ and $\sin^{-1} y = \frac{\pi}{2}$

$\Rightarrow \quad x = \sin\left(\frac{\pi}{2}\right) = 1$ and $y = \sin\left(\frac{\pi}{2}\right) = 1$

Hence, the solutions are $x = 1$ and $y = 1$.

21. Given $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$

It is possible only when each term will provide us the maximum value.

Thus, $\sin^{-1} x = \frac{\pi}{2}, \sin^{-1} y = \frac{\pi}{2}$

and $\quad \sin^{-1} z = \frac{\pi}{2}$

$\Rightarrow \quad x = 1, y = 1$ and $z = 1$

Hence, the value of

$$x^{2013} + y^{2013} + z^{2013} - \frac{9}{x^{2014} + y^{2014} + z^{2014}}$$

$= 1 + 1 + 1 - \frac{9}{1+1+1}$

$= 3 - 3$

$= 0$

22. We have $-\frac{\pi}{2} \le \sin^{-1}(3x+5) \le \frac{\pi}{2}$

$\Rightarrow \quad -\pi \le 2\sin^{-1}(3x+5) \le \pi$

$\Rightarrow \quad -\pi + \frac{\pi}{4} \le 2\sin^{-1}(3x+5) + \frac{\pi}{4} \le \pi + \frac{\pi}{4}$

$\Rightarrow \quad -\frac{3\pi}{4} \le f(x) \le \frac{5\pi}{4}$

Hence, $R_f = \left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

23. We have $\sin^{-1} x > \sin^{-1}(3x-1)$
$\Rightarrow \quad x > (3x-1)$
$\Rightarrow \quad 2x - 1 < 0$
$\Rightarrow \quad x < \frac{1}{2}$
$\Rightarrow \quad x \in \left[-1, \frac{1}{2}\right)$

24. We have $-1 \le 2x + 4 \le 1$
$\Rightarrow \quad -5 \le 2x \le -3$
$\Rightarrow \quad -\frac{5}{2} \le x \le -\frac{3}{2}$
Hence, $D_f = \left[-\frac{5}{2}, -\frac{3}{2}\right]$

25. We have $0 \le \cos^{-1}(3x+5) \le \pi$
$\Rightarrow \quad 0 \le 2\cos^{-1}(3x+5) \le 2\pi$
$\Rightarrow \quad \frac{\pi}{4} \le 2\cos^{-1}(3x+5) + \frac{\pi}{4} \le 2\pi + \frac{\pi}{4}$
Hence, $R_f = \left[\frac{\pi}{4}, \frac{9\pi}{4}\right]$

26. We have $\frac{\pi}{2} \le \cos^{-1}(-x^2) \le \pi$
$\Rightarrow \quad \frac{3\pi}{2} \le 3\cos^{-1}(-x^2) \le 3\pi$
$\Rightarrow \quad \frac{3\pi}{2} - \frac{\pi}{2} \le 3\cos^{-1}(-x^2) - \frac{\pi}{2} \le 3\pi - \frac{\pi}{2}$
$\Rightarrow \quad \pi \le f(x) \le \frac{5\pi}{2}$
Hence, $R_f = \left[\pi, \frac{5\pi}{2}\right]$.

27. Given, $\cos^{-1} x + \cos^{-1} x^2 = 0$
It is possible only when each term will provide us the minimum value.
So, $\quad \cos^{-1} x + \cos^{-1} x^2 = 0$
$\Rightarrow \quad x = 1$ and $x^2 = 1$
$\Rightarrow \quad x = 1$ and $x = \pm 1$
Hence, the solution is $x = 1$.

28. Given, $[\sin^{-1} x] + [\cos^{-1} x]$ and $x \ge 0$
$\Rightarrow \quad [\sin^{-1} x] = 0$ and $[\cos^{-1} x] = 0$
$\Rightarrow \quad x \in [0, \sin 1)$ & $x \in (\cos 1, 1]$
$\Rightarrow \quad x \in (\cos 1, \sin 1)$

29. We have $-1 \le \frac{x^2}{x^2+1} \le 1$
$\Rightarrow \quad \left|\frac{x^2}{x^2+1}\right| \le 1$
$\Rightarrow \quad \frac{|x^2|}{|x^2+1|} \le 1$
$\Rightarrow \quad \frac{x^2}{x^2+1} \le 1$
$\Rightarrow \quad x^2 + 1 \ge x^2$
$\Rightarrow \quad 1 > 0$
Hence, $x \in R$

30. We have $\cos^{-1}(x) > \cos^{-1}(x^2)$
$\Rightarrow \quad x < x^2$
$\Rightarrow \quad x^2 - x > 0$
$\Rightarrow \quad x(x-1) > 0$
$\Rightarrow \quad x \in [-1, 0)$

31. Since $\tan^{-1} x$ is defined for all real values of $x$, so
$9 - x^2 \le 0$
$\Rightarrow \quad x^2 - 9 \le 0$
$\Rightarrow \quad (x+3)(x-3) \le 0$
$\Rightarrow \quad -3 \le x \le 3$
Hence, $D_f = [-3, 3]$

32. We have $-\frac{\pi}{2} \le \tan^{-1}(1-x^2) \le \frac{\pi}{4}$
$\Rightarrow \quad -\pi \le 2\tan^{-1}(1-x^2) \le \frac{\pi}{2}$
$\Rightarrow \quad -\pi + \frac{\pi}{6} \le 2\tan^{-1}(1-x^2) + \frac{\pi}{6} \le \frac{\pi}{2} + \frac{\pi}{6}$
$\Rightarrow \quad -\frac{5\pi}{6} \le f(x) \le \frac{2\pi}{3}$
Hence, $R_f = \left[-\frac{5\pi}{6}, \frac{2\pi}{3}\right]$

33. We have $f(x) = \cot^{-1}(2x - x^2)$
$\Rightarrow \quad f(x) = \cot^{-1}(1 - (x^2 - 2x + 1))$
$\Rightarrow \quad f(x) = \cot^{-1}(1 - (x-1)^2)$
Since $(1 - (x-)^2) \le 1$ and $0 \le \cot^{-1} x \le \pi$ and $\cot^{-1} x$ is strictly decreasing function so,
$\cot^{-1}(1) \le \cot^{-1}(1 - (x-1)^2) \le \cot^{-1}(0)$
$\Rightarrow \quad \frac{\pi}{4} \le f(x) \le \pi$
Hence, $R_f = \left[\frac{\pi}{4}, \pi\right]$
where [,] = GIF

34. We have $[\cot^{-1} x] + [\cos^{-1} x] = 0$
$\Rightarrow \quad [\cot^{-1} x] = 0$ & $[\cos^{-1} x] = 0$
$\Rightarrow \quad 0 \le \cot^{-1} x < 1$ & $0 \le \cos^{-1} x < 1$
$\Rightarrow \quad x \in (\cot 1, \infty)$ & $x \in (\cos 1, 1]$
$\Rightarrow \quad x \in (\cot 1, 1]$

35. We have $\sin\{x\} = \cos\{x\}, \forall x \in [0, 2\pi]$
$\Rightarrow \quad \tan\{x\} = 1$
$\Rightarrow \quad \{x\} = \tan^{-1}(1) = \frac{\pi}{4}$
Hence, the number of solutions = 6
(Since $\{x\}$ is a periodic function with period 1, it has one solution between 0 to 1. So, there is six solutions between 0 to 6.28).

36. We have,
$$-1 \le \left(\frac{|x|-2}{3}\right) \le 1$$

$\Rightarrow \quad -3 \le |x| - 2 \le 3$
$\Rightarrow \quad -1 \le |x| \le 5$
$\Rightarrow \quad -5 \le x \le 5$ …(i)

Also, $-1 \le \left(\frac{1-|x|}{4}\right) \le 1$
$\Rightarrow \quad -4 \le 1 - |x| \le 4$
$\Rightarrow \quad -4 \le |x| - 1 \le 4$
$\Rightarrow \quad -3 \le |x| \le 5$
$\Rightarrow \quad -5 \le x \le 5$ …(ii)

From (i) and (ii), we get
$-5 \le x \le 5$
Thus, $D_f = [-5, 5]$

37. Given, $f(x) = \sin^{-1}(2x^2 - 1)$
So, $-1 \le (2x^2 - 1) \le 1$
$\Rightarrow \quad 0 \le 2x^2 \le 2$
$\Rightarrow \quad 0 \le x^2 \le 1$
$\Rightarrow \quad 0 \le |x| \le 1$
$\Rightarrow \quad -1 \le x \le 1$
Thus, $D_f = [-1, 1]$

38. Given, $f(x) = \sqrt{5\pi \sin^{-1}x - 6(\sin^{-1}x)^2}$
We have $5\pi \sin^{-1} x - 6(\sin^{-1} x) \ge 0$
$\Rightarrow \quad (5\pi - 6(\sin^{-1} x)) \sin^{-1} x \ge 0$
$\Rightarrow \quad (6(\sin^{-1} x) - 5\pi) \sin^{-1} x \le 0$
$\Rightarrow \quad 0 \le \sin^{-1}x \le \frac{5\pi}{6}$
$\Rightarrow \quad 0 \le x \le \sin\left(\frac{5\pi}{6}\right)$
$\Rightarrow \quad 0 \le x \le \frac{1}{2}$
Also, $-1 \le x \le 1$
Thus, $D_f = \left[0, \frac{1}{2}\right]$

39. Given, $f(x) = \log_2\left(\frac{3\tan^{-1}x + \pi}{\pi - 4\tan^{-1}x}\right)$
So, $\frac{3\tan^{-1}x + \pi}{\pi - 4\tan^{-1}x} > 0$
$\Rightarrow \quad \frac{3\tan^{-1}x + \pi}{4\tan^{-1}x - \pi} < 0$
$\Rightarrow \quad -\frac{\pi}{3} < \tan^{-1}x < \frac{\pi}{4}$
$\Rightarrow \quad \tan\left(-\frac{\pi}{3}\right) < x < \tan\left(\frac{\pi}{4}\right)$
$\Rightarrow \quad -\sqrt{3} < x < 1$
Hence, $D_f = (-\sqrt{3}, 1)$

40. Given, $f(x) = \cos^{-1}\left(\frac{3}{2+\sin x}\right)$
$\Rightarrow \quad -1 \le \left(\frac{3}{2+\sin x}\right) \le 1$
$\Rightarrow \quad -1 \le \left(\frac{2+\sin x}{3}\right) \le 1$
$\Rightarrow \quad -3 \le (2 + \sin x) \le 3$
$\Rightarrow \quad -5 \le (\sin x) \le 1$
$\Rightarrow \quad -1 \le (\sin x) \le 1$
$\Rightarrow \quad \sin^{-1}(-1) \le x \le \sin^{-1}(1)$
$\Rightarrow \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$
Thus, $D_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

41. Given, $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$
$\Rightarrow \quad -1 \le \left(\frac{x^2+1}{2x}\right) \le 1$
$\Rightarrow \quad \left|\frac{x^2+1}{2x}\right| \le 1$
$\Rightarrow \quad \frac{x^2+1}{2|x|} \le 1$
$\Rightarrow \quad x^2 + 1 \le 2|x|$
$\Rightarrow \quad |x|^2 + 1 \le 2|x|$
$\Rightarrow \quad |x|^2 - 2|x| + 1 \le 0$
$\Rightarrow \quad (|x| - 1) \le 0$
$\Rightarrow \quad (|x| - 1) = 0$
$\Rightarrow \quad x = \pm 1$
Thus, $D_f = \{-1, 1)$

42. Given, $f(x) = \cos^{-1}\left(\frac{x^2+1}{x^2}\right)$
So, $-1 \le \left(\frac{x^2+1}{x^2}\right) \le 1$
$\Rightarrow \quad -1 \le \left(1 + \frac{1}{x^2}\right) \le 1$
$\Rightarrow \quad -2 \le \left(\frac{1}{x^2}\right) \le 0$
which is not true
Hence, $D_f = \varphi$

43. Given, $f(x) = \sin^{-1}(\log_2(x^2 + 3x + 4))$
So, $-1 \le \log_2(x^2 + 3x + 4) \le 1$
$\Rightarrow \quad \frac{1}{2} \le (x^2 + 3x + 4) \le 2$
when $(x^2 + 3x + 4) \le 2$
$\Rightarrow \quad (x^2 + 3x + 2) \le 0$
$\Rightarrow \quad (x + 1)(x + 2) \le 0$
$\Rightarrow \quad -2 \le x \le -1$
when $x^2 + 3x + 4 \ge \frac{1}{2}$
$\Rightarrow \quad 2x^2 + 6\underline{x} + 7 \ge 0$
Clearly, $D < 0$
So, it is true for all $R$
Hence, $D_f = [-2, -1]$

44. Given, $f(x)=\sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$

So, $-1 \le \log_2\left(\frac{x^2}{2}\right) \le 1$

$\Rightarrow \quad \frac{1}{2} \le \left(\frac{x^2}{2}\right) \le 2$

$\Rightarrow \quad 1 \le x^2 \le 4$

$\Rightarrow \quad 1 \le |x| \le 2$ …(i)

when $|x| \le 2$

$\Rightarrow \quad -2 \le x \le 2$

when $|x| \ge 1$

$\Rightarrow \quad x \ge 1$ and $x \le -1$ …(ii)

From (i) and (ii), we get,

$x \in [-2, -1] \cup [1, 2]$

Thus, $D_f = [-2, -1] \cup [1, 2]$

45. Given, $f(x) = \sin^{-1}[2 - 3x^2]$

$\Rightarrow \quad -1 \le [2 - 3x^2] \le 1$

when $[2 - 3x^2] \ge 1$

$\Rightarrow \quad 1 \le 2 - 3x^2 < 2$

$\Rightarrow \quad -2 < 3x^2 - 2 \le 1$

$\Rightarrow \quad 0 < 3x^2 \le 3$

$\Rightarrow \quad 0 < x^2 \le 1$

$\Rightarrow \quad 0 < |x| \le 1$

$\Rightarrow \quad -1 \le x \le 1 - \{0\}$ …(i)

Also, when $[2 - x^2] \ge -1$

$\Rightarrow \quad 2 - x^2 \ge -1$

$\Rightarrow \quad x^2 \le 3$

$\Rightarrow \quad |x| \le \sqrt{3}$

$\Rightarrow \quad -\sqrt{3} \le x \le \sqrt{3}$ …(ii)

Hence, $D_f = [-1, 0) \cup (0, 1]$

46. Given, $f(x)=\frac{1}{x}+3^{\sin^{-1}x}+\frac{1}{\sqrt{x-2}}$

Let $D_1 = R - \{0\}$

$D_2 = [-1, 1]$

and $D_3 = (2, \infty)$

Thus, $D_f = D_1 \cap D_2 \cap D_3 = [-1, 1]$

47. Given, $f(x) = \sin^{-1}(\log_2 x^2)$

So, $-1 \le \log_2(x^2) \le 1$

$\Rightarrow \quad \frac{1}{2} \le (x^2) \le 2$

$\Rightarrow \quad \frac{1}{\sqrt{2}} \le |x| \le \sqrt{2}$

Thus, $x \in \left[-\sqrt{2}, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$

Hence, $D_f = \left[-\sqrt{2}, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$

48. Given, $f(x)=e^x+\sin^{-1}\left(\frac{x}{2}-1\right)+\frac{1}{x}$

Let $D_1 = R$

$D_2 : -1 \le \left(\frac{x}{2}-1\right) \le 1$

$\Rightarrow \quad 0 \le \left(\frac{x}{2}\right) \le 2$

$\Rightarrow \quad 0 \le x \le 4$

and $D_3 = R - \{0\}$

Hence, $D_f = D_1 \cap D_2 \cap D_3 = (0, 4]$

49. Given, $f(x)=\sqrt{\sin^{-1}(\log_x 2)}$

We have $\sin^{-1}(\log_x 2) \ge 0$

$\Rightarrow \quad (\log_x 2) \ge \sin(0) = 0$

$\Rightarrow \quad 2 \ge x^0 = 1$

$\Rightarrow \quad 2 > 1$

which is true for all $x$ in R

Also, $x \ne 1$ and $x > 0$

Furthermore, $-1 \le \log_x 2 \le 1$ which is also true for $x \ne 1$ and $x > 0$

Hence, $D_f = (0, 1) \cup (1, \infty)$

50. Given, $f(x)=\sqrt{\sin^{-1}(\log_2 x)}$

We have $\sin^{-1}(\log_2 x) \ge 0$

$\Rightarrow \quad (\log_2 x) \ge 0$

$\Rightarrow \quad x \ge 1$

Also, $-1 \le \log_2 x \le 1$

$\Rightarrow \quad \frac{1}{2} \le x \le 2$

Hence, $D_f = [1, 2]$

51. Given, $f(x) = \sin^{-1}(2x - 3)$

$$R_f=\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

52. Given, $f(x)=2\sin^{-1}(2x-1)-\frac{\pi}{4}$

$$-\frac{\pi}{2} \le \sin^{-1}(2x-1) \le \frac{\pi}{2}$$

$$-\pi \le 2\sin^{-1}(2x-1) \le \pi$$

$$-\pi-\frac{\pi}{4} \le 2\sin^{-1}(2x-1)-\frac{\pi}{4} \le \pi-\frac{\pi}{4}$$

$$-\frac{5\pi}{4} \le \left(2\sin^{-1}(2x-1)-\frac{\pi}{4}\right) \le \frac{3\pi}{4}$$

$$-\frac{5\pi}{4} \le f(x) \le \frac{3\pi}{4}$$

So, $R_f=\left[-\frac{5\pi}{4}, \frac{3\pi}{4}\right]$

53. Given, $f(x) = 2\cos^{-1}(-x^2) - \pi$

$= 2(\pi - \cos^{-1}(x^2)) - \pi$

$= \pi - 2\cos^{-1}(x^2)$

54. Given, $f(x)=\frac{1}{2}\tan^{-1}(1-x^2)-\frac{\pi}{4}$

Now, $-\infty < (1 - x^2) \le 1$

$\Rightarrow \quad \tan^{-1}(-\infty) < \tan^{-1}(1 - x^2) \le \tan^{-1}(1)$

$\Rightarrow \quad -\frac{\pi}{2} < \tan^{-1}(1-x^2) \le \frac{\pi}{4}$

$\Rightarrow \quad -\frac{\pi}{4} < \frac{1}{2}\tan^{-1}(1-x^2) \le \frac{\pi}{8}$

$\Rightarrow \quad -\frac{\pi}{4} - \frac{\pi}{4} < \frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4} \le \frac{\pi}{8} - \frac{\pi}{4}$

$\Rightarrow \quad -\frac{\pi}{2} < f(x) \le -\frac{\pi}{8}$

So, $R_f = \left(-\frac{\pi}{2}, -\frac{\pi}{8}\right]$

55. Given, $f(x) = \cot^{-1}(2x - x^2)$
$= \cot^{-1}(1 - (x-1)^2)$
Clearly, $-\infty < (1 - (x-1)^2) \le 1$
$\Rightarrow \quad \cot^{-1}(1) \le \cot^{-1}((1-(x-1)^2)) \le \cot^{-1}(-\infty)$
$\Rightarrow \quad \frac{\pi}{4} \le \cot^{-1}((1-(x-1)^2)) \le \pi$
$\Rightarrow \quad \frac{\pi}{4} \le f(x) \le \pi$
So, $R_f = \left[\frac{\pi}{4}, \pi\right]$

56. Given, $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$
$D_f = [-1, 1]$
So, $R_f = [f(-1), f(1)]$
$= \left[\frac{\pi}{2} - \frac{\pi}{4}, \frac{\pi}{2} + \frac{\pi}{4}\right]$
$= \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

57. Given, $f(x) = \sin^{-1}x + \sec^{-1}x + \tan^{-1}x$
Thus, $D_f = \{-1, 1\}$
So, $R_f = \{f(-1), f(1)\}$
$= \left\{-\frac{\pi}{2} + 0 - \frac{\pi}{4}, \frac{\pi}{2} + \frac{\pi}{4}\right\}$
$= \left\{-\frac{3\pi}{4}, \frac{3\pi}{4}\right\}$

58. Given, $f(x) = 3\cot^{-1}x + 2\tan^{-1}x + \frac{\pi}{4}$
$= 2(\tan^{-1}x + \cot^{-1}x) + \cot^{-1}x + \frac{\pi}{4}$
$= 2 \times \frac{\pi}{2} + \cot^{-1}x + \frac{\pi}{4}$
$= \cot^{-1}x + \frac{5\pi}{4}$
Thus, $0 \le \cot^{-1}x \le \pi$
$\Rightarrow \quad 0 + \frac{5\pi}{4} \le \cot^{-1}x + \frac{5\pi}{4} \le \pi + \frac{5\pi}{4}$
$\Rightarrow \quad \frac{5\pi}{4} \le f(x) \le \frac{9\pi}{4}$
So, $R_f = \left[\frac{5\pi}{4}, \frac{9\pi}{4}\right]$

59. Given, $f(x) = \operatorname{cosec}^{-1}[1 + \sin^2 x]$.
Clearly,
$1 \le (1 + \sin^2 x) \le 2$
$R_f = [\operatorname{cosec}^{-1}(2), \operatorname{cosec}^{-1}(1)]$
$= \left[\sin^{-1}\left(\frac{1}{2}\right), \sin^{-1}(1)\right]$
$= \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

60. Given, $f(x) = \sin^{-1}(\log_2(x^2 + 3x + 4))$
Clearly, $D_f = [-2, -1]$
Thus, $R_f = [f(-2), f(-1)]$
$= \left[\frac{\pi}{2}, \frac{\pi}{2}\right] = \left\{\frac{\pi}{2}\right\}$

61. We have, $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ is defined only when $-1 \le x \le 1$
Now, $f(1) = \sin^{-1}(1) + \cos^{-1}(1) + \tan^{-1}(1)$
$= \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$
and $f(-1) = \sin^{-1}(-1) + \cos^{-1}(-1) + \tan^{-1}(-1)$
$= -\frac{\pi}{2} + \pi - \frac{\pi}{4} = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$
Thus, $R_f = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

62. We have $4\sin^{-1}(x-1) + \cos^{-1}(x-2) = \pi$
$\Rightarrow \quad 3\sin^{-1}(x-2) + \frac{\pi}{2} = \pi$
$\Rightarrow \quad 3\sin^{-1}(x-2) = \frac{\pi}{2}$
$\Rightarrow \quad \sin^{-1}(x-2) = \frac{\pi}{6}$
$\Rightarrow \quad (x-2) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
$\Rightarrow \quad x = 2 + \frac{1}{2} = \frac{5}{2}$
Hence, the solution is $x = \frac{5}{2}$

63. As we know that, if $\sin^{-1}(f(x)) + \cos^{-1}(g(x)) = \frac{\pi}{2}$, then
$f(x) = g(x)$
$\Rightarrow \quad (x^2 - 2x + 1) = (x^2 - x)$
$\Rightarrow \quad 2x - x = 1$
$\Rightarrow \quad x = 1$
Hence, the solution is $x = 1$

64. We have
$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$
$\Rightarrow \quad \cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right) + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$
$\Rightarrow \quad \left(\frac{1}{\sqrt{x^2+x+1}}\right) = \sqrt{x^2+x+1}$
$\Rightarrow \quad x^2 + x + 1 = 1$
$\Rightarrow \quad x^2 + x = 0$
$\Rightarrow \quad x(x+1) = 0$

$\Rightarrow \quad x = 0$ and $-1$

Hence, the number of solutions is 2

65. As we know that, if $\sin^{-1}(f(x)) + \cos^{-1}(g(x)) = \frac{\pi}{2}$, then

$f(x) = g(x)$

$$\Rightarrow \left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\right) = \left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\right)$$

$$\Rightarrow x\left(1 - \frac{x}{2} + \frac{x^2}{4} - \ldots\right) = x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{4} - \ldots\right)$$

$$\Rightarrow x\left(\frac{1}{1+\frac{x}{2}}\right) = x^2\left(\frac{1}{1+\frac{x^2}{2}}\right)$$

$$\Rightarrow \left(\frac{2x}{x+2}\right) = \left(\frac{2x^2}{x^2+2}\right)$$

$$\Rightarrow x\left\{\left(\frac{1}{x+2}\right) - \left(\frac{x}{x^2+2}\right)\right\} = 0$$

$$\Rightarrow x = 0 \text{ and } \left(\frac{1}{x+2}\right) = \left(\frac{x}{x^2+2}\right)$$

$\Rightarrow \quad x =$ and $x = 1$

66. We have $\sin^{-1} x > \cos^{-1} x$

$\Rightarrow \quad 2\sin^{-1} x > \sin^{-1} x + \cos^{-1} x$

$$\Rightarrow 2\sin^{-1}x > \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}x > \frac{\pi}{4}$$

$$\Rightarrow x > \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow x > \frac{1}{\sqrt{2}}$$

$$\Rightarrow x \in \left(\frac{1}{\sqrt{2}}, 1\right]$$

67. $(\sin^{-1} x)^2 - 3\sin^{-1} x + 2 = 0$

$\Rightarrow \quad (\sin^{-1} x - 1)(\sin^{-1} x - 2) = 0$

$\Rightarrow \quad (\sin^{-1} x - 1) = 0, (\sin^{-1} x - 2) = 0$

$\Rightarrow \quad \sin^{-1} x = 1, 2$

$\Rightarrow \quad \sin^{-1} x = 1$

$\Rightarrow \quad x \sin(1)$

68. Given equation is $\sin^{-1}x + \sin^{-1}2y = \pi$. It is possible only when

$$\Rightarrow \sin^{-1}x = \frac{\pi}{2}, \sin^{-1}(2y) = \frac{\pi}{2}$$

$\Rightarrow \quad x = 1, 2y = 1$

$$\Rightarrow x = 1, y = \frac{1}{2}$$

69. Given equation is $\cos^{-1} x + \cos^{-1} x^2 = 2\pi$. It is possible only when

$\Rightarrow \quad \cos^{21} x = \pi, \cos^{-1}(x^2) = \pi$

$\Rightarrow \quad x = -1, x^2 = -1$

$\Rightarrow \quad x = \varphi$

70. Given equation is $\cos^{-1} x + \cos^{-1} x^2 = 0$

It is possible only when

$\Rightarrow \quad \cos^{-1} x = 0, \cos^{-1}(x^2) = 0$

$\Rightarrow \quad x = 1$ and $x^2 = 1$

$\Rightarrow \quad x = 1$

71. Given equation is

$4\sin^{-1}(x - 1) + \cos^{-1}(x - 1) = \pi$

$$\Rightarrow 3\sin^{-1}(x-1) + \frac{\pi}{2} = \pi$$

$$\Rightarrow 3\sin^{-1}(x-1) = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}(x-1) = \frac{\pi}{6}$$

$$\Rightarrow (x-1) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\Rightarrow x = \frac{3}{2}$$

Hence, the solution is $x = \frac{3}{2}$

72. Given equation is

$$\cot^{-1}\left(\frac{1}{x^2-1}\right) + \tan^{-1}(x^2-1) = \frac{\pi}{2}$$

It is possible only when

$$\Rightarrow \frac{1}{x^2-1} = x^2 - 1$$

$\Rightarrow \quad (x^2 - 1)^2 = 1$

$\Rightarrow \quad (x^2 - 1) = \pm 1$

$\Rightarrow \quad x^2 = 1 \pm 1 = 2, 0$

$\Rightarrow \quad x = \{-\sqrt{2}, 0, \sqrt{2}\}$

73. Given equation is

$$\cot^{-1}\left(\frac{x^2-1}{2x}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{x^2-1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow 2\tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2}\right) = -\frac{\pi}{3}$$

$$\Rightarrow 2\tan^{-1}x = -\frac{\pi}{3}$$

$$\Rightarrow \tan^{-1}x = -\frac{\pi}{6}$$

$$\Rightarrow x = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

Hence, the solution is $x = -\frac{1}{\sqrt{3}}$.

74. Given equation is

$$4\sin^{-1}x + \cos^{-1}x = \frac{3\pi}{4}$$

$\Rightarrow \quad 3\sin^{-1}x + \frac{\pi}{2} = \frac{3\pi}{4}$

$\Rightarrow \quad 3\sin^{-1}x = \frac{\pi}{4}$

$\Rightarrow \quad \sin^{-1}x = \frac{\pi}{12}$

$\Rightarrow \quad x = \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

75. Given equation is

$$5\tan^{-1}x + 3\cot^{-1}x = \frac{7\pi}{4}$$

$\Rightarrow \quad 2\tan^{-1}x + \frac{3\pi}{2} = \frac{7\pi}{4}$

$\Rightarrow \quad 2\tan^{-1}x = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{8}$

$\Rightarrow \quad x = \tan\left(\frac{\pi}{8}\right) = (\sqrt{2}-1)$

76. Given equation is

$5\tan^{-1}x + 4\cot^{-1}x = 2\pi$

$\Rightarrow \quad \tan^{-1}x + 2\pi = 2\pi$

$\Rightarrow \quad \tan^{-1}x = 0$

$\Rightarrow \quad x = \tan(0) = 0$

Hence, the solution is $x = 0$.

77. Given equation is

$$\cot^{-1}x - \cot^{-1}(x+1) = \frac{\pi}{2}$$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right) - \tan^{-1}\left(\frac{1}{x+1}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{x} - \frac{1}{x+1}}{1 + \frac{1}{x} \times \frac{1}{x+1}}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{x^2+x+1}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \left(\frac{1}{x^2+x+1}\right) = \tan\left(\frac{\pi}{2}\right) = \infty$

$\Rightarrow \quad x^2 + x + 1 = \frac{1}{\infty} = 0$

$\Rightarrow \quad x^2 + x = 1 = 0$

So, no real values of $x$ satisfies the above equation.

Hence, the solution is $x = \varphi$

78. Given equation is

$[\sin^{-1}x] + [\cos^{-1}x] = 0$

It is possible only when

$[\sin^{-1}x] = 0, [\cos^{-1}x] = 0$

$\Rightarrow \quad 0 \le \sin^{-1}x \le 1$ and $0 \le \cos^{-1}x \le 1$

$\Rightarrow \quad 0 \le x \le \sin(1)$ and $\cos(1) \le x \le 1$

$\Rightarrow \quad x \in [\cos(1), \sin(1)]$

79. Given equation is

$[\tan^{-1}x] + [\cot^{-1}x] = 0$

It is possible only when

$[\tan^{-1}x] = 0$ and $[\cot^{-1}x] = 0$

$\Rightarrow \quad 0 \le \tan^{-1}x \le 1$ and $0 \le \cot^{-1}x \le 1$

$\Rightarrow \quad \cot(1) \le x \le \tan(1)$

Hence, $x \in [\cot(1), \tan(1)]$

80. Given equation is

$[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 0$

$\Rightarrow \quad 0 \le \sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1}x))) < 1$

$\Rightarrow \quad 0 \le (\cos^{-1}(\sin^{-1}(\tan^{-1}x))) < \sin(1)$

$\Rightarrow \quad \cos(\sin(1)) < (\sin^{-1}(\tan^{-1}x)) \le 1$

$\Rightarrow \quad \sin(\cos(\sin(1))) < (\tan^{-1}x) \le \sin(1)$

$\Rightarrow \quad \tan(\sin(\cos(\sin(1)))) < x \le \tan(\sin(1))$

81. Do yourself.

82. Given equation is

$$(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$$

$\Rightarrow \quad (\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x \cdot \cot^{-1}x = \frac{5\pi^2}{8}$

$\Rightarrow \quad \frac{\pi^2}{4} - 2a\left(\frac{\pi}{2} - a\right) = \frac{5\pi^2}{8}, a = \tan^{-1}x$

$\Rightarrow \quad 2a\left(\frac{\pi}{2} - a\right) + \frac{3\pi^2}{8} = 0$

$\Rightarrow \quad a\pi - 2a^2 + \frac{3\pi^2}{8} = 0$

$\Rightarrow \quad 8a\pi - 16a^2 + 3\pi^2 = 0$

$\Rightarrow \quad 16a^2 - 8a\pi - 3\pi^2 = 0$

$\Rightarrow \quad 16a^2 - 12a\pi + 4a\pi - 3\pi^2 = 0$

$\Rightarrow \quad 4a(4a - 3\pi) + \pi(4a - 3\pi) = 0$

$\Rightarrow \quad (4a + \pi)(4a - 3\pi) = 0$

$\Rightarrow \quad a = \frac{3\pi}{4}, -\frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}x = \frac{3\pi}{4}, -\frac{\pi}{4}$

$\Rightarrow \quad x = \tan\left(\frac{3\pi}{4}\right), \tan\left(-\frac{\pi}{4}\right)$

$x = -1$

83. Let $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right) = \theta$

$\Rightarrow \quad \cos^{-1}\left(\frac{3}{5}\right) = 2\theta$

$\Rightarrow \quad \cos(2\theta) = \frac{3}{5}$

$\Rightarrow \quad 2\cos^2\theta - 1 = \frac{3}{5}$

$\Rightarrow \quad 2\cos^2\theta = 1 + \frac{3}{5} = \frac{8}{5}$

$\Rightarrow \quad \cos^2\theta = \frac{4}{5}$

$\Rightarrow \quad \cos\theta = \frac{2}{\sqrt{5}}$

84. We have

$\sin\left(\frac{\pi}{4} + \sin^{-1}\left(\frac{1}{2}\right)\right)$

$= \sin\left(\frac{\pi}{4} + \theta\right), \theta = \sin^{-1}\left(\frac{1}{2}\right)$

$= \sin\left(\frac{\pi}{4} + \theta\right), \sin\theta = \frac{1}{2}$

$= \sin\left(\frac{\pi}{4}\right)\cos(\theta) + \cos\left(\frac{\pi}{4}\right)\sin(\theta)$

$= \frac{1}{\sqrt{2}}\cos(\theta) + \frac{1}{\sqrt{2}}\sin(\theta)$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

85. Let $m_1$ and $m_2$ be the roots of

$x^2 + 3x + 1 = 0$

Thus, $m_1 + m_2 = -3 < 0$

and $m_1 \cdot m_2 = 1$

It is possible only when both are negative.

Thus, $\tan^{-1}(m) + \tan^{-1}\left(\frac{1}{m}\right)$

$= \tan^{-1}(m) - \pi + \cot^{-1}(m)$

$= \tan^{-1}(m) + \cot^{-1}(m) - \pi$

$= \frac{\pi}{2} - \pi$

$= -\frac{\pi}{2}$

86. We have $\cos(\tan^{-1}(\sin(\cot^{-1} x)))$

$= \cos(\tan^{-1}(\sin\theta)), \cot\theta = x$

$= \cos\left(\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)$

$= \cos\varphi, \tan\varphi = \left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \sqrt{\frac{x^2+1}{x^2+2}}$

87. Given, $6(\sin^{-1} x)^2 - \pi\sin^{-1} x \le 0$

$\Rightarrow \quad \sin^{-1} x(6\sin^{-1} x - \pi) \le 0$

$\Rightarrow \quad 0 \le \sin^{-1} x \le \frac{\pi}{6}$

$\Rightarrow \quad 0 \le x \le \frac{1}{2}$

88. Given, in-equation is

$\frac{2\tan^{-1} x + \pi}{4\tan^{-1} x - \pi} \le 0$

$\Rightarrow \quad -\frac{\pi}{2} \le \tan^{-1} x \le \frac{\pi}{4}$

$\Rightarrow \quad -\infty < x < 1$

89. Given inequation is

$\Rightarrow \quad \sin^{-1} x < \sin^{-1} x^2$

$\Rightarrow \quad x^2 > x$

$\Rightarrow \quad x(x-1) > 0$

$\Rightarrow \quad x > 1$ and $x < 0$

$\Rightarrow \quad x \in [-1, 0)$

90. Given in-equation is

$\Rightarrow \quad \cos^{-1} x > \cos^{-1} x^2$

$\Rightarrow \quad x^2 > x$

$\Rightarrow \quad x^2 - x > 0$

$\Rightarrow \quad x(x-1) > 0$

$\Rightarrow \quad -1 \le x < 0$

91. Given in-eqution is

$\log_2(\tan^{-1} x) > 1$

$\Rightarrow \quad \tan^{-1} x > 2$

$\Rightarrow \quad x > \tan(2)$

Hence, the solution is

$(\tan 2, \infty)$

92. Given in-equation is

$(\cot^{-1} x)^2 - 5\cot^{-1} x + 6 > 0$

$\Rightarrow \quad (\cot^{-} x - 2)(\cot^{-1} x - 3) > 0$

$\Rightarrow \quad (\cot^{-1} x - 2) < 0, (\cot^{-1} x - 3) > 0$

$\Rightarrow \quad x > \cot(2), x < \cot(3)$

$\Rightarrow \quad x \in (\cot 2, \cot 3)$

93. Given in-equation is

$\sin^{-1} x < \cos^{-1} x$

$\Rightarrow \quad 2\sin^{-1} x < \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1} x < \frac{\pi}{4}$

$\Rightarrow \quad x < \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x \in \left[-1, \frac{1}{\sqrt{2}}\right)$

94. Given in-equation is

$\sin^{-1} x > \sin^{-1}(1-x)$

$x > (1-x)$

$2x > 1$

$x > \frac{1}{2}$

Hence, the solution is $x \in \left(\frac{1}{2}, 1\right]$

95. Given in-equation is

$\sin^{-1} 2x > \operatorname{cosec}^{-1} x$

$\Rightarrow \quad \sin^{-1}(2x) > \sin^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \quad 2x > \frac{1}{x}$

$$\Rightarrow \quad 2x - \frac{1}{x} > 0$$

$$\Rightarrow \quad \frac{2x^2 - 1}{x} > 0$$

$$\Rightarrow \quad \frac{(\sqrt{2}x+1)(\sqrt{2}x-1)}{x} > 0$$

$$\Rightarrow \quad x \in \left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right]$$

96. Given in-equation is

$$\tan^{-1} 3x < \cot^{-1} x$$

$$\Rightarrow \quad \tan^{-1}(3x) < \tan^{-1}\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad (3x) - \left(\frac{1}{x}\right) < 0$$

$$\Rightarrow \quad \frac{(\sqrt{3}x+1)(\sqrt{3}x-1)}{x} < 0$$

$$\Rightarrow \quad x \in \left(-\frac{1}{\sqrt{3}}, 0\right) \cup \left(0 \frac{1}{\sqrt{3}}\right)$$

97. Given in-equation is

$$\cos^{-1} 2x \geq \sin^{-1} x$$

$$\Rightarrow \quad \sin^{-1}\sqrt{1-4x^2} > \sin^{-1}x$$

$$\Rightarrow \quad \sqrt{1-4x^2} > x$$

$$\Rightarrow \quad (1 - 4x^2) > x^2$$

$$\Rightarrow \quad 5x^2 - 1 < 0$$

$$\Rightarrow \quad (\sqrt{5}x+1)(\sqrt{5}x-1) < 0$$

$$\Rightarrow \quad -\frac{1}{\sqrt{5}} < x < \frac{1}{\sqrt{5}}$$

98. Given in-equation is

$$x^2 - 2x < \sin^{-1}(\sin 2)$$

$$\Rightarrow \quad x^2 - 2x < (\pi - 2)$$

$$\Rightarrow \quad (x-1)^2 < (\pi - 1)$$

$$\Rightarrow \quad |(x-1)| < \sqrt{(\pi-1)}$$

$$\Rightarrow \quad -\sqrt{(\pi-1)} < (x-1) < \sqrt{(\pi-1)}$$

$$\Rightarrow \quad 1-\sqrt{(\pi-1)} < x < 1+\sqrt{(\pi-1)}$$

99. Given in-equation is

$$\sin^{-1}\left(\frac{x}{2}\right) < \cos^{-1}(x+1)$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{2}\right) < \sin^{-1}(\sqrt{1-(x+1)^2})$$

$$\Rightarrow \quad \left(\frac{x}{2}\right) < (\sqrt{1-(x+1)^2})$$

$$\Rightarrow \quad \left(\frac{x}{2}\right)^2 < 1-(x+1)^2$$

$$\Rightarrow \quad \frac{x^2}{4} < -2x - x^2$$

$$\Rightarrow \quad \frac{5x^2}{4} + 2x < 0$$

$$\Rightarrow \quad 5x^2 + 8x < 0$$

$$\Rightarrow \quad x(5x+8) < 0$$

$$\Rightarrow \quad -\frac{8}{5} < x < 0$$

100. Given in-equation is

$$\tan^{-1} 2x > 2\tan^{-1} x$$

$$\Rightarrow \quad \tan^{-1}(2x) > \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow \quad (2x) > \left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow \quad (2x)\left(1 - \frac{1}{1-x^2}\right) > 0$$

$$\Rightarrow \quad x\left(\frac{1-x^2-1}{1-x^2}\right) > 0$$

$$\Rightarrow \quad x\left(\frac{x^2}{x^2-1}\right) > 0$$

$$\Rightarrow \quad \left(\frac{x^3}{(x-1)(x+1)}\right) > 0$$

$$\Rightarrow \quad x \in (-1, 0) \cup (1, \infty)$$

101. Given in-equation is

$$\tan(\cos^{-1}x) \leq \sin\left(\cot^{-1}\left(\frac{1}{2}\right)\right)$$

$$\Rightarrow \quad \frac{\sqrt{1-x^2}}{x} \leq \frac{1}{\sqrt{5}}$$

$$\Rightarrow \quad \left(\frac{1-x^2}{x^2}\right) \leq \frac{1}{5}$$

$$\Rightarrow \quad \left(\frac{1-x^2}{x^2} - \frac{1}{5}\right) \leq 0$$

$$\Rightarrow \quad \frac{5-5x^2-x^2}{5x^2} \leq 0$$

$$\Rightarrow \quad \frac{6x^2-5}{x^2} \leq 0$$

$$\Rightarrow \quad \frac{6x^2-5}{x^2} \geq 0$$

$$\Rightarrow \quad x \in R - \left(-\sqrt{\frac{5}{6}}, \sqrt{\frac{5}{6}}\right)$$

102. As we know that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, for every $x$ in $[-1, 1]$

(i) Since $0 < \frac{1}{m^2+1} \leq 1$

$$\text{so, } f\left(\frac{1}{m^2+1}\right) = \frac{\pi}{2}$$

(ii) Since, $0 \le \dfrac{m^2}{m^2+1} < 1$,

so $f\left(\dfrac{m^2}{m^2+1}\right) = \dfrac{\pi}{2}$.

(iii) Since, $-\dfrac{1}{2} \le \dfrac{m}{m^2+1} \le \dfrac{1}{2}$

so, $f\left(\dfrac{m}{m^2+1}\right) = \dfrac{\pi}{2}$

(iv) Since $m^2 - 2m + 6 = (m-1)^2 + 5$

Thus, $4 \le (m-1)^2 + 5 < \infty$

Hence, $f((m-1)^2 + 5)$ is not defined.

(v) Also, $1 \le (m-1)^2 + 5 < \infty$

So, $f(m^2 + 1)$ is not defined.

103. Given, $\cos^{-1}x + \cos^{-1}y = \dfrac{2\pi}{3}$

Now, $\sin^{-1} x + \sin^{-1} y$

$= \dfrac{\pi}{2} - \cos^{-1}x + \dfrac{\pi}{2} - \cos^{-1}y$

$= \pi - (\cos^{-1} x + \cos^{-1} y)$

$= \pi - \dfrac{2\pi}{3}$

$= \dfrac{\pi}{3}$

104. Let $m_1$ and $m_2$ be the two roots of the given equation.

Now, $m_1 + m_2 = -3$ and $m_1 \cdot m_2 = 1$

$\Rightarrow$ $m_1$ and $m_2$ are two negative roots.

Now, $\tan^{-1}(m) + \tan^{-1}\left(\dfrac{1}{m}\right)$

$= \tan^{-1}(m) - \pi + \cot^{-1}(m)$

$= -\pi + \tan^{-1}(m) + \cot^{-1}(m)$

$= -\pi + \dfrac{\pi}{2}$

$= -\dfrac{\pi}{2}$

105. Let $m = \dfrac{2x^2+5}{x^2+2} = 2 + \dfrac{1}{x^2+2}$

Thus, $m \in \left[2, \dfrac{5}{2}\right]$

now, $\sin^{-1}\left(\sin\left(\dfrac{2x^2+5}{x^2+2}\right)\right) > \sin^{-1}(\sin 3)$

$\Rightarrow \sin^{-1}\left(\sin\left(\pi - \dfrac{2x^2+5}{x^2+2}\right)\right)$

$< \sin^{-1}(\sin(\pi - 3))$

$\Rightarrow \pi - \left(\dfrac{2x^2+5}{x^2+2}\right) > \pi - 3$

$\Rightarrow \left(\dfrac{2x^2+5}{x^2+2}\right) < 3$

$\Rightarrow \left(\dfrac{2x^2+5}{x^2+2} - 3\right) < 0$

$\Rightarrow \left(\dfrac{-x^2-5}{x^2+2}\right) < 0$

$\Rightarrow \left(\dfrac{x^2+5}{x^2+2}\right) > 0$

$\Rightarrow x \in R$

106. (i) $\sin^{-1}(\sin 3)$

$= \sin^{-1}(\sin(\pi - 3))$

$= (\pi - 3)$

(ii) $\sin^{-1}(\sin 5)$

$= \sin^{-1}(\sin(5 - 2\pi))$

$= (5 - 2\pi)$

(iii) $\sin^{-1}(\sin 7)$

$= \sin^{-1}(\sin(7 - 2\pi))$

$= (7 - 2\pi)$

(iv) $\sin^{-1}(\sin 10)$

$= \sin^{-1}(\sin(3\pi - 10))$

$= (3\pi - 10)$

(v) $\sin^{-1}(\sin 20)$

$= \sin^{-1}(\sin(20 - 6\pi))$

$= (20 - 6\pi)$

107. (i) $\cos^{-1}(\cos 2) = 2$

(ii) $\cos^{-1}(\cos 3) = 2$

(iii) $\cos^{-1}(\cos 5)$

$= \cos^{-1}(\cos(2\pi - 5))$

$= (2\pi - 5)$

(iv) $\cos^{-1}(\cos 7)$

$= \cos^{-1}(\cos(7 - 2\pi))$

$= (7 - 2\pi)$

(v) $\cos^{-1}(\cos 10)$

$= \cos^{-1}(\cos(4\pi - 10))$

$= (4\pi - 10)$

108. (i) $\tan^{-1}(\tan 3)$

$= \tan^{-1}(\tan(3 - \pi))$

$= (3 - \pi)$

(ii) $\tan^{-1}(\tan 5)$

$= \tan^{-1}(\tan(5 - 2\pi))$

$= (5 - 2\pi)$

(iii) $\tan^{-1}(\tan 7)$

$= \tan^{-1}(\tan(7 - 2\pi))$

$= (7 - 2\pi)$

(iv) $\tan^{-1}(\tan 10)$

$= \tan^{-1}(\tan(10 - 3\pi))$

$= (10 - 3\pi)$

(v) $\tan^{-1}(\tan 15)$

$= \tan^{-1}(\tan(15 - 5\pi))$

$= (15 - 5\pi)$

109. We have

$\cos^{-1}(\sin(-5))$

$= \cos^{-1}(-\sin 5)$

$= \pi - \cos^{-1}(\sin 5)$

$$= \pi - \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 5\right)\right)$$
$$= \pi - \left(\frac{\pi}{2} - 5\right)$$
$$= \frac{\pi}{2} + 5$$

110. We have
$f(x) = \sin^{-1}(\sin x)$
$= x + 2\pi - \pi - x + x + \pi - x$
$= 2\pi$
$\Rightarrow f(x) = 0$

111. We have
$f(x) = \cos^{-1}(\cos x)$
$= -x + x + 2\pi = x$
$= 2\pi - x$
$\Rightarrow f(x) = -1$

112. We have
$$\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+1}\right)\right) < \pi - 3$$
$$\Rightarrow \sin^{-1}\left(\sin\left(\pi - \left(\frac{2x^2+5}{x^2+1}\right)\right)\right) > \pi - 3$$
$$\Rightarrow \left(\pi - \left(\frac{2x^2+5}{x^2+1}\right)\right) < \pi - 3$$
$$\Rightarrow -\left(\frac{2x^2+5}{x^2+1}\right) < -3$$
$$\Rightarrow \left(\frac{2x^2+5}{x^2+1}\right) > 3$$
$$\Rightarrow \left(\frac{2x^2+5}{x^2+1} - 3\right) > 0$$
$$\Rightarrow \left(\frac{2x^2+5-3x^2-3}{x^2+1}\right) > 0$$
$\Rightarrow x^2 < 2$
$\Rightarrow -\sqrt{2} < x < \sqrt{2}$

113. We have
$x^2 - 3x < \sin^{-1}(\sin 2)$
$\Rightarrow x^2 - 3x < \sin^{-1}(\sin(\pi - 2))$
$\Rightarrow x^2 - 3x < (\pi - 2)$
$\Rightarrow x^2 - 3x + (2 - \pi) < 0$
$$\Rightarrow \left(x - \frac{3+\sqrt{1+4\pi}}{2}\right)\left(x - \frac{3-\sqrt{1+4\pi}}{2}\right) < 0$$
$$\Rightarrow \frac{3-\sqrt{1+4\pi}}{2} < x < \frac{3+\sqrt{1+4\pi}}{2}$$

114. We have
$= \sin^{-1}(\sin 50) + \cos^{-1}(\cos 50) + \tan^{-1}(\tan 50)$
$= \sin^{-1}(\sin(50 - 16\pi)) + \cos^{-1}(\cos(16\pi - 50)) + \tan^{-1}(\tan(50 - 16\pi)$
$= (50 - 16\pi) + (16\pi - 50) + (50 - 16\pi)$
$= (50 - 16\pi)$

115. $\sin^{-1}(\sin 1) + \sin^{-1}(\sin 2) + \sin^{-1}(\sin 3)$
$= 1 + (\pi - 2) + (\pi - 3)$
$= (2\pi - 4)$

116. $\sin^{-1}(\sin 10) + \sin^{-1}(\sin 20) + \sin^{-1}(\sin 30) + \sin^{-1}(\sin 40)$
$= (3\pi - 10) + (20 - 6\pi) + (30 - 10\pi) + (13\pi - 40)$
$= 0$

117. $\cos^{-1}(\cos 1) + \cos^{-1}(\cos 2) + \cos^{-1}(\cos 3) + \cos^{-1}(\cos 4)$
$= 1 + 2 + 3 + (2\pi - 4)$
$= 2(\pi + 1)$

118. $\cos^{-1}(\cos 10) + \cos^{-1}(\cos 20) + \cos^{-1}(\cos 30) + \cos^{-1}(\cos 40)$
$= (4\pi + 10) + (20 + 6\pi) + (10\pi + 30) + (40 - 12\pi)$
$= (20 - 4\pi)$

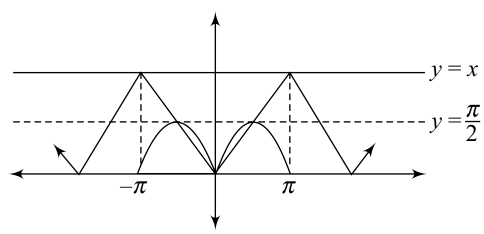
119. $\sin^{-1}(\sin 10) + \cos^{-1}(\cos 10)$
$= (3\pi - 10) + (4\pi - 10)$
$= (7\pi - 20)$

120. $\sin^{-1}(\sin 50) + \cos^{-1}(\cos 50)$
$= (50 - 16\pi) + (16\pi - 50)$
$= 0$

121. $\sin^{-1}(\sin 100) + \cos^{-1}(\cos 100)$
$= (100 - 32\pi) + (32\pi - 100)$
$= 0$

122. $\cos^{-1}(\sin(-5)) + \sin^{-1}(\cos(-5))$
$= \pi - \cos^{-1}(\sin 5) + \sin^{-1}(\cos 5)$
$$= \pi - \cos^{-1}\left(\cos\left(5 - \frac{3\pi}{2}\right)\right) + \sin^{-1}\left(\sin\left(5 - \frac{3\pi}{2}\right)\right)$$
$$= \pi - \left(5 - \frac{3\pi}{2}\right) + \left(5 - \frac{3\pi}{2}\right)$$
$= \pi$

123.




Hence, the number of solutions is 3.
Thus, the ordered pairs are
$$\left(-\frac{\pi}{2}, 1\right), (0, 0), \left(\frac{\pi}{2}, 1\right)$$

124. Given,
$f(x) = \cos^{-1}(\cos x) - \sin^{-1}(\sin x)$
$= x - (x + \pi - x)$
$= x - \pi$
Hence, the required area
$$= \frac{1}{2} \times \pi \times \pi = \frac{\pi^2}{2}$$

125. $\tan^{-1}(\tan 1) + \tan^{-1}(\tan 2) + \tan^{-1}(\tan 3) + \tan^{-1}(\tan 4)$
$= 1 + (2 - \pi) + (3 - \pi) + (4 - \pi)$
$= (10 - 3\pi)$

126. $\tan^{-1}(\tan 20) + \tan^{-1}(\tan 40) + \tan^{-1}(\tan 60) + \tan^{-1}(\tan 80)$
$= (20 - 6\pi) + (40 - 13\pi) + (60 - 21\pi) + (80 - 26\pi)$
$= (200 - 66\pi)$

127. $\sin^{-1}(\sin 15) + \cos^{-1}(\cos 15) + \tan^{-1}(\tan 15)$
$= (5\pi - 15) + (15 - 4\pi) + (15 - 5\pi)$
$= (15 - 4\pi)$

128. $\sin^{-1}(\sin 50) + \cos^{-1}(\cos 50) - \tan^{-1}(\tan 50)$
$= (50 - 16\pi) + (16\pi - 50) - (50 - 16\pi)$
$= (16\pi - 50)$

129. $3x^2 + 8x < 2\sin^{-1}(\sin 4) - \cos^{-1}(\cos 4)$
$\Rightarrow \quad 3x^2 + 2x < 2(\pi - 4) - (2\pi - 4)$
$\Rightarrow \quad 3x^2 + 2x < -4$
$\Rightarrow \quad 3x^2 + 8x + 4 < 0$
$\Rightarrow \quad 3x^2 + 6x + 2x + 4 < 0$
$\Rightarrow \quad 3x(x + 2) + 2(x + 2) < 0$
$\Rightarrow \quad (3x + 2)(x + 2) < 0$
$\Rightarrow \quad -2 < x < -\frac{2}{3}$

130. We have

$$\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right) < \pi - 3$$

$$\Rightarrow \quad \sin^{-1}\left(\sin\left(\pi - \left(\frac{2x^2+4}{x^2+1}\right)\right)\right) < \pi - 3$$

$$\Rightarrow \quad \left(\pi - \left(\frac{2x^2+4}{x^2+1}\right)\right) < \pi - 3$$

$$\Rightarrow \quad -\left(\frac{2x^2+4}{x^2+1}\right) < -3$$

$$\Rightarrow \quad \left(\frac{2x^2+4}{x^2+1}\right) > 3$$

$$\Rightarrow \quad \left(\frac{2x^2+4}{x^2+1} - 3\right) > 0$$

$$\Rightarrow \quad \left(\frac{2x^2+4-3x^2-3}{x^2+1}\right) > 0$$

$\Rightarrow \quad x^2 < 1$
$\Rightarrow \quad -1 < x < 1$

131. We have $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)$$

$$= \tan^{-1}\left(\frac{5/6}{1-1/6}\right)$$

$$= \tan^{-1}\left(\frac{5/6}{5/6}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

132. We have

$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$$

$$= \frac{\pi}{4} + \pi + \tan^{-1}\left(\frac{2+3}{1-2.3}\right)$$

$$= \frac{\pi}{4} + \pi + \tan^{-1}(-1)$$

$$= \frac{\pi}{4} + \pi - \frac{\pi}{4}$$

$$= \pi$$

133. We have

$$\tan^{-1}(9) + \tan^{-1}\left(\frac{5}{4}\right)$$

$$= \pi + \tan^{-1}\left(\frac{9+\frac{5}{4}}{1-9\cdot\frac{5}{4}}\right)$$

$$= \pi + \tan^{-1}\left(\frac{\frac{41}{4}}{-\frac{41}{4}}\right)$$

$$= \pi + \tan^{-1}(-1)$$

$$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

134. We have

$$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5}\cdot\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\cdot\sqrt{1-\left(\frac{4}{5}\right)^2}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5}\cdot\frac{12}{13} + \frac{5}{13}\cdot\frac{3}{13}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{48}{65} + \frac{15}{65}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

$$= 0$$

135. We have

$$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{2\cdot\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{25}{25}\right)$$

$$= \frac{\pi}{4}$$

136. Let $\sin^{-1} x = A$, $\sin^{-1} y = B$, $\sin^{-1} z = C$
Then $x = \sin A$, $y = \sin B$, $z = \sin C$

we have, $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$

$= \sin A \cdot \cos A + \sin B \cdot \cos B + \sin C \cdot \cos C$

$= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$

$= \frac{1}{2}(4\sin A \cdot \sin B \cdot \sin C)$

$= 2 \sin A \cdot \sin B \cdot \sin C$

$= 2xyz$

137. We have

$\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$

$\Rightarrow \cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z$

$\Rightarrow \cos^{-1} x + \cos^{-1} y = \cos^{-1} (-z)$

$\Rightarrow \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos(-z)$

$\Rightarrow (xy + z)^2 = (1 - x^2)(1 - y^2)$

$\Rightarrow x^2y^2 + 2xyz + z^2 = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

138. Given, $\cos^{-1}\left(\frac{x}{2}\right) + \cos^{-1}\left(\frac{y}{3}\right) = \theta$

$\Rightarrow \cos^{-1}\left(\frac{x}{2}\cdot\frac{y}{3} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right) = \theta$

$\Rightarrow \left(\frac{xy}{6} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right) = \cos\theta$

$\Rightarrow \left(\frac{xy}{6} - \cos\theta\right)^2 = \left(1-\frac{x^2}{4}\right)\left(1-\frac{y^2}{9}\right)$

$\frac{x^2y^2}{36} - \frac{xy}{3}\cos\theta + \cos^2\theta$

$\Rightarrow = 1 - \frac{x^2}{4} - \frac{y^2}{9} + \frac{x^2y^2}{36}$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{9} - \frac{xy}{3}\cos\theta = 1 - \cos^2\theta$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{9} - \frac{xy}{3}\cos\theta = \sin^2\theta$

$\Rightarrow 9x^2 + 4y^2 - 12\,xy\cos\theta = 36\sin^2\theta$

139. We have

$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$

and $\cot^{-1}(1) + \cot^{-1}(2) + \cot^{-1}(3)$

$= \tan^{-1}(1) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$= \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\pi}{2}$

Hence, $m = \frac{\pi}{\pi/2} = 2$

140. **Case I:** When $x \le 0$
Then, $\tan^{-1}(2x) \le 0$, $\tan^{-1}(3x) \le 0$,

$\Rightarrow x \le 0$

So, it has no solution.

**Case II:** When $x > 0$, $2x \cdot 3x = 6x^2 < 1$

$\Rightarrow x < \frac{1}{\sqrt{6}}$

Then $\frac{3\pi}{4} = \tan^{-1}(2x) + \tan^{-1}(3x) < \frac{\pi}{2}$

So, it is not possible.

**Case III:** When $x > 0$, $2x \cdot 3x > 1$

$\Rightarrow x > \frac{1}{\sqrt{6}}$

Then $\frac{3\pi}{4} = \tan^{-1}(2x) + \tan^{-1}(3x)$

$\Rightarrow \pi + \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{3\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{3\pi}{4} - \pi = -\frac{\pi}{4}$

$\Rightarrow \left(\frac{5x}{1-6x^2}\right) = -1$

$\Rightarrow 6x^2 - 5x - 1 = 0$

$\Rightarrow x = 1, -1/6$

Thus, $x = 1$ is a solution.

141. We have

$\sin^{-1}(x) + \sin^{-1}(2x) = \frac{\pi}{3}$

$\Rightarrow \sin^{-1}x + \sin^{-1}(2x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \sin^{-1}x - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\sin^{-1}(2x)$

$\Rightarrow \sin^{-1}\left(\frac{x}{2} - \frac{\sqrt{3}}{2}\sqrt{1-x^2}\right) = \sin^{-1}(-2x)$

$\Rightarrow \left(\frac{x}{2} - \frac{\sqrt{3}}{2}\sqrt{1-x^2}\right) = -2x$

$\Rightarrow 5x = \sqrt{3}\sqrt{1-x^2}$

$\Rightarrow 25\,x^3 = 3(1 - x^2)$

$\Rightarrow 28x^2 = 3$

$\Rightarrow x = \pm\frac{\sqrt{3}}{2\sqrt{7}}$

$\Rightarrow \quad x = \dfrac{\sqrt{3}}{2\sqrt{7}}$, negative value of $x$ does not satisfy the given equation.

142. We have

$$f(x) = \cos^{-1}(x) + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right)$$

$$= \cos^{-1}(x) + \cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(x)$$

$$= \frac{\pi}{3}$$

Now, $f(2013) = \dfrac{\pi}{3}$

143. $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$

$\Rightarrow \quad \sin^{-1}(1-x) = \dfrac{\pi}{2} - 2\sin^{-1}x$

$\Rightarrow \quad (1-x) = \sin\left(\dfrac{\pi}{2} - 2\sin^{-1}x\right)$

$\Rightarrow \quad (1-x) = \cos(2\sin^{-1} x)$

$\Rightarrow \quad (1-x) = 1 - 2x^2$

$\Rightarrow \quad x(2x-1) = 0$

$\Rightarrow \quad x = 0, (2x-1) = 0$

$\Rightarrow \quad x = 0, \dfrac{1}{2}$

144. $x^2 - 4x > \sin^{-1}(\sin[\pi^{3/2}]) + \cos^{-1}(\cos[\pi^{3/2}])$

$\Rightarrow \quad x^2 - 4x > \sin^{-1}(\sin 5.5) + \cos^{-1}(\cos 5.5)$

$\Rightarrow \quad x^2 - 4x > (5.5 - 2\pi) + (2\pi - 5.5)$

$\Rightarrow \quad x^2 - 4x > 0$

$\Rightarrow \quad x(x-4) > 0$

$\Rightarrow \quad x < 0$ and $x > 4$

$\Rightarrow \quad x \in (-\infty, 0) \cup (4, \infty)$

145. $\cos(\tan^{-1} x) = x$

$\Rightarrow \quad \dfrac{1}{\sqrt{x^2+1}} = x$

$\Rightarrow \quad x^2(x^2+1) = 1$

$\Rightarrow \quad x^4 + x^2 - 1 = 0$

$\Rightarrow \quad x^2 = \dfrac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad x^2 = \dfrac{\sqrt{5}-1}{2}$

$\Rightarrow \quad x = \pm\sqrt{\dfrac{\sqrt{5}-1}{2}}$

146. $\sin(\tan^{-1} x) = \cos(\cot^{-1}(x+1))$

$\Rightarrow \quad \dfrac{x}{\sqrt{x^2+1}} = \dfrac{x+1}{\sqrt{1+(x+1)^2}}$

$\Rightarrow \quad \dfrac{x^2}{x^2+1} = \dfrac{(x+1)^2}{1+(x+1)^2}$

$\Rightarrow \quad \dfrac{x^2}{x^2+1} - 1 = \dfrac{(x+1)^2}{1+(x+1)^2} - 1$

$\Rightarrow \quad \dfrac{-1}{x^2+1} = \dfrac{-1}{1+(x+1)^2}$

$\Rightarrow \quad x^2 + 1 = 1 + (x+1)^2$

$\Rightarrow \quad x^2 = (x+1)^2$

$\Rightarrow \quad 2x = -1$

$\Rightarrow \quad x = -\dfrac{1}{2}$

147. $\sec^{-1}\left(\dfrac{x}{2}\right) - \sec^{-1}x = \sec^{-1}2$

$\Rightarrow \quad \cos^{-1}\left(\dfrac{x}{2}\right) = \cos^{-1}\left(\dfrac{1}{x}\right) + \cos^{-1}\left(\dfrac{1}{2}\right)$

$\Rightarrow \quad \cos^{-1}\left(\dfrac{x}{2}\right) = \cos^{-1}\left(\dfrac{1}{2}\cdot\dfrac{1}{\mathrm{x}} - \sqrt{1-\dfrac{1}{x^2}}\cdot\dfrac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad \left(\dfrac{x}{2}\right) = \left(\dfrac{1}{2}\cdot\dfrac{1}{\mathrm{x}} - \sqrt{1-\dfrac{1}{x^2}}\cdot\dfrac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad x - \dfrac{1}{x} = -\sqrt{3}\sqrt{1-\dfrac{1}{x^2}}$

$\Rightarrow \quad \left(x - \dfrac{1}{x}\right)^2 = 3\left(1 - \dfrac{1}{x^2}\right)$

$\Rightarrow \quad x^2 + \dfrac{1}{x^2} - 2 = 3 - \dfrac{3}{x^2}$

$\Rightarrow \quad x^2 + \dfrac{4}{x^2} - 5 = 0$

$\Rightarrow \quad x^4 - 5x^2 + 4 = 0$

$\Rightarrow \quad (x^2-1)(x^2-4) = 0$

$\Rightarrow \quad x = \pm 2, \pm 1$

148. $\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(\mathrm{x} + \dfrac{3}{2}\right)\right)\right)\right) + \tan(\sec^{-1}x) = 0$

$\Rightarrow \quad \left(\dfrac{2x+3}{2}\right) = \sqrt{x^2-1}$

$\Rightarrow \quad (2x+3)^2 = 4(x^2-1)$

$\Rightarrow \quad 12x + 9 = -4$

$\Rightarrow \quad x = -\dfrac{13}{12}$

149. Given equation is

$$\tan\left(\tan^{-1}\left(\frac{x}{10}\right) + \tan^{-1}\left(\frac{1}{x+1}\right)\right) = \tan\left(\frac{\pi}{4}\right)$$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{\dfrac{x}{12} + \dfrac{1}{x+1}}{1 - \dfrac{x}{12}\cdot\dfrac{1}{x+1}}\right) = \dfrac{\pi}{4}$

$\Rightarrow \quad \dfrac{x^2+x+12}{11x+12} = 1$

$\Rightarrow \quad x^2 + x + 12 = 11x + 12$

$\Rightarrow \quad x^2 - 10x = 0$

$\Rightarrow \quad x(x-10) = 0$

$\Rightarrow \quad x = 0, 10$

150. $(x-2)x^2 + 8x + k + 4$
$> \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$
$\Rightarrow (k-2)x^2 + 8x + k + 4$
$> (12 - 4\pi) + (4\pi - 12)$
$\Rightarrow (k-2)x^2 + 8x + (k+4) > 0$
For all $x$ in $R$, $D \geq 0$
$\Rightarrow 64 - 4(k-2)(k+4) \geq 0$
$\Rightarrow 16 - (k-2)(k+4) \geq 0$
$\Rightarrow (k-2)(k+4) - 16 \leq 0$
$\Rightarrow k^2 + 2k - 24 \leq 0$
$\Rightarrow (k+6)(k-4) \leq 0$
$\Rightarrow -6 \leq k \leq 4$
Thus, the least integral value of $k$ is $-6$

151. Do yourself.

152. Given,
$f(x) = \sin^{-1}(\sin x), \forall x \in [-\pi, 2\pi]$.
$= (-\pi - x) - x + (\pi - x) + (x - 2\pi)$
$= -2\pi - 2x$
Thus, $f'(x) = -2$

153. Given,
$f(x) = \cos^{-1}(\cos x), \forall x \in [-2\pi, \pi]$
$= (x + 2\pi) - x + x$
$= (x + 2\pi)$
Thus, $f'(x) = 1$

154. Given,
$f(x) = \tan^{-1}(\tan x), \ \forall x \in \left[-\frac{3\pi}{2}, \frac{5\pi}{2}\right]$
$= (x + \pi) + x + (x - \pi) + (x - 2\pi)$
$= (4x - 2\pi)$
Thus, $f'(x) = 4$

155. We have

$$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cot^{-1}(3)$$
$$= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$
$$= \tan^{-1}\left(\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}}\right)$$
$$= \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

156. We have $2\tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{12}{5}\right)$

$$= \tan^{-1}\left(\frac{2\cdot\frac{3}{2}}{1 - \frac{9}{4}}\right) + \tan^{-1}\left(\frac{12}{5}\right)$$
$$= \tan^{-1}\left(-\frac{12}{5}\right) + \tan^{-1}\left(\frac{12}{5}\right)$$
$$= \pi - \tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{12}{5}\right)$$
$$= \pi$$

157. $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$

$$= \cot^{-1}\left(\frac{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)}\right)$$
$$= \cot^{-1}\left(\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)$$
$$= \cot^{-1}\left(\cot\left(\frac{x}{2}\right)\right) = \frac{x}{2}$$

158. $\sin^{-1}(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2})$

$$= \sin^{-1}(x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2})$$
$$= \sin^{-1}(x) - \sin^{-1}(\sqrt{x})$$

159. $\sin^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$
$$= \sin^{-1}\left(\sin\left(x + \frac{\pi}{4}\right)\right)$$
$$= \left(x + \frac{\pi}{4}\right)$$

160. $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), \frac{\pi}{4} < x < \frac{5\pi}{4}$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$
$$= \cos^{-1}\left(\cos\left(x - \frac{\pi}{4}\right)\right)$$
$$= \left(x - \frac{\pi}{4}\right)$$

161. $\tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right)$

Put $x^2 = \cos 2\pi$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}\right)$$
$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$
$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$
$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \theta\right)\right)$$
$$= \left(\frac{\pi}{4} + \theta\right)$$

$$= \left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}(x^2)\right)$$

162. $\sin^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$

$= \sin^{-1}(\sin\alpha\cos x + \cos\alpha\sin x)$

$= \sin^{-1}(\sin(x+\alpha))$

$= (x+\alpha)$

$= x + \tan^{-1}\left(\frac{3}{4}\right)$

163. Let $\sin^{-1}\left(\frac{1}{4}\right) = \theta$

$\Rightarrow \quad \sin\theta = \frac{1}{4}$

Now,

$$\sin(2\theta) = 2\sin\theta\cdot\cos\theta = 2\times\frac{1}{4}\times\sqrt{1-\frac{1}{16}} = \frac{\sqrt{15}}{8}$$

164. Let $\cos^{-1}\left(\frac{1}{3}\right) = \theta$

$\Rightarrow \quad \cos\theta = \frac{1}{3}$

Now,

$$\cos(2\theta) = 2\cos^2\theta - 1 = \frac{2}{9} - 1 = -\frac{7}{9}$$

165. Let $\tan^{-1}\left(\frac{1}{3}\right) = \theta$

$\Rightarrow \quad \tan\theta = \frac{1}{3}$

Now,

$$\cos(2\theta) = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{3}\right)^2} = \frac{9-1}{9+1} = \frac{8}{10} = \frac{4}{5}$$

166. Let $\cot^{-1}\left(\frac{3}{4}\right) = \theta$

Now, $\sin\left(\frac{\theta}{2}\right)$

$$= \sqrt{\frac{1-\cos\theta}{2}}$$

$$= \sqrt{\frac{1-\frac{\cos\theta}{\sin\theta\cdot\operatorname{cosec}\theta}}{2}}$$

$$= \sqrt{\frac{1-\frac{\cot\theta}{\operatorname{cosec}\theta}}{2}}$$

$$= \sqrt{\frac{1-\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}}{2}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1-\frac{3/4}{\sqrt{1+9/16}}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1-\frac{3}{5}}$$

$$= \frac{2}{\sqrt{5}}$$

167. Let $\tan^{-1}\left(\frac{3}{4}\right) = \theta$

$\Rightarrow \quad \tan\theta = \frac{3}{4}$

We have

$$\tan\left(\frac{3\pi}{4} - 2\theta\right) = \frac{\tan\left(\frac{3\pi}{4}\right) - \tan(2\theta)}{1+\tan\left(\frac{3\pi}{4}\right)\cdot\tan(2\theta)}$$

$$= \frac{-1-\tan(2\theta)}{1-\tan(2\theta)}$$

$$= \frac{-1-\frac{2\tan\theta}{1-\tan^2\theta}}{1-\frac{2\tan\theta}{1-\tan^2\theta}}$$

$$= \frac{\tan^2\theta - 2\tan\theta - 1}{1-\tan^2\theta - 2\tan\theta}$$

$$= \frac{\frac{9}{16}-\frac{6}{4}-1}{1-\frac{9}{16}-\frac{6}{4}}$$

$$= \frac{41}{17}$$

168. Let $\sin^{-1}\left(\frac{1}{2}\right) = \theta$

Then $\sin\theta = \frac{1}{2}$

Now, $\sin\left(2\sin^{-1}\left(\frac{1}{2}\right)\right)$

$= \sin(2\theta)$
$= 2\sin\theta\cos\theta$
$= 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

169. Let $\sin^{-1}\left(\frac{1}{3}\right) = \theta$

Then $\sin\theta = \frac{1}{3}$

Now, $\sin\left(3\sin^{-1}\left(\frac{1}{3}\right)\right)$
$= \sin(3\theta)$
$= 3\sin\theta - 4\sin^3\theta$
$= 3 \cdot \frac{1}{3} - 4 \cdot \left(\frac{1}{3}\right)^3$
$= 1 - \frac{4}{27} = \frac{23}{27}$

170. Let $\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right) = \theta$

Then $\cos^{-1}\left(\frac{1}{8}\right) = 2\theta$

$\Rightarrow \cos(2\theta) = \frac{1}{8}$

$\Rightarrow 2\cos^2\theta - 1 = \frac{1}{8}$

$\Rightarrow 2\cos^2\theta = \frac{9}{8}$

$\Rightarrow \cos^2\theta = \frac{9}{16}$

$\Rightarrow \cos\theta = \frac{3}{4}$

$\Rightarrow \cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right) = \frac{3}{4}$

171. Let $\frac{1}{2}\cos^{-1}\left(-\frac{1}{10}\right) = \theta$

Then $\cos^{-1}\left(-\frac{1}{10}\right) = 2\theta$

$\Rightarrow \cos(2\theta) = -\frac{1}{10}$

$\Rightarrow 2\cos^2\theta - 1 = -\frac{1}{10}$

$\Rightarrow 2\cos^2\theta = 1 - \frac{1}{10} = \frac{9}{10}$

$\Rightarrow \cos^2\theta = \frac{9}{20}$

$\Rightarrow \cos\theta = \frac{3}{2\sqrt{5}}$

$\Rightarrow \cos\theta = \frac{3}{2\sqrt{5}} = \frac{3\sqrt{5}}{2\sqrt{5} \times \sqrt{5}} = \frac{3\sqrt{5}}{10}$

$\Rightarrow \cos\left(\frac{1}{2}\cos^{-1}\left(-\frac{1}{10}\right)\right) = \frac{3\sqrt{5}}{10}$

172. Let $\frac{1}{2}\cos^{-1}\left(\frac{1}{9}\right) = \theta$

$\Rightarrow \cos^{-1}\left(\frac{1}{9}\right) = 2\theta$

$\Rightarrow \cos(2\theta) = \frac{1}{9}$

$\Rightarrow 2\cos^2(\theta) - 1 = \frac{1}{9}$

$\Rightarrow 2\cos^2(\theta) = 1 + \frac{1}{9} = \frac{10}{9}$

$\Rightarrow \cos^2(\theta) = \frac{5}{9}$

$\Rightarrow \sin^2(\theta) = 1 - \frac{5}{9} = \frac{4}{9}$

$\Rightarrow \sin(\theta) = \frac{2}{3}$

$\Rightarrow \sin\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{9}\right)\right) = \frac{2}{3}$

173. Let $\frac{1}{4}\tan^{-1}(\sqrt{63}) = \theta$

$\Rightarrow \tan^{-1}(\sqrt{63}) = 4\theta$

$\Rightarrow \tan(4\theta) = \sqrt{63}$

$\Rightarrow \tan(4\theta) = \sqrt{63}$

$\Rightarrow \frac{\sin(4\theta)}{\cos(4\theta)} = \sqrt{63}$

$\Rightarrow \frac{\sin(4\theta)}{\sqrt{63}} = \frac{\cos(4\theta)}{1} = \frac{1}{8}$

Now, $\cos(4\theta) = \frac{1}{8}$

$\Rightarrow 2\cos^2(2\theta) - 1 = \frac{1}{8}$

$\Rightarrow 2\cos^2(2\theta) = 1 + \frac{1}{8} = \frac{9}{8}$

$\Rightarrow \cos^2(2\theta) = \frac{9}{16}$

$\Rightarrow \cos(2\theta) = \frac{3}{4}$ …(i)

$\Rightarrow 2\cos^2(\theta) - 1 = \frac{3}{4}$

$\Rightarrow 2\cos^2(\theta) = \frac{7}{4}$

$\Rightarrow \cos^2(\theta) = \frac{7}{8}$

$\Rightarrow \cos(\theta) = \frac{\sqrt{7}}{2\sqrt{2}}$ …(ii)

Also, $\sin(4\theta) = \frac{\sqrt{63}}{8}$

$\Rightarrow \quad 2\sin(2\theta)\cos(2\theta) = \frac{\sqrt{63}}{8}$

$\Rightarrow \quad 2\sin(2\theta) \times \frac{3}{4} = \frac{\sqrt{63}}{8}$, from (i)

$\Rightarrow \quad \sin(2\theta) = \frac{\sqrt{63}}{12}$

$\Rightarrow \quad 2\sin(\theta)\cos(\theta) = \frac{\sqrt{63}}{12}$, from (ii)

$\Rightarrow \quad 2\sin(\theta) \times \frac{\sqrt{7}}{2\sqrt{2}} = \frac{\sqrt{63}}{12}$

$\Rightarrow \quad \sin(\theta) \times \frac{\sqrt{7}}{\sqrt{2}} = \frac{\sqrt{63}}{12}$

$\Rightarrow \quad \sin(\theta) = \frac{3\sqrt{2}}{12}$

$\Rightarrow \quad \sin(\theta) = \frac{\sqrt{2}}{4}$

$\Rightarrow \quad \sin(\theta) = \frac{1}{2\sqrt{2}}$

$\Rightarrow \quad \sin\left(\frac{1}{4}\tan^{-1}(\sqrt{63})\right) = \frac{1}{2\sqrt{2}}$

174. Let $\frac{1}{4}\tan^{-1}\left(\frac{24}{7}\right) = \theta$

$\Rightarrow \quad \tan^{-1}\left(\frac{24}{7}\right) = 4\theta$

$\Rightarrow \quad \tan(4\theta) = \frac{24}{7}$

$\Rightarrow \quad \frac{\sin(4\theta)}{24} = \frac{\cos(4\theta)}{7} = \frac{1}{25}$

Now, $\cos(4\theta) = \frac{7}{25}$

$\Rightarrow \quad 2\cos^2(2\theta) - 1 = \frac{7}{25}$

$\Rightarrow \quad 2\cos^2(2\theta) = 1 + \frac{7}{25} = \frac{32}{25}$

$\Rightarrow \quad \cos^2(2\theta) = \frac{32}{50}$

$\Rightarrow \quad \cos(2\theta) = \sqrt{\frac{32}{50}}$

$\Rightarrow \quad 2\cos^2(\theta) - 1 = \sqrt{\frac{32}{50}}$

$\Rightarrow \quad 2\cos^2(\theta) = 1 + \frac{8}{10} = \frac{18}{10}$

$\Rightarrow \quad \cos^2(\theta) = \frac{9}{10}$

$\Rightarrow \quad \cos(\theta) = \frac{3}{\sqrt{10}}$

$\Rightarrow \quad \cos\left(\frac{1}{4}\left(\tan^{-1}\left(\frac{24}{7}\right)\right)\right) = \frac{3}{\sqrt{10}}$

175. Let $\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right) = \theta$

$\Rightarrow \quad \cos^{-1}\left(\frac{2}{3}\right) = 2\theta$

$\Rightarrow \quad \cos(2\theta) = \frac{2}{3}$

$\Rightarrow \quad \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{2}{3}$

$\Rightarrow \quad 2 + 2\tan^2\theta = 3 - 3\tan^2\theta$

$\Rightarrow \quad 5\tan^2\theta = 1$

$\Rightarrow \quad \tan^2\theta = \frac{1}{5}$

$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{5}}$

$\Rightarrow \quad \tan\left(\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right)\right) = \frac{1}{\sqrt{5}}$

176. We have $\tan\left(2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right) = -\frac{7}{17}$

Let $2\tan^{-1}\left(\frac{1}{5}\right) = \theta$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{2}{5}}{1 - \frac{1}{25}}\right) = \theta$

$\Rightarrow \quad \tan\theta = \left(\frac{\frac{2}{5}}{1 - \frac{1}{25}}\right) = \frac{10}{24} = \frac{5}{12}$

Now, $\tan\left(\theta - \frac{\pi}{4}\right)$

$= \frac{\tan\theta - 1}{1 + \tan\theta}$

$= \frac{\frac{5}{12} - 1}{\frac{5}{12} + 1} = -\frac{7}{17}$

177. Let $\frac{1}{4}\sin^{-1}\left(-\frac{4}{5}\right) = \theta$

$\Rightarrow \quad \sin^{-1}\left(-\frac{4}{5}\right) = 4\theta$

$\Rightarrow \quad \sin(4\theta) = -\frac{4}{5}$

$\Rightarrow \quad \dfrac{2\tan(2\theta)}{1+\tan^2(2\theta)} = -\dfrac{4}{5}$

$\Rightarrow \quad \dfrac{\tan(2\theta)}{1+\tan^2(2\theta)} = -\dfrac{2}{5}$

$\Rightarrow \quad 2\tan^2(2\theta) + 5\tan(2\theta) + 2 = 0$

$\Rightarrow \quad \tan(2\theta) = -\dfrac{1}{2}, -2$

when $\tan(2\theta) = -\dfrac{1}{2}$

$\Rightarrow \quad \dfrac{2\tan\theta}{1-\tan^2\theta} = -\dfrac{1}{2}$

$\Rightarrow \quad \tan^2\theta - 4\tan\theta - 1 = 0$

$\Rightarrow \quad \tan\theta = 2 - \sqrt{5}$

Now, $\tan\left(\dfrac{3\pi}{4} - \dfrac{1}{4}\sin^{-1}\left(-\dfrac{4}{5}\right)\right)$

$= \tan\left(\dfrac{3\pi}{4} - \theta\right)$

$= \tan\left(\pi - \left(\dfrac{\pi}{4} + \theta\right)\right)$

$= -\tan\left(\dfrac{\pi}{4} + \theta\right)$

$= -\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$

$= -\left(\dfrac{1+2-\sqrt{5}}{1-2+\sqrt{5}}\right)$

$= \left(\dfrac{3-\sqrt{5}}{1-\sqrt{5}}\right)$

$= \left(\dfrac{(3-\sqrt{5})(1+\sqrt{5})}{-4}\right)$

$= -\dfrac{1}{4}(3 + 2\sqrt{5} - 5)$

$= -\dfrac{1}{4}(-2 + 2\sqrt{5})$

$= \left(\dfrac{1-\sqrt{5}}{2}\right)$

178. Given equation is

$x^2 - 3x < \sin^{-1}(\sin 2)$

$\Rightarrow \quad x^2 - 3x < (\pi - 2)$

$\Rightarrow \quad \left(x - \dfrac{3}{2}\right)^2 < \left(\pi - 2 + \dfrac{9}{4}\right) = \dfrac{4\pi+1}{4}$

$\Rightarrow \quad \left|\left(x - \dfrac{3}{2}\right)\right| < \dfrac{\sqrt{4\pi+1}}{2}$

$\Rightarrow \quad -\dfrac{\sqrt{4\pi+1}}{2} < \left(x - \dfrac{3}{2}\right) < \dfrac{\sqrt{4\pi+1}}{2}$

$\Rightarrow \quad \dfrac{3}{2} - \dfrac{\sqrt{4\pi+1}}{2} < x < \dfrac{\sqrt{4\pi+1}}{2} + \dfrac{3}{2}$

$\Rightarrow \quad -0.2 < x < 3.3$

Thus, the integral values of $x$ are 0, 1, 2, 3.

179. Given in-equation is

$3x^2 + 8x < 2\sin^{-1}(\sin 4) - \cos^{-1}(\cos 4)$

$\Rightarrow \quad 3x^2 + 8x < 2(\pi - 4) - (2\pi - 4)$

$\Rightarrow \quad 3x^2 + 8x + 4 < 0$

$\Rightarrow \quad 3x^2 + 6x + 2x + 4 < 0$

$\Rightarrow \quad 3x(x+2) + 2(x+2) < 0$

$\Rightarrow \quad (3x+2)(x+2) < 0$

$\Rightarrow \quad -2 < x < -\dfrac{2}{3}$

180. We have

$f(x) = \cos^{-1}x + \cos^{-1}\left\{\dfrac{x}{2} + \dfrac{\sqrt{1-3x^2}}{2}\right\}$

$= \cos^{-1}(x) + \cos^{-1}\left(\dfrac{1}{2}x + \sqrt{1-x^2}\sqrt{1-\dfrac{1}{4}}\right)$

$= \cos^{-1}(x) + \cos^{-1}\left(\dfrac{1}{2}\right) - \cos^{-1}(x)$

$= \cos^{-1}\left(\dfrac{1}{2}\right)$

$= \dfrac{\pi}{3}$

= constant function.

Hence, the result.

181. We have $f(x)$

$= \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) + 2\tan^{-1}(x)$

$= \pi - 2\tan^{-1}(x) + 2\tan^{-1}(x)$

$= \pi$

Hence, the value of $f(2013) = \pi$

182. We have

$f(x) = 2\tan^{-1}\left(\dfrac{1+x}{1-x}\right) + \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$= 2(\tan^{-1}(1) + \tan^{-1}(x)) + \dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$= 2\left(\dfrac{\pi}{4} + \tan^{-1}(x)\right) + \dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$= \left(\dfrac{\pi}{2} + 2\tan^{-1}(x)\right) + \left(\dfrac{\pi}{2} - 2\tan^{-1}(x)\right)$

$= \pi$

Hence, the value of $f\left(\dfrac{1}{2014}\right) = \pi$

183. We have

$$f(x) = \sin^{-1}\left(\frac{6x}{x^2+9}\right) + 2\tan^{-1}\left(-\frac{x}{3}\right)$$

$$= \sin^{-1}\left(\frac{2\cdot\left(\frac{x}{3}\right)}{1+\left(\frac{x}{3}\right)^2}\right) - 2\tan^{-1}\left(\frac{x}{3}\right)$$

$$= 2\tan^{-1}\left(\frac{x}{3}\right) - 2\tan^{-1}\left(\frac{x}{3}\right)$$

$$= 0$$

It will happen when $\left|\frac{x}{3}\right| \le 1$

$\Rightarrow \quad |x| \le 3$

$\Rightarrow \quad -3 \le x \le 3$

184. We have

$$f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\tan^{-1}(x)$$

$$= -2\tan^{-1}(x) + 2\tan^{-1}(x),\ x \le 0$$

$$= 0$$

It is possible only when $x \le 0$

$\Rightarrow \quad x \in (-\infty, 0]$

185. We have

$$f(x) = 3\cos^{-1}(2x^2-1) + 2\cos^{-1}(4x^3-3x)$$

$$= 3(2\cos^{-1}x) + 2(2\pi - 3\cos^{-1}x)$$

$$(\text{for } 0 \le x < 1)\ \left(\text{for } -\frac{1}{2} \le x \le \frac{1}{2}\right)$$

It is possible only when $0 \le x \le \frac{1}{2}$

$\Rightarrow \quad x \in \left[0, \frac{1}{2}\right]$

Also,

$$f(x) = 3\cos^{-1}(2x^2-1) + 2\cos^{-1}(4x^3-3x)$$

$$= 3(2\pi - 2\cos^{-1}x) + 2(-2\pi + 3\cos^{-1}x)$$

$$(\text{for } -1 \le x \le 0)\ \left(\text{for } -1 \le x < -\frac{1}{2}\right)$$

$$= 2\pi$$

It is possible only when $x \in \left[-1, -\frac{1}{2}\right)$

Hence, the value of $x$ is

$$\left[-1, -\frac{1}{2}\right) \cup \left[0, \frac{1}{2}\right]$$

186. We have

$$\tan^{-1}y = 4\tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}y = \tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)$$

$$\Rightarrow \quad y = \frac{4x(1-x^2)}{1-6x^2+x^4}$$

Which is a function of $x$.

Let $\tan^{-1}x = \frac{\pi}{8}$

$$\Rightarrow \quad x = \tan\left(\frac{\pi}{8}\right)$$

$$\Rightarrow \quad \tan^{-1}y = 4\tan^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{4x(1-x^2)}{1-6x^2+x^4} \to \infty$$

$$\Rightarrow \quad 1 - 6x^2 + x^4 = 0$$

$\Rightarrow \quad x = \tan\left(\frac{\pi}{8}\right)$ is a root of $1 + x^4 = 6x^2$

187. Do yourself.

188. We have

$$\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right)$$

$$= \tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right),\ \theta = \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)$$

$$= \frac{1+\tan\theta}{1-\tan\theta} + \frac{1-\tan\theta}{1+\tan\theta},\ \cos(2\theta) = \frac{a}{b}$$

$$= \frac{2(1+\tan^2\theta)}{1-\tan^2\theta}$$

$$= \frac{2}{\cos(2\theta)}$$

$$= \frac{2}{\frac{a}{b}} = \frac{2b}{a}$$

189. Do yourself.

190. We have

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right)$$

$$= \tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right)$$

$$= (\tan^{-1}a - \tan^{-1}b) + (\tan^{-1}b - \tan^{-1}c) + (\tan^{-1}c - \tan^{-1}a)$$

$$= 0$$

191. We have

$$\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right)$$

$$= \tan\left(\frac{1}{2}\cdot 2\tan^{-1}x + \frac{1}{2}\cdot 2\tan^{-1}y\right)$$

$$= \tan(\tan^{-1}x + \tan^{-1}y)$$

$$= \tan\left(\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right)$$

$$= \left(\frac{x+y}{1-xy}\right),\ xy < 1$$

192. We have

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{1-y}{1+y}\right)$$

$$= \{\tan^{-1}(1) - \tan^{-1}(x)\} - \{\tan^{-1}(1) - \tan^{-1}(y)\}$$

$$= \tan^{-1}(y) - \tan^{-1}(x)$$

$$= \tan^{-1}\left(\frac{y-x}{1+xy}\right)$$

$$= \sin^{-1}\left(\frac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}\right)$$

193. We have

$$\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{\cot A + \cot^3 A}{1-\cot^4 A}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{\cot A}{1-\cot^2 A}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{\tan A}{\tan^2 A - 1}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(-\frac{\tan A}{1-\tan^2 A}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) - \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right)$$

$$= 0$$

194. We have

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{\theta}{2}\right)\right)$$

$$= \cos^{-1}\left(\frac{1-\left(\frac{a-b}{a+\mathrm{b}}\right)\tan^2\left(\frac{\theta}{2}\right)}{1+\left(\frac{a-b}{a+b}\right)\tan^2\left(\frac{\theta}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{(a+b)-(a-b)\tan^2\left(\frac{\theta}{2}\right)}{(a+b)+(a-b)\tan^2\left(\frac{\theta}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)+b\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}{a\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)+b\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\left(\dfrac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)+b}{a+b\left(\dfrac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\cos\theta + b}{a + b\cos\theta}\right)$$

195. We have

$2\tan(\tan^{-1} a + \tan^{-1} a^3)$

$$= 2\tan\left(\tan^{-1}\left(\frac{a+a^3}{1-a^4}\right)\right)$$

$$= 2\tan\left(\tan^{-1}\left(\frac{a}{1-a^2}\right)\right)$$

$$= \left(\frac{2a}{1-a^2}\right)$$

$$= \tan(2\tan^{-1} a)$$

196. Do yourself

197. Given, $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$

Let $A = \sin^{-1} x, B = \sin^{-1} y, C = \sin^{-1} z$

$\Rightarrow \quad x = \sin A, y = \sin B, z = \sin C$

$\Rightarrow \quad \cos A = \sqrt{1-x^2}, \cos B = \sqrt{1-y^2}$

and $\cos C = \sqrt{1-z^2}$

Now, $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$

$= \sin A\cos A + \sin B\cos B + \sin C\cos C$

$= \frac{1}{2}[2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C]$

$= \frac{1}{2}(\sin(2A) + \sin(2B) + \sin(2C))$

$= \frac{1}{2}(4\sin A\sin B\sin C)$

$= (2\sin A\sin B\sin C)$

$= 2xyz$

198. We have

$\cos^{-1} x \cos^{-1} y + \cos^{-1} z = \pi,$

$\Rightarrow \quad \cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z$

$\Rightarrow \quad \cos^{-1}(x \cdot y - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos^{-1}(-z)$

$\Rightarrow \quad (xy+z)^2 = (\sqrt{(1-x^2)(1-y^2)})^2$

$\Rightarrow \quad x^2y^2 + 2xyz + z^2 = 1 - x^2 - x^2 - y^2 + x^2y^2$

$\Rightarrow \quad 2xyz + z^2 = 1 - x^2 - y^2$

$\Rightarrow \quad x^2 + y^2 + z^2 + 2xyz = 1$

199. We have

$$\cos^{-1}\left(\frac{x}{2}\right) + \cos^{-1}\left(\frac{y}{3}\right) = \theta$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{xy}{6} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right) = \theta$$

$$\Rightarrow \quad \left(\frac{xy}{6} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right) = \cos\theta$$

$$\Rightarrow \quad \left(\frac{xy}{6} - \cos\theta\right)^2 = \left(1-\frac{x^2}{4}\right)\left(1-\frac{y^2}{9}\right)$$

$$\Rightarrow \quad \frac{x^2y^2}{36} - \frac{xy}{3}\cos\theta + \cos^2\theta = 1 - \frac{x^2}{4} - \frac{y^2}{9} + \frac{x^2y^2}{36}$$

$$\Rightarrow \quad -\frac{xy}{3}\cos\theta + \cos^2\theta = 1 - \frac{x^2}{4} - \frac{y^2}{9}$$

$\Rightarrow \quad \frac{x^2}{4}+\frac{y^2}{9}-\frac{xy}{3}\cos\theta = 1-\cos^2\theta$

$\Rightarrow \quad \frac{x^2}{4}+\frac{y^2}{9}-\frac{xy}{3}\cos\theta = \sin^2\theta$

$\Rightarrow \quad 9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

Hence, the result.

200. Given, $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

It is possible only when each term will provide us the maximum value

Thus, $\sin^{-1}x=\frac{\pi}{2}=\sin^{-1}y=\sin^{-1}z$

So, $x=1, y=1, z=1$

Hence, the value of

$x^2+y^2+z^2-2xyz$

$= 1+1+1-2$

$= 1$

201. Given, $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

It is possible only when each term will provide us the maximum value

Thus, $\sin^{-1}x=\frac{\pi}{2}=\sin^{-1}y=\sin^{-1}z$

So, $x=1, y=1, z=1$

Hence, the value of

$xy+yz+zx = 1+1+1$

$= 3$

202. Given, $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

It is possible only when each term will provide us the maximum value

Thus, $\sin^{-1}x=\frac{\pi}{2}=\sin^{-1}y=\sin^{-1}z$

So, $x=1, y=1, z=1$

Hence, the value of

$x^{2012}+y^{2012}+z^{2012}-\frac{9}{x^{2013}+y^{2013}+z^{2013}}$

$= 1+1+1-\frac{9}{1+1+1}$

$= 3-3$

$= 0$

203. Given, $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$

It is possible only when, each term will provide us the maximum value.

Thus, $\cos^{-1}x=\pi, \cos^{-1}y=\pi, \cos^{-1}z=\pi$

$x=-1, y=-1, z=-1$

Hence, the value of

$xy+yz+zx = 1+1+1$

$= 3$

204. Given, $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$

It is possible only when, each term will provide us the maximum value.

Thus, $\cos^{-1}x=\pi, \cos^{-1}y=\pi, \cos^{-1}z=\pi$

$x=-1, y=-1, z=-1$

Hence, the value of

$\left(\frac{x^{2013}+y^{2013}+z^{2013}+6}{x^{2014}+y^{2014}+z^{2014}}\right)$

$=\frac{-1-1-1+6}{1+1+1}$

$=\frac{3}{3}=1$

205. Given, $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y=\frac{\pi}{2}-\tan^{-1}z$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right)=\cot^{-1}(z)$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}\left(\frac{1}{z}\right)$

$\Rightarrow \quad \left(\frac{x+y}{1-xy}\right)=\left(\frac{1}{z}\right)$

$\Rightarrow \quad xz+yz=1-xy$

$\Rightarrow \quad xy+yz+zx=1$

206. Given, $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right)=\frac{\pi}{4}$

$\Rightarrow \quad \left(\frac{x+y}{1-xy}\right)=\tan\left(\frac{\pi}{4}\right)=1$

$\Rightarrow \quad x+y=1-xy$

$\Rightarrow \quad x+y+xy=1$

207. Given, $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)=\pi$

$\Rightarrow \quad \left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)=\tan(\pi)=0$

$\Rightarrow \quad \left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)=0$

$\Rightarrow \quad x+y+z-xyz=0$

$\Rightarrow \quad x+y+z=xyz$

Hence, the result.

208. Given, $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)=\alpha$

Put $x^2=\sin(2\theta)$

Now, $\tan^{-1}\left(\frac{\sqrt{1+\sin 2\theta}-\sqrt{1-\sin 2\theta}}{\sqrt{1+\sin 2\theta}+\sqrt{1-\sin 2\theta}}\right)=\alpha$

$\Rightarrow \quad \tan^{-1}\left(\frac{(\cos\theta+\sin\theta)-(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)+(\cos\theta-\sin\theta)}\right)=\alpha$

$\Rightarrow \quad \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)=\alpha$

$\Rightarrow \quad \tan^{-1}(\tan\theta) = \alpha$
$\Rightarrow \quad \theta = \alpha$
$\Rightarrow \quad 2\theta = 2\alpha$
$\Rightarrow \quad \sin(2\theta) = \sin(2\alpha)$
$\Rightarrow \quad x^2 = \sin(2\alpha)$

209. We have

$m = \tan^2(\sec^{-1} 2) + \cot^2(\text{cosec}^{-1} 3)$
$= \tan^2(\theta) + \cot^2(\varphi)$, where

$\sec\theta = 2$ and $\text{cosec}\,\varphi = 3$

$= 1 + \sec^2\theta + 1 + \text{cosec}^2\varphi$
$= 1 + 4 + 1 + 9$
$= 15$

Hence, the value of $(m^2 + m + 10)$

$= 225 + 15 + 10$
$= 250$

210. Given, $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) = \frac{\pi}{4}$

$\Rightarrow \quad \sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right) = 1$

$\Rightarrow \quad 3\sin 2\theta = 5 + 4\cos 2\theta$

$\Rightarrow \quad \frac{3.2\tan\theta}{1+\tan^2\theta} = 5 + 4\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\Rightarrow \quad \frac{6\tan\theta}{1+\tan^2\theta} = \frac{5+5\tan^2\theta+4-4\tan^2\theta}{1+\tan^2\theta}$

$\Rightarrow \quad 6\tan\theta = \tan^2\theta + 9$
$\Rightarrow \quad \tan^2\theta - 6\tan\theta + 9 = 0$
$\Rightarrow \quad (\tan\theta - 3)^2 = 0$
$\Rightarrow \quad (\tan\theta - 3) = 0$
$\Rightarrow \quad \tan\theta = 3$

211. Given, $m = \frac{(\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3)}{(\cot^{-1}1 + \cot^{-1}2 + \cot^{-1}3)}$

$= \frac{\pi}{\frac{\pi}{2}} = 2$

Hence, the value of

$(m+2)^{m+1} = (2+2)^3 = 4^3 = 64$

212. Given equation is

$$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{3\pi}{4}$$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right) = \frac{\pi}{4}$

$\Rightarrow \quad \left(\frac{5x}{1-6x^2}\right) = 1$

$\Rightarrow \quad 5x = 1 - 6x^2$
$\Rightarrow \quad 6x^2 + 5x - 1 = 0$
$\Rightarrow \quad 6x^2 + 6x - x - 1 = 0$
$\Rightarrow \quad 6x(x+1) - 1(x+1) = 0$
$\Rightarrow \quad (6x-1)(x+1) = 0$
$\Rightarrow \quad x = -1, \frac{1}{6}$

213. $\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}(-7)$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{x+1}{x-1} + \frac{x-1}{x}}{1 - \frac{x+1}{x-1} \times \frac{x-1}{x}}\right) = \tan^{-1}(-7)$

$\Rightarrow \quad \tan^{-1}\left(\frac{x^2+x+x^2-2x+1}{x^2-x-x^2+1}\right) = \tan^{-1}(-7)$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x^2-x+1}{1-x}\right) = \tan^{-1}(-7)$

$\Rightarrow \quad \left(\frac{2x^2-x+1}{1-x}\right) = (-7)$

$\Rightarrow \quad 2x^2 - x + 1 = -7 + 7x$
$\Rightarrow \quad 2x^2 - 8x + 8 = 0$
$\Rightarrow \quad x^2 - 4x + 4 = 0$
$\Rightarrow \quad (x-2)^2 = 0$
$\Rightarrow \quad x = 2$

Hence, the solution is $x = 2$

214. Given equation is

$$\sin^{-1}(2x) + \sin^{-1}(x) = \frac{\pi}{3}$$

$\Rightarrow \quad \sin^{-1}(2x) = \frac{\pi}{3} - \sin^{-1}x$

$\Rightarrow \quad \sin^{-1}(2x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}x$

$\Rightarrow \quad \sin^{-1}(2x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right)$

$\Rightarrow \quad (2x) = \left(\frac{\sqrt{3}}{2}\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right)$

$\Rightarrow \quad (2x) = \left(\frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{x}{2}\right)$

$\Rightarrow \quad \left(2x + \frac{x}{2}\right) = \frac{\sqrt{3}}{2}\sqrt{1-x^2}$

$\Rightarrow \quad \left(\frac{5x}{2}\right) = \frac{\sqrt{3}}{2}\sqrt{1-x^2}$

$\Rightarrow \quad \left(\frac{25x^2}{4}\right) = \frac{3(1-x^2)}{4}$

$\Rightarrow \quad 25x^2 = 3 - 3x^2$
$\Rightarrow \quad 28x^2 = 3$

$\Rightarrow \quad x^2 = \frac{3}{28}$

$\Rightarrow \quad x = \pm\sqrt{\frac{3}{28}} = \pm\frac{\sqrt{3}}{2\sqrt{7}}$

215. Given equation is

$$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cos^{-1}x = \frac{\pi}{4}$$

$$\Rightarrow \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) + \cos^{-1}x = \frac{\pi}{4}$$

$$\Rightarrow \cos^{-1}x = \frac{\pi}{4} - \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \cos^{-1}x = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) - \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \cos^{-1}x = \cos^{-1}\left(\frac{1}{\sqrt{2}}\cdot\frac{2}{\sqrt{5}} + \sqrt{1-\frac{1}{2}}\sqrt{1-\frac{4}{5}}\right)$$

$$\Rightarrow x = \left(\frac{1}{\sqrt{2}}\cdot\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{5}}\right)$$

$$\Rightarrow x = \frac{3}{\sqrt{10}}$$

216. Given equation is

$$\sin^{-1}(x) + \sin^{-1}(3x) = \frac{\pi}{3}$$

$$\Rightarrow \sin^{-1}(3x) = \frac{\pi}{3} - \sin^{-1}(x)$$

$$\Rightarrow \sin^{-1}(3x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}(x)$$

$$\Rightarrow \sin^{-1}(3x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right)$$

$$\Rightarrow (3x) = \left(\frac{\sqrt{3}}{2}\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right)$$

$$\Rightarrow \left(3x + \frac{x}{2}\right) = \frac{\sqrt{3}}{2}\sqrt{1-x^2}$$

$$\Rightarrow \frac{7x}{2} = \frac{\sqrt{3}}{2}\sqrt{1-x^2}$$

$$\Rightarrow 49x^2 = 3(1-x^2)$$

$$\Rightarrow 52x^2 = 3$$

$$\Rightarrow x^2 = \frac{3}{52}$$

$$\Rightarrow x = \pm\sqrt{\frac{3}{52}}$$

Hence, the solutions are

$$\left\{-\sqrt{\frac{3}{52}}, \sqrt{\frac{3}{52}}\right\}$$

217. Given equation is

$$\tan^{-1}\left(\frac{1}{1+2x}\right) + \tan^{-1}\left(\frac{1}{1+4x}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{1}{1+2x}+\frac{1}{1+4x}}{1-\frac{1}{1+2x}\times\frac{1}{1+4x}}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \left(\frac{\frac{1}{1+2x}+\frac{1}{1+4x}}{1-\frac{1}{1+2x}\times\frac{1}{1+4x}}\right) = \left(\frac{2}{x^2}\right)$$

$$\Rightarrow \left(\frac{1+2x+1+4x}{1+6x+8x^2-1}\right) = \left(\frac{2}{x^2}\right)$$

$$\Rightarrow \left(\frac{2+6x}{6x+8x^2}\right) = \left(\frac{2}{x^2}\right)$$

$$\Rightarrow \frac{1+3x}{3x+4x^2} = \frac{2}{x^2}$$

$$\Rightarrow 3x^3 + x^2 = 6x + 8x^2$$

$$\Rightarrow 3x^3 - 7x^2 - 6x = 0$$

$$\Rightarrow (3x^2 - 7x - 6)\,x = 0$$

$$\Rightarrow (3x^2 - 9x + 2x - 6)\,x = 0$$

$$\Rightarrow x(x-3)(2x+3) = 0$$

$$\Rightarrow x = 0, 3, -\frac{3}{2}$$

Hence, the solutions are

$$\left\{0, 3, -\frac{3}{2}\right\}$$

218. Given equation is

$$2\tan^{-1}(2x+1) = \cos^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2(2x+1)}{1-(2x+1)^2}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$\Rightarrow \frac{2(2x+1)}{1-(2x+1)^2} = \left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$\Rightarrow -\frac{2(2x+1)}{(4x^2+4x)} = \left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$\Rightarrow -\frac{(2x+1)}{(2x^2+2x)} = \left(\frac{\sqrt{1-x^2}}{x}\right), x = 0$$

$$\Rightarrow -\frac{(2x+1)}{(2x+2)} = \sqrt{1-x^2}, x = 0$$

$$\Rightarrow \frac{(2x+1)^2}{(2x+2)^2} = (1-x^2), x = 0$$

$$\Rightarrow \frac{4x^2+4x+1}{4x^2+8x+4} = 1-x^2, x = 0$$

$$\Rightarrow 3 - 4x^4 - 8x^3 - 4x^2 + 4x = 0, x = 0$$

$$\Rightarrow 4x^4 + 8x^3 + 4x^2 - 4x - 3 = 0, x = 0$$

Clearly, it has 3 solutions.

219. Given equation is

$$\cos^{-1}x - \sin^{-1}x = \cos^{-1}(x\sqrt{3})$$

$$\Rightarrow \cos^{-1}x - \frac{\pi}{2} + \cos^{-1}x = \cos^{-1}(x\sqrt{3})$$

$$\Rightarrow 2\cos^{-1}x - \frac{\pi}{2} = \cos^{-1}(x\sqrt{3})$$

$\Rightarrow \quad 2\cos^{-1}x = \frac{\pi}{2} + \cos^{-1}(x\sqrt{3})$

$\Rightarrow \quad \cos^{-1}(2x^2 - 1) = \frac{\pi}{2} + \cos^{-1}(x\sqrt{3})$

$\Rightarrow \quad (2x^2 - 1) = \cos\left(\frac{\pi}{2} + \cos^{-1}(x\sqrt{3})\right)$

$\Rightarrow \quad (2x^2 - 1) = -\sin(\cos^{-1}(x\sqrt{3}))$

$\Rightarrow \quad (2x^2 - 1) = -\sqrt{1 - 3x^2}$

$\Rightarrow \quad (2x^2 - 1)^2 = (1 - 3x^2)$

$\Rightarrow \quad 4x^4 - 4x^2 + 1 = (1 - 3x^2)$

$\Rightarrow \quad 4x^4 - x^2 = 0$

$\Rightarrow \quad x^2(4x^2 - 1) = 0$

$\Rightarrow \quad x^2(2x - 1)(2x + 1) = 0$

$\Rightarrow \quad x = 0, \pm\frac{1}{2}$

Hence, the solutions are

$\left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$

220. We have, $\tan^{-1} y = 4\tan^{-1} x$

$\Rightarrow \quad \tan^{-1}y = \tan^{-1}\left(\frac{4x - 4x^3}{1 - 6x^2 + x^4}\right)$

$\Rightarrow \quad y = \frac{4x(1 - x^2)}{1 - 6x^2 + x^4}$

Which is a function of $x$.

Let $\tan^{-1}x = \frac{\pi}{8}$

$\Rightarrow \quad x = \tan\left(\frac{\pi}{8}\right)$

$\Rightarrow \quad \tan^{-1}y = 4\tan^{-1}x = \frac{\pi}{2}$

$\Rightarrow \quad \frac{4x(1 - x^2)}{1 - 6x^2 + x^4} \to \infty$

$\Rightarrow \quad 1 - 6x^2 + x^4 = 0$

$\Rightarrow \quad x = \tan\left(\frac{\pi}{8}\right)$ is a root of $1 + x^4 = 6x^2$

## Level IIA

1. We have $\sin^{-1}(\sin 10)$

$= \sin^{-1}(\sin(3\pi - 10))$

$= (3\pi - 10)$

2. We have $\cos^{-1}(\cos 5)$

$= \cos^{-1}(\cos(2\pi - 5))$

$= (2\pi - 5)$

3. We have $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$

$= \tan^{-1}(1) + \pi + \tan^{-1}\left(\frac{2 + 3}{1 - 2.3}\right)$

$= \frac{\pi}{4} + \pi + \tan^{-1}\left(-\frac{5}{5}\right)$

$= \frac{\pi}{4} + \pi + \tan^{-1}(-1)$

$= \frac{\pi}{4} + \pi - \tan^{-1}(1)$

$= \frac{\pi}{4} + \pi - \frac{\pi}{4}$

$= \pi$

4. Given, $\sin^{-1} x > \cos^{-1} x$

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}x > \sin^{-1} > \sin^{-1}x + \cos^{-1}x$

$\Rightarrow \quad 2\sin^{-1}x > \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}x > \frac{\pi}{4}$

$\Rightarrow \quad x > \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Also, the domain of $\sin^{-1} x$ is $[-1, 1]$

Thus, the solution set is $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$

5. Given, $\sin^{-1} x < \cos^{-1} x$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} x < \sin^{-1} x + \cos^{-1} x$

$\Rightarrow \quad 2\sin^{-1}x < \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}x < \frac{\pi}{4}$

$\Rightarrow \quad x < \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Also, the domain of $\sin^{-1} x$ is $[-1, 1]$

Thus, the solution set is $x \in \left[-1, \frac{1}{\sqrt{2}}\right)$

6. Given, $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1 - x^2})$

Now, the range of $\sin^{-1}(2x\sqrt{1 - x^2})$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Thus, $-\frac{\pi}{2} \le 2\sin^{-1}x \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{4} \le \sin^{-1}x \le \frac{\pi}{4}$

$\Rightarrow \quad \sin\left(-\frac{\pi}{4}\right) \le x \le \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad -\sin\left(\frac{\pi}{4}\right) \le x \le \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

Hence, the solution set is $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

7. Given, $3\sin^{-1}x = \pi + \sin^{-1}(3x - 4x^3)$

Now, the range of $\pi + \sin^{-1}(3x - 4x^3)$

is $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Thus, $\frac{\pi}{2} \le 3\sin^{-1}x \le \frac{3\pi}{2}$

$\Rightarrow \quad \frac{\pi}{6} \le \sin^{-1}x \le \frac{\pi}{2}$

$\Rightarrow \quad \sin\left(\frac{\pi}{6}\right) \le x \le \sin\left(\frac{\pi}{2}\right)$

$\Rightarrow \quad \frac{1}{2} \le x \le 1$

Hence, the solution set is $\left[\frac{1}{2}, 1\right]$

8. Given, $2\tan^{-1}x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Now, the range of $\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Thus, $\frac{\pi}{2} \le 2\tan^{-1}x \le \frac{3\pi}{2}$

$\Rightarrow \quad \frac{\pi}{4} \le \tan^{-1}x \le \frac{3\pi}{4}$

$\Rightarrow \quad \tan\left(\frac{\pi}{4}\right) \le x < \tan\left(\frac{\pi}{2}\right)$

and $\quad \tan\left(\frac{\pi}{2}\right) < x \le \tan\left(\frac{3\pi}{4}\right)$

$\Rightarrow \quad 1 \le x < \infty$ and $-\infty < x \le -1$

Therefore, the solution set is

$x \in (-\infty, -1] \cup [1, \infty)$

9. We have

$$\cos\left(\frac{\pi}{6} + \cos^{-1}\left(-\frac{1}{2}\right)\right)$$

$$= \cos\left(\frac{\pi}{6} + \pi - \cos^{-1}\left(\frac{1}{2}\right)\right)$$

$$= \cos\left(\frac{\pi}{6} + \pi - \frac{\pi}{3}\right)$$

$$= \cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$$

$$= \cos\left(\frac{5\pi}{6}\right)$$

$$= \cos\left(\pi - \frac{\pi}{6}\right)$$

$$= -\cos\left(\frac{\pi}{6}\right)$$

$$= -\frac{1}{2}$$

10. We have

$$\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2}-1)))$$

$$= \cos^{-1}\left(\cos\left(2\cos^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2(\sqrt{2}-1)^2}{(4-2\sqrt{2})} - 1\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2(3-2\sqrt{2}) - 4 + 2\sqrt{2}}{(4-2\sqrt{2})}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2-2\sqrt{2}}{(4-2\sqrt{2})}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2(1-\sqrt{2})}{2\sqrt{2}(\sqrt{2}-1)}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\pi - \frac{\pi}{4}\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)$$

$$= \left(\frac{3\pi}{4}\right)$$

11. We have

$$\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)$$

$$= \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r(r+1)}\right)$$

$$= \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{(r+1)-\mathrm{r}}{1+r(r+1)}\right)$$

$$= \sum_{r=0}^{\infty} (\tan^{-1}(r+1) - \tan^{-1}(r))$$

$$= \sum_{r=0}^{n} (\tan^{-1}(r+1) - \tan^{-1}(r)), n \to \infty$$

$= (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(3) - \tan^{-1}(2))$
$\quad + (\tan^{-1}(4) - \tan^{-1}(3) + (\tan^{-1}(5) - \tan^{-1}(4))$
$\quad + \ldots + ) \tan^{-1}(n+1) - \tan^{-1}(n))$

$= (\tan^{-1}(n+1) - \tan^{-1}(1)), n \to \infty$

$$= \tan^{-1}\left(\frac{(n+1)-1}{1+(n+1)\cdot 1}\right), n \to \infty$$

$$= \tan^{-1}\left(\frac{n}{n+2}\right), n \to \infty$$

$= \tan^{-1}(1)$, when $n \to \infty$

$= \dfrac{\pi}{4}$

12. We have

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right)$$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r-1}}{1+2^{r}\cdot 2^{r-1}}\right)$$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r}-2^{r-1}}{1+2^{r}\cdot 2^{r-1}}\right)$$

$$= \sum_{r=1}^{n} (\tan^{-1}(2^{r}) - \tan^{-1}(2^{r-1}))$$

$= (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(2^2) - \tan^{-1}(2))$
$+ (\tan^{-1}(2^3) - \tan^{-1}(2^2)) + \ldots$
$+ (\tan^{-1}(2^n) - \tan^{-1}(2^{n-1}))$

$= \tan^{-1}(2^n) - \tan^{-1}(1)$

$= \tan^{-1}(2^n) - \dfrac{\pi}{4}$

13. We have

$$\sum_{r=1}^{n} \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r}\cdot\sqrt{r-1}}\right)$$

$$= \sum_{r=1}^{n} (\tan^{-1}(\sqrt{r}) - \tan^{-1}(\sqrt{r-1}))$$

$= (\tan^{-1}(1) - \tan^{-1}(0) + (\tan^{-1}(2) - \tan^{-1}(1))$
$+ (\tan^{-1}(3) - \tan^{-1}(2)) + (\tan^{-1}(4) - \tan^{-1}(3))$
$+ \ldots + (\tan^{-1}(n) - \tan^{-1}(n-1))$

$= \tan^{-1}(n)$

14. We have

$$\tan^{-1}\left(\frac{a_1 x - y}{a_1 y + x}\right) + \tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\frac{a_3 - \mathrm{a}_2}{1 + a_3 a_2}\right) + \ldots + \tan^{-1}\left(\frac{a_n - a_{n-1}}{1 + a_n a_{n-1}}\right) + \tan^{-1}\left(\frac{1}{a_n}\right)$$

$$= \tan^{-1}\left(\frac{a_1 - \frac{y}{x}}{1 + a_1 \cdot \frac{y}{x}}\right) + \tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1 + a_3 a_2}\right) + \ldots + \tan^{-1}\left(\frac{a_n - a_{n-1}}{1 + a_n a_{n-1}}\right) + \tan^{-1}\left(\frac{1}{a_n}\right)$$

$$= \tan^{-1}(a_1) - \tan^{-1}\left(\frac{y}{x}\right) + \tan^{-1}(a_2) - \tan^{-1}(a_1)$$

$\tan^{-1}(a_3) - \tan^{-1}(a_2) + \ldots + \tan^{-1}(a_n)$
$- \tan^{-1}(a_{n-1}) + \cot^{-1}(a_n)$

$$= \tan^{-1}(a_n) + \cot^{-1}(a_n) - \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \cot^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{x}{y}\right)$$

15. We have $(\tan^{-1}x)^2 + (\cot^{-1}\mathrm{x})^2 = \dfrac{\pi^2}{8}$

$\Rightarrow \quad (\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x \cdot \cot^{-1}x = \dfrac{\pi^2}{8}$

$\Rightarrow \quad \left(\dfrac{\pi}{2}\right)^2 - 2\tan^{-1}x \cdot \cot^{-1}x = \dfrac{\pi^2}{8}$

$\Rightarrow \quad \dfrac{\pi^2}{4} - 2\tan^{-1}x \cdot \cot^{-1}x = \dfrac{\pi^2}{8}$

$\Rightarrow \quad 2\tan^{-1}x \cdot \cot^{-1}x = \dfrac{\pi^2}{8}$

$\Rightarrow \quad \tan^{-1}x \cdot \cot^{-1}x = \dfrac{\pi^2}{16}$

$\Rightarrow \quad a\left(\dfrac{\pi}{2} - a\right) = \dfrac{\pi^2}{16}$, where $a = \tan^{-1}x$

$\Rightarrow \quad 16a\left(\dfrac{\pi}{2} - a\right) = \pi^2$

$\Rightarrow \quad 16a^2 - 8a\pi + \pi^2 = 0$

$\Rightarrow \quad (4a - \pi)^2 = 0$

$\Rightarrow \quad (4a - \pi) = 0$

$\Rightarrow \quad a = \dfrac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}x = \dfrac{\pi}{4}$

$\Rightarrow \quad x = 1$

Hence, the solution is $x = 1$

16. Given,

$f(x) = (\sec^{-1}x)^2 + (\mathrm{cosec}^{-1}x)^2$

$= (\sec^{-1}x + \mathrm{cosec}^{-1}x)^2 - 2\sec^{-1}x \cdot \mathrm{cosec}^{-1}x$

$= \left(\dfrac{\pi}{2}\right)^2 - 2\sec^{-1}x\left(\dfrac{\pi}{2} - \sec^{-1}x\right)$

$= \dfrac{\pi^2}{4} - \pi \cdot \sec^{-1}x + 2(\sec^{-1}x)^2$

$= 2\left((\sec^{-1}x)^2 - \dfrac{\pi}{2} \cdot \sec^{-1}x + \dfrac{\pi^2}{8}\right)$

$$= 2\left((\sec^{-1}x)^2 - 2\cdot\sec^{-1}x\cdot\frac{\pi}{4} + \left(\frac{\pi}{4}\right)^2 + \frac{\pi^2}{8} - \frac{\pi^2}{16}\right)$$

$$= 2\left(\sec^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{8}$$

Maximum value of $f(x)$ is $\frac{\pi^2}{4}$ at $x = 1$.

17. Given,

$$f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$$
$$= (\sin^{-1} x + \cos^{-1} x)((\sin^{-1} x)^2 + (\cos^{-1} x)^2 - \sin^{-1} x \cos^{-1} x)$$
$$= (\sin^{-1} x + \cos^{-1} x)((\sin^{-1} x + \cos^{-1} x)^2 - 3\sin^{-1} x \cos^{-1} x)$$
$$= \frac{\pi}{2}\left(\left(\frac{\pi}{2}\right)^2 - 3\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)\right)$$
$$= \frac{\pi}{2}\left(\left(\frac{\pi}{2}\right)^2 - 3a\left(\frac{\pi}{2} - a\right)\right)$$
$$= \frac{\pi}{2}\left(\frac{\pi^2}{4} - \frac{3a\pi}{2} + 3a^2\right)$$
$$= \frac{\pi}{2}\left(3\left(a^2 - \frac{a\pi}{2}\right) + \frac{\pi^2}{4}\right)$$
$$= \frac{\pi}{2}\left(3\left(a^2 - 2\cdot\frac{\pi}{4}\cdot a + \left(\frac{\pi}{6}\right)^2\right) + \frac{\pi^2}{4} - \frac{3\pi^2}{16}\right)$$
$$= \frac{\pi}{2}\left(3\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{4} - \frac{3\pi^2}{16}\right)$$
$$= \frac{\pi}{2}\left(3\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right)$$
$$= \left(\frac{3\pi}{2}\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^3}{32}\right)$$
$$= \frac{3\pi}{2}\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^3}{32} \ge \frac{\pi^3}{32}$$

Hence, the minimum value of $f(x)$ is $\frac{\pi^3}{32}$

18. Given,

$$[\cot^{-1} x] + [\cos^{-1} x] = 0$$

It is possible only when

$\Rightarrow$ $\cot^{-1} x = 0$ and $\cos^{-1} x = 0$

$\Rightarrow$ $0 \le \cot^{-1} x < 1$ and $0 \le \cos^{-1} x < 1$

$\Rightarrow$ $x \in (\cot 1, \infty)$ and $x \in (\cos 1, 1]$

Thus, the solution is $x$ $(\cot 1, 1]$

19. Given,

$$[\sin^{-1} x] + [\cos^{-1} x] = 0$$

It is possible only when

$\Rightarrow$ $[\sin^{-1} x] = 0$ and $[\cos^{-1} x] = 0$

$\Rightarrow$ $0 \le \sin^{-1} x < 1$ and $0 \le \cos^{-1} x < 1$

$\Rightarrow$ $x \in [0, \sin 1)$ and $x \in (\cos 1, 1]$

Thus, the solution is $x \in (\cos 1, \sin 1)$

20. Given,

$$[\tan^{-1} x] + [\cot^{-1} x] = 2$$

The range of $[\tan^{-1} x]$ is $\{-2, -1, 0, 1\}$ and $[\cot^{-1} x]$ is $\{0, 1, 2, 3\}$

**Case I:** When $[\cot^{-1} x] = 1$ and $[\tan^{-1} x] = 1$

$\Rightarrow$ $1 \le \cot^{-1} x < 2$ and $1 \le \tan^{-1} x < 2$

$\Rightarrow$ $x \in (\cot 2, \cot 1]$ and $x \in [\tan 1, \tan 2)$

$\Rightarrow$ $x \in \varphi$ ($\because \cot 1 < \tan 1$)

**Case II:** When $[\cot^{-1} x] = 3$ and $[\tan^{-1} x] = -1$

$\Rightarrow$ $3 \le \cot^{-1} x < 4$ and $-1 \le \tan^{-1} x < 0$

$\Rightarrow$ $x \in (\cot 4, \cot 3]$ and $x \in [-\tan 1, 0)$

$\Rightarrow$ $x \in \varphi$, ($\because \cot 3 < -\tan 1$)

**Case III:** When $[\cot^{-1} x] = 2$ and $[\tan^{-1} x] = 0$

$\Rightarrow$ $2 \le \cot^{-1} x < 3$ and $0 \le \tan^{-1} x < 1$

$\Rightarrow$ $x \in (3, \cot 2]$ and $x \in [0 \tan 1)$

$\Rightarrow$ $x \in \varphi$, ($\because \cot 2 < \tan 1$)

Thus, there is no such value of $x$, where the equation is valid.

21. Given,

$$[\sin^{-1} (\cos^{-1}(\sin^{-1}(\tan^{-1} x)))] = 1$$

$\Rightarrow$ $0 \le \sin^{-1} (\cos^{-1}(\sin^{-1}(\tan^{-1} x))) < 1$

$\Rightarrow$ $0 \le (\cos^{-1}(\sin^{-1}(\tan^{-1} x))) < \sin 1$

$\Rightarrow$ $\cos (\sin 1) < (\sin^{-1}(\tan^{-1} x)) \le 1$

$\Rightarrow$ $\sin (\cos (\sin 1)) < (\tan^{-1} x) \le \sin 1$

$\Rightarrow$ $\tan (\sin (\cos (\sin 1))) < x \le \tan (\sin 1)$

$\Rightarrow$ $x \le (\tan (\sin (\cos (\sin 1))), \tan (\sin 1)]$

22. Given,

$$f(x) = \sin^{-1} x + \tan^{-1} x + \cot^{-1} x$$

It is defined for $-1 \le x \le 1$

Thus,

$$f(-1) = \sin^{-1} (-1) + \tan^{-1} (-1) + \cot^{-1} (-1)$$
$$= -\frac{\pi}{2} - \frac{\pi}{4} + \pi - \frac{\pi}{4}$$
$$= -\pi + \pi = 0$$

and

$$f(1) = \sin^{-1}(1) + \tan^{-1}(1) + \cot^{-1}(1)$$
$$= \frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{4}$$
$$= \pi$$

Thus, range of $f(x)$ is $= [f(-1), f(1)] = [0, \pi]$

23. Given,

$$f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$$
$$= \frac{\pi}{2} + \tan^{-1}x$$

As we know that, range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, the range of $f(x)$ is $(0, \pi)$

24. Given, $f(x) = \sin^{-1} x + \sec^{-1} x + \tan^{-1} x$

The domain of $f(x)$ is $\{-1, 1\}$

Now, $f(1) = \sin^{-1}(1) + \sec^{-1}(1) + \tan^{-1}(1)$

$$= \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$$

and $f(-1) = \sin^{-1}(-1) + \sec^{-1}(-1) + \tan^{-1}(-1)$

$$= -\frac{\pi}{2} + \pi - 0 - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Thus, the range of $f(x)$ is $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

25. Given, $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x+3x}{1-2x\cdot 3x}\right) = \frac{\pi}{4}$

$\Rightarrow \quad \left(\frac{5x}{1-6x^2}\right) = 1$

$\Rightarrow \quad 6x^2 + 5x - 1 = 0$

$\Rightarrow \quad 6x^2 + 6x - x - 1 = 0$

$\Rightarrow \quad (6x - 1)(x + 1) = 0$

$\Rightarrow \quad x = -1, \frac{1}{6}$

Hence, the solution set is $\left\{-1, \frac{1}{6}\right\}$

26. Given, $\cos^{-1}x = \cot^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \quad \cos^{-1}x = \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \quad \cos^{-1}x = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$

$\Rightarrow \quad \cos^{-1}x = \tan^{-1}\left(\frac{25}{25}\right) = \tan^{-1}(1)$

$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{4}$

$\Rightarrow \quad x = \frac{1}{\sqrt{2}}$

Hence, the solution is $x = \frac{1}{\sqrt{2}}$

27. Given, $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$

$\Rightarrow \quad \frac{n}{\pi} < \cot\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad \frac{n}{\pi} < \sqrt{3}$

$\Rightarrow \quad n < \frac{\pi}{\sqrt{3}}$

$\Rightarrow \quad n < \frac{\pi}{\sqrt{3}} = 3.14 \times 1.732 = 5.43848$

Hence, the max. value of $n$ is 5.

28. Given $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$

$$\sin^{-1}\left(\frac{5}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{12}{x}\right)$$

$\Rightarrow \quad \sin^{-1}\left(\frac{5}{x}\right) = \cos^{-1}\left(\frac{12}{x}\right)$

$\Rightarrow \quad \sin^{-1}\left(\frac{5}{x}\right) = \sin^{-1}\sqrt{1-\left(\frac{12}{x}\right)^2}$

$\Rightarrow \quad \left(\frac{5}{x}\right)^2 = 1 - \left(\frac{12}{x}\right)^2$

$\Rightarrow \quad \left(\frac{169}{x^2}\right) = 1$

$\Rightarrow \quad x^2 = 169$

$\Rightarrow \quad x = \pm 13$

29. We have

$$x = \sin^{-1}(b^6 + 1) + \cos^{-1}(b^4 + 1) + \tan^{-1}(a^2 + 1)$$

It is possible only when $a = 0$

Thus, $x = \sin^{-1}(1) + \cos^{-1}(1) + \tan^{-1}(1)$

$$= \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$$

Therefore, $\sin\left(x + \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)$

$$= \sin\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{4} + \frac{\pi}{4}\right)$$

$$= \sin\pi + \cos\pi$$

$$= 0 - 1 = -1$$

30. Given,

$$\sin^{-1}x + \tan^{-1}x = 2k + 1$$

Let $g(x) = \sin^{-1}x + \tan^{-1}x$

Domain of $g = [-1, 1]$

Now, $g(-1) = \sin^{-1}(-1) + \tan^{-1}(-1)$

$$= -\frac{\pi}{2} - \frac{\pi}{4} = -\frac{3\pi}{4}$$

$$g(1) = \sin^{-1}(1) + \tan^{-1}(1)$$

$$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

Range of $g = \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$

Thus, $-\frac{3\pi}{4} \le 2k + 1 \le \frac{3\pi}{4}$

$\Rightarrow \quad -\frac{3\pi}{4} - 1 \le 2k \le \frac{3\pi}{4} - 1$

$\Rightarrow \quad -\frac{3\times 3.14}{4} - 1 \le 2k \le \frac{3\times 3.14}{4} - 1$

$\Rightarrow \quad -2.35 - 1 \le 2k \le 2.35 - 1$

$\Rightarrow \quad -3.35 \le 2k \le 1.35$

$\Rightarrow \quad -\frac{3.35}{2} \le k \le \frac{1.35}{2}$

$\Rightarrow \quad -1.67 \le k \le 6.7$

Thus, the integral values of $k$ are $-1$ and 0

31. Given,

$$\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$$

$\Rightarrow \quad \sin^{-1}x = \frac{\pi}{2} - \sin^{-1}y$

$\Rightarrow \quad \sin^{-1}x = \cos^{-1}y$

$\Rightarrow \quad \sin^{-1}x = \sin^{-1}\sqrt{1-y^2}$

$\Rightarrow \quad x = \sqrt{1-y^2}$

$\Rightarrow \quad x^2 = 1 - y^2$

$\Rightarrow \quad x^2 = y^2 = 1$

Now,

$$\left(\frac{1+x^4+y^4}{x^2-x^2y^2+y^2}\right) = \left(\frac{1+(x^2+y^2)^2-2x^2y^2}{(x^2+y^2-x^2y^2)}\right)$$
$$= \left(\frac{1+1-2x^2y^2}{(1-x^2y^2)}\right)$$
$$= \frac{2(1-x^2y^2)}{(1-x^2y^2)}$$
$$= 2$$

32. Given, $\cos^{-1}x + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$

$\Rightarrow \quad \cos^{-1}2x + \cos^{-1}3x = \pi - \cos^{-1}x$

$\Rightarrow \quad \cos^{-1}2x + \cos^{-1}3x = \cos^{-1}(-x)$

$\Rightarrow \quad \cos^{-1}(2x\cdot 3x - \sqrt{1-4x^2}\sqrt{1-9x^2}) = \cos^{-1}(-x)$

$\Rightarrow \quad (6x^2 - \sqrt{1-4x^2}\sqrt{1-9x^2}) = -x$

$\Rightarrow \quad (6x^2+x)^2 = (\sqrt{1-4x^2}\sqrt{1-9x^2})^2$

$\Rightarrow \quad 36x^4 + 12x^3 + x^2 = 1 - 4x^2 - 9x^2 + 36x^4$

$\Rightarrow \quad 12x^3 + 14x^2 = 1$

Also, $ax^3 + bx^2 + cx = 1$

Thus, $a = 12$, $b = 14$ and $c = 0$

Hence, the value of $a^2 + b^2 + c^2 + 10$

$= 144 + 196 + 10$

$= 350$

33. The domain of $\sin^{-1}x + \tan^{-1}x$ is $[-1, 1]$

Now, $f(1) = \sin^{-1}(1)\tan^{-1}(1) + 1 + 4 + 5$

$= \frac{3\pi}{4} + 10$

and $f(-1) = \sin^{-1}(-1) + \tan^{-1}(-1) + 1 - 4 + 5$

$= -\frac{3\pi}{4} + 2$

Therefore, $a + b + 5 = 10 + 2 + 5 = 17$.

34. Clearly, $x = \frac{\pi}{2}$.

Thus, $\sin x = 1$

Ans. (a)

35. As we know that, domain of $\sin^{-1}x$ is $[-1, 1]$

Therefore, $-1 \le \log_3\left(\frac{x}{3}\right) \le 1$

$\Rightarrow \quad 3^{-1} \le \left(\frac{x}{3}\right) \le 3$

$\Rightarrow \quad \frac{1}{3} \le \left(\frac{x}{3}\right) \le 3$

$\Rightarrow \quad 1 \le x \le 9$

Thus the domain of $f(x)$ is $[1, 9]$

36. As we know that, the range of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore, $-\frac{\pi}{2} \le 2\sin^{-1}a \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{4} \le \sin^{-1}a \le \frac{\pi}{4}$

$\Rightarrow \quad -\frac{1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}}$

$\Rightarrow \quad |a| \le \frac{1}{\sqrt{2}}$

Ans. (c)

37. Now, $\sin^{-1}(x-3)$ is defined for

$-1 \le (x-3) \le 1$

$\Rightarrow \quad 2 \le x \le 4$

Also, the function $\frac{1}{\sqrt{9-x^2}}$ is defined for

$\Rightarrow \quad 9 - x^2 > 0$

$\Rightarrow \quad x^2 - 9 < 0$

$\Rightarrow \quad (x+3)(x-3) < 0$

$\Rightarrow \quad -3 < x < 3$

Thus, the solution is $x \in [2, 3)$

Hence, the domain is $[2, 3)$

Ans. (b)

38. Since the $f$ is ont, so the range of $f$ is co-domain.

i.e., range $= B$

Clearly, range of $f$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, $B = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Ans. (c)

39. Given, $\cos^{-1}x - \cos^{-1}\left(\frac{y}{2}\right) = \alpha$

$\Rightarrow \quad \cos^{-1}\left(x\cdot\frac{y}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \alpha$

$\Rightarrow \quad \left(\frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \cos\alpha$

$\Rightarrow \quad \left(\cos\alpha - \frac{xy}{2}\right)^2 = \left(\sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right)^2$

$\Rightarrow \quad \left(\cos^2\alpha - xy\cos\alpha + \left(\frac{xy}{2}\right)^2\right)$

$\Rightarrow \quad = 1 - x^2 - \frac{y^2}{4} + \frac{x^2y^2}{4}$

$\Rightarrow \quad x^2 - xy\cos\alpha + \frac{y^2}{4} = 1 - \cos^2\alpha$

$\Rightarrow \quad x^2 - xy\cos\alpha + \frac{y^2}{4} = \sin^2\alpha$

$\Rightarrow \quad 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

Ans. (d)

***Note:*** No question asked in 2006.

40. Given, $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$

$\Rightarrow \quad x = 3$

Ans. (d)

41. Given, $\cot\left(\operatorname{cosec}^{-1}\left(\frac{5}{3}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$

$= \cot\left(\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$

$= \cot\left(\cot^{-1}\left(\frac{6}{17}\right)\right)$

$= \frac{6}{17}$

Ans. (b)

***Note:*** No questions asked in between 2009 to 2014.

## Level III

1. Let $D_1: -1 \le \left(\frac{|x|-2}{3}\right) \le 1$

$\Rightarrow \quad -3 \le (|x| - 2) \le 3$

$\Rightarrow \quad -1 \le |x| \le 5$

$\Rightarrow \quad -5 \le x \le 5$

and $D_2$: $-1 \le \left(\frac{1-|x|}{4}\right) \le 1$

$\Rightarrow \quad -4 \le 1 - |x| \le 4$

$\Rightarrow \quad -5 \le -|x| \le 3$

$\Rightarrow \quad -3 \le |x| \le 5$

$\Rightarrow \quad -5 \le x \le 5$

Thus, $D_f = D_1 \cap D_2 = [-5, 5]$

2. The function $f$ is defined for

$\Rightarrow \quad 5\pi\sin^{-1}x - 6(\sin^{-1}x)^2 \le 0$

$\Rightarrow \quad 6(\sin^{-1}x)^2 - 5\pi\sin^{-1}x \le 0$

$\Rightarrow \quad (\sin^{-1}x)(6(\sin^{-1}x) - 5\pi) \le 0$

$\Rightarrow \quad 0 \le \sin^{-1}x \le \frac{5\pi}{6}$

$\Rightarrow \quad 0 \le x \le \frac{1}{2}$

Also, $\sin^{-1}x$ is defined for $[-1, 1]$

Thus, $D_f = \left[0, \frac{1}{2}\right]$

3. $f$ is defined for

$-1 \le \log_2(x^2 + 3x + 4) \le 1$

$\Rightarrow \quad 2^{-1} \le (x^2 + 3x + 4) \le 2$

when $(x^2 + 3x + 4) \le 2$

$\Rightarrow \quad (x^2 + 3x + 2) \le 0$

$\Rightarrow \quad (x + 1)(x + 2) \le 0$

$\Rightarrow \quad -2 \le x \le -1$

when $x^2 + 3x + 4 \ge \frac{1}{2}$

$\Rightarrow \quad 2x^2 + 6x + 7 \ge 0$

$\Rightarrow \quad x \in R$

Thus, $D_f = [-2, -1]$

4. Given, $\cos^{-1}x + \cos^{-1}x^2 = 2\pi$

It is possible only when

$\cos^{-1}x = \pi$ and $\cos^{-1}x^2 = \pi$

$x = \cos\pi = -1$ and $x^2 = -1$ and $x^2 = -1$

Thus, no such value of $x$ is exist.

5. It is true only when $\frac{1}{x^2-1} = x^2 - 1$

$\Rightarrow \quad (x^2 - 1)^2 = 1$

$\Rightarrow \quad (x^2 - 1) = \pm 1$

$\Rightarrow \quad x^2 = 1 \pm 1 = 2, 0$

$\Rightarrow \quad x = 0, \pm\sqrt{2}$

6. Given, $\cot^{-1}\left(\frac{x^2-1}{2x}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \left(\frac{2x}{1-x^2}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\Rightarrow \quad \frac{2x}{\sqrt{3}} = 1 - x^2$

$\Rightarrow \quad x^2 + \frac{2}{\sqrt{3}}x - 1 = 0$

$\Rightarrow \quad \left(x + \frac{1}{\sqrt{3}}\right)^2 = 1 + \frac{1}{3} = \frac{4}{3}$

$\Rightarrow \quad \left(x+\frac{1}{\sqrt{3}}\right)=\pm\frac{2}{\sqrt{3}}$

$\Rightarrow \quad x=-\frac{1}{\sqrt{3}}\pm\frac{2}{\sqrt{3}}$

$\Rightarrow \quad x=\frac{1}{\sqrt{3}},-\sqrt{3}$

Hence, the solution set is $\left\{-\sqrt{3},\frac{1}{\sqrt{3}}\right\}$

7. Given, $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(\pi-\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3$

$\Rightarrow \quad \left(\pi-\frac{2x^2+4}{x^2+1}\right)<\pi-3$

$\Rightarrow \quad \frac{2x^2+4}{x^2+1}>3$

$\Rightarrow \quad \frac{2x^2+4}{x^2+1}-3>0$

$\Rightarrow \quad \frac{2x^2+4-3x^2-3}{x^2+1}>0$

$\Rightarrow \quad \frac{-x^2+1}{x^2+1}>0$

$\Rightarrow \quad \frac{x^2-1}{x^2+1}<0$

$\Rightarrow \quad \frac{(x-1)(x+1)}{x^2+1}<0$

$\Rightarrow \quad -1<x<1$

8. We have

$\sin^{-1}(\sin[\pi^{3/2}])+\cos^{-1}(\cos[\pi^{3/2}])$

$=\sin^{-1}(\sin[\pi\sqrt{\pi}])+\cos^{-1}(\cos[\pi\sqrt{\pi}])$

$=\sin^{-1}(\sin[5.56])+\cos^{-1}(\cos[5.56])$

$=\sin^{-1}(\sin 5)+\cos^{-1}(\cos 5)$

$=\sin^{-1}(\sin(5-2p))+\cos^{-1}(\cos(2\pi-5))$

$=(5-2\pi)+(2\pi-5)=0$

Thus, the given expression reduces to

$x^2-4x<0$

$x(x-4)<0$

$0<x<4$

9. Now, $\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(x+\frac{3}{2}\right)\right)\right)\right)$

$=\cos\left(\tan^{-1}\left(\cot\left(\cot^{-1}\left(\frac{\sqrt{1-\left(x+\frac{3}{2}\right)^2}}{\left(x+\frac{3}{2}\right)}\right)\right)\right)\right)$

$=\cos\left(\tan^{-1}\left(\frac{\sqrt{1-\left(x+\frac{3}{2}\right)^2}}{\left(x+\frac{3}{2}\right)}\right)\right)$

$=\cos\left(\cos^{-1}\left(\frac{x+\frac{3}{2}}{1}\right)\right)$

$=\left(x+\frac{3}{2}\right)$

Thus, the given equation reduces to

$\left(x+\frac{3}{2}\right)+\tan(\sec^{-1}x)=0$

$\left(x+\frac{3}{2}\right)+\tan\left(\tan^{-1}\left(\frac{\sqrt{x^2-1}}{1}\right)\right)=0$

$\Rightarrow \quad \left(x+\frac{3}{2}\right)+\left(\frac{\sqrt{x^2-1}}{1}\right)=0$

$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2=(-\sqrt{x^2-1})^2$

$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2=x^2-1$

$\Rightarrow \quad x^2+3x+\frac{9}{4}=x^2-1$

$\Rightarrow \quad 3x+\frac{9}{4}=-1$

$\Rightarrow \quad 3x=-1-\frac{9}{4}=-\frac{13}{4}$

$\Rightarrow \quad x=-\frac{13}{12}$

Hence, the solution is $x=-\frac{13}{12}$

10. We have

$\Rightarrow \quad \tan\left(\tan^{-1}\left(\frac{\frac{x}{10}+\frac{1}{x+1}}{1-\frac{x}{10}\cdot\frac{1}{x+1}}\right)\right)=1$

$\Rightarrow \quad \left(\frac{\frac{x}{10}+\frac{1}{x+1}}{1-\frac{x}{10}\cdot\frac{1}{x+1}}\right)=1$

$\Rightarrow \quad \left(\frac{x}{10}+\frac{1}{x+1}\right)=\left(1-\frac{x}{10}\cdot\frac{1}{x+1}\right)$

$\Rightarrow \quad x(x+1)+10=10(x+1)-x$

$\Rightarrow \quad x^2+x+10=10x+10-x$

$\Rightarrow \quad x^2-8x=0$

$\Rightarrow \quad x=0,8$

Hence, the solution set is $\{0, 8\}$.

11. Given

$$\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$$
$$= 2(\tan^{-1} 1 + \tan^{-1} x)$$
$$= 2\left(\frac{\pi}{4} + \tan^{-1} x\right)$$
$$= \frac{\pi}{2} + 2\tan^{-1} x$$

Also, $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$= \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$
$$= \frac{\pi}{2} - 2\tan^{-1} x$$

Thus,

$$\alpha + \beta = \frac{\pi}{2} + 2\tan^{-1} x + \frac{\pi}{2} - 2\tan^{-1} x$$
$$= \pi$$

12. As we know that

$$-\frac{\pi}{2} \le \sin^{-1}(2x-3) \le \frac{\pi}{2}$$
$$\Rightarrow -\frac{2\pi}{2} \le 2\sin^{-1}(2x-3) \le \frac{2\pi}{2}$$
$$\Rightarrow -\pi \le f(x) \le \pi$$

Thus, $R_f = [-\pi, \pi]$

13. We have

$$-\frac{\pi}{2} \le \sin^{-1}(2x-1) \le \frac{\pi}{2}$$
$$\Rightarrow -\pi \le 2\sin^{-1}(2x-1) \le \pi$$
$$\Rightarrow -\pi - \frac{\pi}{4} \le 2\sin^{-1}(2x-1) - \frac{\pi}{4} \le \pi - \frac{\pi}{4}$$
$$\Rightarrow -\frac{5\pi}{4} \le f(x) \le \frac{3\pi}{4}$$

Thus, $R_f = \left[-\frac{5\pi}{4}, \frac{3\pi}{4}\right]$

14. We have

$$f(x) = 2\cos^{-1}(-x^2) - \pi$$
$$= 2(\pi - \cos^{-1}(x^2)) - \pi$$
$$= \pi - 2\cos^{-1}(x^2)$$

As we know that, $0 \le \cos^{-1}(x^2) \le \pi$

$$\Rightarrow 0 \le 2\cos^{-1}(x^2) \le 2\pi$$
$$\Rightarrow -2\pi \le -2\cos^{-1}(x^2) \le 0$$
$$\Rightarrow -2\pi + \pi \le \pi - 2\cos^{-1}(x^2) \le \pi$$
$$\Rightarrow -\pi \le f(x) \le \pi$$

Thus, $r_f = [-\pi, \pi]$

15. We have

$$-\infty < 1 - x^2 \le 1$$
$$\Rightarrow \tan^{-1}(-\infty) < \tan^{-1}(1-x^2) \le \tan^{-1}(1)$$
$$\Rightarrow -\frac{\pi}{2} < \tan^{-1}(1-x^2) \le \frac{\pi}{4}$$
$$\Rightarrow -\frac{\pi}{4} < \frac{1}{2}\tan^{-1}(1-x^2) \le \frac{\pi}{8}$$
$$\Rightarrow -\frac{\pi}{4} - \frac{\pi}{4} < \frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4} \le \frac{\pi}{8} - \frac{\pi}{4}$$
$$\Rightarrow -\frac{\pi}{2} < \frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4} \le -\frac{\pi}{8}$$

Hence, the range of the function

$$= \left[-\frac{\pi}{2}, -\frac{\pi}{8}\right]$$

16. We have

$$2x - x^2 = -(x^2 - 2x + 1) + 1$$
$$= 1 - (x-1)^2$$

Thus, $-\infty < 1 - (x-1)^2 \le 1$

$$\Rightarrow \cot^{-1}(-\infty) < \cot^{-1}(1-(x-1)^2) \le \cot^{-1}(1)$$
$$\Rightarrow 0 < \cot^{-1}(1-(x-1)^2) \le \frac{\pi}{4}$$

Hence, the range of the function $= \left(0, \frac{\pi}{4}\right]$.

17. The function $f$ is defined for $-1 \le x \le 1$

We have

$$f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$$
$$= \frac{\pi}{2} + \tan^{-1} x$$

Thus, $R_f = [f(-1), f(1)]$

$$= \left[\frac{\pi}{2} - \frac{\pi}{4}, \frac{\pi}{2} + \frac{\pi}{4}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

18. The function $f$ is defined for $x = \pm 1$

Now, $f(1) = \sin^{-1}(1) + \sec^{-1}(1) + \tan^{-1}(1)$

$$= \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$$

Also, $f(-1) = \sin^{-1}(-1) + \sec^{-1}(-1) + \tan^{-1}(-1)$

$$= -\frac{\pi}{2} + \pi - 0 - \frac{\pi}{4}$$
$$= \frac{3\pi}{4}$$

Thus, $R_f = \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

19. We have

$$f(x) = 2(\tan^{-1} x + \cot^{-1} x) + \cot^{-1} x + \frac{\pi}{4}$$
$$= 2 \cdot \frac{\pi}{2} + \cot^{-1} x + \frac{\pi}{4}$$
$$= \cot^{-1} x + \frac{5\pi}{4}$$

Also, $0 < \cot^{-1} x < \pi$

$$\Rightarrow \frac{5\pi}{4} < \cot^{-1} x + \frac{5\pi}{4} < \pi + \frac{5\pi}{4}$$

$$\Rightarrow \quad \frac{5\pi}{4} < f(x) < \frac{9\pi}{4}$$

Thus, $R_f = \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right)$

20. We have

sin (cot$^{-1}$(tan (cos$^{-1}$ $x$)))

$$= \sin\left(\cot^{-1}\left(\tan\left(\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right)\right)\right)$$

$$= \sin\left(\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\frac{x}{1}\right)\right)$$

$$= x$$

21. We have

sin (cosec$^{-1}$(cot(tan$^{-1}$ $x$)))

$$= \sin\left(\operatorname{cosec}^{-1}\left(\cot\left(\cot^{-1}\left(\frac{1}{x}\right)\right)\right)\right)$$

$$= \sin\left(\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)\right)$$

$$= \sin(\sin^{-1}(x))$$

$$= x$$

22. We have

$$\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) - \tan^{-1}(\tan 6) + \pi - \cot^{-1}(\cot 10)$$

$$= \sin^{-1}(\sin(5-2\pi)) + \cos^{-1}(\cos(4\pi - 10)) + \pi - \tan^{-1}(\tan(6-2\pi)) - \cot^{-1}(\cot(10-4\pi))$$

$$= (5-2\pi) + (4\pi - 10) + \pi + (6-2\pi) - (10-4\pi)$$

$$= 5\pi - 9$$

23. Given, $U = \cot^{-1}(\sqrt{\cos 2\theta}) - \tan^{-1}(\sqrt{\cos 2\theta})$

$$\Rightarrow \quad U = \tan^{-1}\left(\frac{1}{\sqrt{\cos 2\theta}}\right) - \tan^{-1}(\sqrt{\cos 2\theta})$$

$$\Rightarrow \quad U = \tan^{-1}\left(\frac{\frac{1}{\sqrt{\cos 2\theta}} - \sqrt{\cos 2\theta}}{1 + \frac{1}{\sqrt{\cos 2\theta}} \cdot \sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow \quad U = \tan^{-1}\left(\frac{1-\cos 2\theta}{2\sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow \quad U = \tan^{-1}\left(\frac{2\sin^2\theta}{2\sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow \quad U = \tan^{-1}\left(\frac{\sin^2\theta}{\sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow \quad U = \sin^{-1}\left(\frac{\sin^2\theta}{\sqrt{\sin^4\theta + 1 - 2\sin^2\theta}}\right)$$

$$\Rightarrow \quad \sin U = \sin\left(\sin^{-1}\left(\frac{\sin^2\theta}{\sqrt{\sin^4\theta + 1 - 2\sin^2\theta}}\right)\right)$$

$$\Rightarrow \quad \sin U = \left(\frac{\sin^2\theta}{\sqrt{\sin^4\theta + 1 - 2\sin^2\theta}}\right)$$

$$\Rightarrow \quad \sin U = \left(\frac{\sin^2\theta}{\sqrt{(\sin^2\theta - 1)^2}}\right) = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

Hence, the result.

24. Let $\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right) = \theta$

$$\cos(2\theta) = \frac{a}{b}$$

The given expression reduces to

$$\tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right)$$

$$= \frac{1+\tan\theta}{1-\tan\theta} + \frac{1-\tan\theta}{1+\tan\theta}$$

$$= \frac{(1+\tan\theta)^2 + (1-\tan\theta)^2}{(1-\tan^2\theta)}$$

$$= \frac{2(1+\tan^2\theta)}{(1-\tan^2\theta)}$$

$$= \frac{2}{\cos 2\theta}$$

$$= \frac{2b}{a}$$

25. RHS $= 2\tan^{-1}\left(\tan\left(\frac{x}{2}\right)\tan\left(\frac{y}{2}\right)\right)$

$$= \cos^{-1}\left(\frac{1-\tan^2\left(\frac{x}{2}\right)\tan^2\left(\frac{y}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)\tan^2\left(\frac{y}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{\cos^2\left(\frac{x}{2}\right)\cos^2\left(\frac{y}{2}\right) - \sin^2\left(\frac{x}{2}\right)\sin^2\left(\frac{y}{2}\right)}{\cos^2\left(\frac{x}{2}\right)\cos^2\left(\frac{y}{2}\right) + \sin^2\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{(1+\cos x)(1+\cos y) - (1-\cos x)(1-\cos y)}{(1+\cos x)(1+\cos y) + (1-\cos x)(1-\cos y)}\right)$$

$$= \cos^{-1}\left(\frac{(2\cos x + 2\cos y)}{(2 + 2\cos x\cos y)}\right)$$

$$= \cos^{-1}\left(\frac{(\cos x + \cos y)}{(1 + 1\cos x\cos y)}\right)$$

Hence, the result.

26. We have $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{x}{2}\right)\right)$

$$= \cos^{-1}\left(\frac{1-\left(\frac{a-b}{a+b}\right)\tan^2\left(\frac{x}{2}\right)}{1+\left(\frac{a-b}{a+b}\right)\tan^2\left(\frac{x}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{(a+b)-(a-b)\tan^2\left(\frac{x}{2}\right)}{(a+b)+(a-b)\tan^2\left(\frac{x}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\left(1-\tan^2\left(\frac{x}{2}\right)\right)+b\left(1+\tan^2\left(\frac{x}{2}\right)\right)}{a\left(1+\tan^2\left(\frac{x}{2}\right)\right)+b\left(1-\tan^2\left(\frac{x}{2}\right)\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{\left(1+\tan^2\left(\frac{x}{2}\right)\right)}\right)+b}{a+b\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{\left(1+\tan^2\left(\frac{x}{2}\right)\right)}\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)$$

27. Given,

$\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$

$\Rightarrow \tan^{-1} x + \tan^{-1} z = 2\tan^{-1} y$

$\Rightarrow \tan^{-1}\left(\frac{x+z}{1-xz}\right) = \tan^{-1}\left(\frac{2y}{1-y^2}\right)$

$\Rightarrow \left(\frac{x+z}{1-xz}\right) = \left(\frac{2y}{1-y^2}\right)$

$\Rightarrow (x+z)(1-y^2) = 2y(1-xz)$

$\Rightarrow y^2(x+z) + 2y(1-xz) = (x+z)$

28. We have

$$\sin^{-1}\left(\sin\frac{33\pi}{7}\right)+\cos^{-1}\left(\cos\frac{46\pi}{7}\right)$$
$$+\tan^{-1}\left(-\tan\frac{13\pi}{8}\right)+\cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$$
$$=\sin^{-1}\left(\sin\left(4\pi+\frac{5\pi}{7}\right)\right)+\cos^{-1}\left(\cos\left(6\pi+\frac{4\pi}{7}\right)\right)$$
$$+\tan^{-1}\left(-\tan\left(2\pi-\frac{3\pi}{8}\right)\right)+\cot^{-1}\left(-\cot\left(2\pi+\frac{3\pi}{8}\right)\right)$$
$$=\sin^{-1}\left(\sin\left(\frac{5\pi}{7}\right)\right)+\cos^{-1}\left(\cos\left(\frac{4\pi}{7}\right)\right)$$
$$+\tan^{-1}\left(\tan\left(\frac{3\pi}{8}\right)\right)+\cot^{-1}\left(-\cot\left(\frac{3\pi}{8}\right)\right)$$
$$=\sin^{-1}\left(\sin\left(\pi-\frac{2\pi}{7}\right)\right)+\cos^{-1}\left(\cos\left(\frac{4\pi}{7}\right)\right)$$
$$+\tan^{-1}\left(\tan\left(\frac{3\pi}{8}\right)\right)+\cot^{-1}\left(\cot\left(\frac{5\pi}{8}\right)\right)$$
$$=\frac{2\pi}{7}+\frac{4\pi}{7}+\frac{3\pi}{8}+\frac{5\pi}{8}$$
$$=\frac{6\pi}{7}+\pi$$
$$=\frac{13\pi}{7}$$

29. Given equations are

$\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$, …(i)

$\cos^{-1}x-\cos^{-1}y=\frac{\pi}{3}$ …(ii)

(i) reduces to

$\Rightarrow \frac{\pi}{2}-\cos^{-1}x+\frac{\pi}{2}-\cos^{-1}y=\frac{2\pi}{3}$

$\pi-(\cos^{-1}x+\cos^{-1}y)=\frac{2\pi}{3}$

$\Rightarrow (\cos^{-1}x+\cos^{-1}y)=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$ …(iii)

Adding (ii) and (iii), we get

$2\cos^{-1}x=\frac{2\pi}{3}$

$\Rightarrow \cos^{-1}x=\frac{\pi}{3}$

$\Rightarrow x=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

when $x=\frac{1}{2}$, $y=0$

Hence, the solutions are $x = 1/2$ and $y = 0$.

30. Given, $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$

Put $x^2 = \cos(2\theta)$

Thus, $y=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$

$$=\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$$
$$=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$
$$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)$$
$$=\left(\frac{\pi}{4}-\theta\right)$$

$\Rightarrow \quad \theta = \frac{\pi}{4} - y\ y$

$\Rightarrow \quad \frac{1}{2}\cos^{-1}(x^2) = \frac{\pi}{4} - y$

$\Rightarrow \quad \cos^{-1}(x^2) = \frac{\pi}{2} - 2y$

$\Rightarrow \quad (x^2) = \cos\left(\frac{\pi}{2} - 2y\right)$

$\Rightarrow \quad x^2 = \sin(2y)$

31. We have

$$\frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$= \frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{\tan^{-1}\left(\frac{\beta}{\alpha}\right)}{2}\right) + \frac{\alpha^3}{2}\sec^2\left(\frac{\tan^{-1}\left(\frac{\alpha}{\beta}\right)}{2}\right)$$

$$= \frac{\beta^3}{2\sin^2\left(\frac{\tan^{-1}\left(\frac{\beta}{\alpha}\right)}{2}\right)} + \frac{\alpha^3}{2\cos^2\left(\frac{\tan^{-1}\left(\frac{\alpha}{\beta}\right)}{2}\right)}$$

$$= \frac{\beta^3}{1-\cos\left(\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)} + \frac{\alpha^3}{1+\cos\left(\tan^{-1}\left(\frac{\alpha}{\beta}\right)\right)}$$

$$= \frac{\beta^3}{1-\cos\left(\cos^{-1}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)\right)}$$

$$+ \frac{\alpha^3}{1+\cos\left(\cos^{-1}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)\right)}$$

$$= \frac{\beta^3}{1-\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)} + \frac{\alpha^3}{1+\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}$$

$$= \frac{\beta^3(\sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}-\alpha} + \frac{\alpha^3(\sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}+\beta}$$

$$= (\sqrt{\alpha^2+\beta^2})\left(\frac{\beta^3}{\sqrt{\alpha^2+\beta^2}-\alpha} + \frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}+\beta}\right)$$

$$= (\sqrt{\alpha^2+\beta^2})\left(\frac{\beta^3(\sqrt{\alpha^2+\beta^2}+\alpha)}{\beta^2} + \frac{\alpha^3(\sqrt{\alpha^2+\beta^2}-\beta)}{\alpha^2}\right)$$

$$= (\sqrt{\alpha^2+\beta^2})(\beta(\sqrt{\alpha^2+\beta^2}+\alpha)+\alpha(\sqrt{\alpha^2+\beta^2}-\beta))$$

$$= (\sqrt{\alpha^2+\beta^2})\sqrt{\alpha^2+\beta^2}\,(\alpha+\beta)$$

$$= (\alpha^2+\beta^2)(\alpha+\beta)$$

Hence, the result.

32. Given, $\cot^{-1}\left(\frac{n^2-10n+21.6}{\pi}\right) > \frac{\pi}{6}, n \in N$

$\Rightarrow \quad \left(\frac{n^2-10n+21.6}{\pi}\right) < \cot\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad (n^2 - 10n + 21.6 < \pi\sqrt{3}$

$\Rightarrow \quad (n^2 - 10n) + 21.6 < 5.6$

$\Rightarrow \quad (n^2\ 10n) + 16 < 0$

$\Rightarrow \quad (n-2)(n-8) < 0$

$\Rightarrow \quad 2 < n < 8$

Thus, the minimum value of $n$ is 2.

33. We have

$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right)+\cos^{-1}\left(\frac{\sqrt{12}}{4}\right)+\sec^{-1}(\sqrt{2})\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{4-2\sqrt{3}}{8}}\right)\right.\right.$$

$$\left.\left. + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sec^{-1}(\sqrt{2})\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{(\sqrt{3}-1)^2}{8}}\right) + \frac{\pi}{6} + \frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + \frac{\pi}{6} + \frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left(\frac{\pi}{2}\right)\right\}$$

$= \sin^{-1}(0)$

$= 0$

34. Given, $[\sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1} x)))] = 1$

$\Rightarrow \quad 1 \le \sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1} x))) < 2$

$\Rightarrow \quad \sin(1) \le \cos^{-1}(\sin^{-1}(\tan^{-1} x)) < \sin(2)$

$\Rightarrow \quad \cos(\sin(2)) < \sin^{-1}(\tan^{-1} x) \le \cos(\sin(11)$

$\Rightarrow \quad \sin(\cos(\sin(2))) < (\tan^{-1} x) \le \sin(\cos(\sin(1)))$

$\Rightarrow \quad \tan(\sin(\cos(\sin(2)))) < x \le \tan(\sin(\cos(\sin(1))))$

35. Given, $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$= 2\tan^{-1}x + \pi - 2\tan^{-1}x, x > 1$

$= \pi, x > 1$

Thus, $x \in (1, \infty)$

36. Given, $x = \operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a)))))$

$$= \operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$$

$$= \operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cos^{-1}\left(\frac{1}{a}\right)\right)\right)\right)$$

$$= \operatorname{cosec}\left(\tan^{-1}\left(\frac{1}{a}\right)\right)$$

$$= \operatorname{cosec}\left(\operatorname{cosec}^{-1}(\sqrt{a^2+1})\right)$$

$$= (\sqrt{a^2+1})$$

and $y = \sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1} a)))))$

$$= \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{a^2-1}}\right)\right)\right)\right)$$

$$= \sec\left(\cot^{-1}\left(\frac{1}{a}\right)\right)$$

$$= \sec(\sec^{-1}(\sqrt{a^2+1}))$$

$$= (\sqrt{a^2+1})$$

Thus $x = y$

37. We have

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\ldots$$

$$= \tan^{-1}\left(\frac{1}{1+2}\right)+\tan^{-1}\left(\frac{1}{1+6}\right)+\tan^{-1}\left(\frac{1}{1+12}\right)+\ldots$$

$$= \tan^{-1}\left(\frac{2-1}{1+2.1}\right)+\tan^{-1}\left(\frac{3-2}{1+3.2}\right)+\tan^{-1}\left(\frac{4-3}{1+4.3}\right)$$

$$+\ldots+\tan^{-1}\left(\frac{n+1-n}{1+(n+1)\cdot n}\right)$$

$= \tan^{-1}(2) - \tan^{-1}(1) + \tan^{-1}(3) - \tan^{-1}(2)$

$\quad + \tan^{-1}(4) - \tan^{-1}(3) + \ldots + \tan^{-1}(n+1) - \tan^{-1}(n)$

$= \tan^{-1}(n+1) - \tan^{-1}(1)$

$$= \tan^{-1}\left(\frac{n+1-1}{1+(n+1)\cdot 1}\right)$$

$$= \tan^{-1}\left(\frac{n}{n+2}\right), n \to \infty$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

38. Let I $t_n = \sin^{-1}\left(\dfrac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}\times\sqrt{n+1}}\right)$

$$= \tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r(r-1)}}\right)$$

$$= \tan^{-1}(\sqrt{r}) - \tan^{-1}(\sqrt{r-1})$$

Now, $\displaystyle\sum_{r=1}^{n} \sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}\sqrt{n+1}}\right)$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r}\sqrt{r-1}}\right)$$

$$= \sum_{r=1}^{n} (\tan^{-1}\sqrt{r} - \tan^{-1}\sqrt{r-1})$$

$= (\tan^{-1} 1 - \tan^{-1} 0) + (\tan^{-1}\sqrt{2} - \tan^{-1} 1)$

$\quad + (\tan^{-1}\sqrt{3} - \tan^{-1}\sqrt{2}) + (\tan^{-1}\sqrt{4} - \tan^{-1}\sqrt{3})$

$\quad + \ldots + (\tan^{-1}\sqrt{n} - \tan^{-1}\sqrt{n-1})$

$= \tan^{-1}\sqrt{n} - \tan^{-1} 0$

$= \tan^{-1}\sqrt{n}$

When $n \to \infty$, the sum is $\tan^{-1}(\infty) = \dfrac{\pi}{2}$.

39. We have

$\cot^{-1}(2.1^2) + \cot^{-1}(2.2^2) + \cot^{-1}(2.3^2) + \ldots$

$$= \tan^{-1}\left(\frac{1}{2.1^2}\right)+\tan^{-1}\left(\frac{1}{2.2^2}\right)+\tan^{-1}\left(\frac{1}{2.3^2}\right)+\ldots$$

$$= \tan^{-1}\left(\frac{2}{4}\right)+\tan^{-1}\left(\frac{2}{16}\right)+\tan^{-1}\left(\frac{2}{36}\right)+\ldots$$

$$= \tan^{-1}\left(\frac{2}{1+3}\right)+\tan^{-1}\left(\frac{2}{1+15}\right)+\tan^{-1}\left(\frac{2}{1+35}\right)+\ldots$$

$$= \tan^{-1}\left(\frac{3-1}{1+3.1}\right)+\tan^{-1}\left(\frac{5-3}{1+5.3}\right)$$

$$+\tan^{-1}\left(\frac{7-5}{1+7.5}\right)+\ldots+\tan^{-1}\left(\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right)$$

$= (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3)$

$\quad + (\tan^{-1}7 - \tan^{-1}5) + (\tan^{-1}9 - \tan^{-1}7)$

$\quad + \ldots + (\tan^{-1}(2n+1) - \tan^{-1}(2n-1))$

$= (\tan^{-1}(2n+1) - \tan^{-1} 1)$

$$= \left(\tan^{-1}\frac{(2n+1-1)}{1+(2n+1)\cdot 1}\right)$$

$$= \left(\tan^{-1}\left(\frac{n}{n+1}\right)\right) = \frac{\pi}{4}, \text{ when } n \to \infty$$

40. Given, $\cos^{-1}\left(\dfrac{x}{2}\right)+\cos^{-1}\left(\dfrac{y}{3}\right) = \theta$

$$\Rightarrow \quad \cos^{-1}\left(\frac{x}{2}\cdot\frac{y}{3} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right) = \theta$$

$$\Rightarrow \quad \cos\theta = \left(\frac{x}{2}.\frac{y}{3} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right)$$

$$\Rightarrow \quad \left(\cos\theta - \frac{xy}{6}\right)^2 = \left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{9}\right)$$

$$\Rightarrow \quad \left(\cos^2\theta - 2\cdot\frac{xy}{6}\cdot\cos\theta + \frac{x^2y^2}{36}\right)$$

$$= 1 - \frac{x^2}{4} - \frac{y^2}{9} + \frac{x^2y^2}{36}$$

$\Rightarrow \quad \cos^2\theta - \frac{xy}{3}\cos\theta = 1 - \frac{x^2}{4} - \frac{y^2}{9}$

$\Rightarrow \quad \frac{x^2}{4} - \frac{xy}{3} \cdot \cos\theta + \frac{y^2}{9} = 1 - \cos^2\theta$

$\Rightarrow \quad 9x^2 - 12xy\cos\theta + 4y^2 = 12\sin^2\theta$

Hence, the result.

***Note*** No questions asked in 1985.

41. Let $\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right) = \theta$

$\Rightarrow \quad \cos^{-1}\left(\frac{\sqrt{5}}{3}\right) = 2\theta$

$\Rightarrow \quad \cos 2\theta = \frac{\sqrt{5}}{3}$

$\Rightarrow \quad \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{\sqrt{5}}{3}$

$\Rightarrow \quad \frac{1 - \tan^2\theta + 1 + \tan^2\theta}{1 - \tan^2\theta - 1 - \tan^2\theta} = \frac{\sqrt{5}+3}{\sqrt{5}-3}$

$\Rightarrow \quad \frac{2}{-2\tan^2\theta} = \frac{\sqrt{5}+3}{\sqrt{5}-3}$

$\Rightarrow \quad \frac{1}{\tan^2\theta} = \frac{\sqrt{5}+3}{3-\sqrt{5}}$

$\Rightarrow \quad \tan^2\theta = \frac{3-\sqrt{5}}{3+\sqrt{5}} = \frac{(3-\sqrt{5})^2}{4}$

$\Rightarrow \quad \tan\theta = \pm\sqrt{\frac{(3-\sqrt{5})^2}{4}} = \pm\frac{(3-\sqrt{5})}{2}$

42. Given, $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}0 = 0$

Let $\tan^{-1}x = \theta \Rightarrow x = \tan\theta$

$\Rightarrow \quad \sin(2\cos^{-1}(\cot(2\theta))) = 0$

$\Rightarrow \quad \sin\left(2\cos^{-1}\left(\frac{1-\tan^2\theta}{2\tan\theta}\right)\right) = 0$

$\Rightarrow \quad \sin\left(2\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right) = 0$

$\Rightarrow \quad \sin\left(\cos^{-1}\left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)\right) = 0$

$\Rightarrow \quad \sin\left(\cos^{-1}\left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)\right) = 0$

$\Rightarrow \quad \sin\left(\sin^{-1}\left(\sqrt{1 - \left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)^2}\right)\right) = 0$

$\Rightarrow \quad \sqrt{1 - \left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)^2} = 0$

$\Rightarrow \quad 1 - \left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)^2 = 0$

$\Rightarrow \quad \left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)^2 = 1$

$\Rightarrow \quad \left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right) = \pm 1$

$\Rightarrow \quad 2\left(\frac{1-x^2}{2x}\right)^2 = 1 \pm 1 = 2, 0$

$\Rightarrow \quad 2\left(\frac{1-x^2}{2x}\right)^2 = 2$ and $2\left(\frac{1-x^2}{2x}\right)^2 = 0$

$\Rightarrow \quad \left(\frac{1-x^2}{2x}\right)^2 = 1$ and $1 - x^2 = 0$

$\Rightarrow \quad \left(\frac{1-x^2}{2x}\right) = \pm 1$ and $x = \pm 1$

$\Rightarrow \quad x^2 + 2x - 1 = 0,\ x^2 - 2x - 1 = 0$ and $x = \pm 1$

$\Rightarrow \quad (x+1)^2 = (\sqrt{2})^2,\ (x-1)^2 = (\sqrt{2})^2$ and $x = \pm 1$

$\Rightarrow \quad x = -1 \pm \sqrt{2}, 1 \pm \sqrt{2}, \pm 1$

43. Given, $\tan^{-1}x + \cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right) = \sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$

$\Rightarrow \quad \tan^{-1}x + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}3$

$\Rightarrow \quad \tan^{-1}\left(\frac{x + \frac{1}{y}}{1 - x \cdot \frac{1}{y}}\right) = \tan^{-1}(3)$

$\Rightarrow \quad \tan^{-1}\left(\frac{xy+1}{y-x}\right) = \tan^{-1}(3)$

$\Rightarrow \quad \left(\frac{xy+1}{y-x}\right) = 3$

$\Rightarrow \quad xy + 1 = 3y - 3x$

$\Rightarrow \quad 3x + 1 = y(3 - x)$

$\Rightarrow \quad y = \frac{3x+1}{3-x}$

when $x = 1, y = 2$

Also, when $x = 2, y = 7$

Hence, the positive integral solutions are 2.

44. We have $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y = \cos^{-1}(-z)$

$\Rightarrow \quad \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos(-z)$

$\Rightarrow \quad (xy + z)^2 = (1 - x^2)(1 - y^2)$

$\Rightarrow \quad x^2y^2 + 2xyz + z^2 = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow \quad x^2 + y^2 + z^2 + 2xyz = 1$

45. See solutions of Ex-42.

46. Given, $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right)$

Now, $\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right)$

$$= \left(\frac{\frac{6\tan\theta}{1+\tan^2\theta}}{5 + 4\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)}\right)$$

$$= \left(\frac{6\tan\theta}{5(1+\tan^2\theta) + 4(1-\tan^2\theta)}\right)$$

$$= \left(\frac{6\tan\theta}{9+\tan^2\theta}\right)$$

$$= \left(\frac{\frac{2}{3}\tan\theta}{1 + \left(\frac{\tan\theta}{3}\right)^2}\right)$$

$$= \left(\frac{2\cdot\frac{\tan\theta}{3}}{1 + \left(\frac{\tan\theta}{3}\right)^2}\right)$$

Also, $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\cdot 2\tan^{-1}\left(\frac{\tan\theta}{3}\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{\tan\theta}{3}\right)$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2\tan^2\theta - \frac{\tan\theta}{3}}{1 + 2\tan^2\theta\cdot\frac{\tan\theta}{3}}\right)$

$\Rightarrow \quad \tan\theta = \left(\frac{6\tan^2\theta - \tan\theta}{3 + 2\tan^3\theta}\right)$

$\Rightarrow \quad 3\tan\theta + 2\tan^4\theta - 6\tan^2\theta + \tan\theta = 0$

$\Rightarrow \quad 2\tan^4\theta - 6\tan^2\theta + 4\tan\theta = 0$

$\Rightarrow \quad \tan^4\theta - 3\tan^2\theta + 2\tan\theta = 0$

$\Rightarrow \quad \tan\theta(\tan 3\theta - 3\tan\theta + 2) = 0$

$\Rightarrow \quad \tan\theta(\tan\theta - 1)^2(\tan\theta + 2) = 0$

$\Rightarrow \quad \tan\theta = 0, 1, -2$

when $\tan\theta = 0 \Rightarrow \theta = n\pi, n \notin I$

when $\tan\theta = 1 \Rightarrow \theta = m\pi + \frac{\pi}{4}, m \in I$

when $\tan q = -2 \Rightarrow q = p\pi + \tan^{-1}(-2), p \in I$

47. We have

$3\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)$

Now, $3\tan^{-1}\left(\frac{1}{2}\right)$

$$= \tan^{-1}\left(\frac{3\cdot\frac{1}{2} - \left(\frac{1}{2}\right)^3}{1 - 3\left(\frac{1}{2}\right)^2}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{2} - \frac{1}{8}}{1 - \frac{3}{4}}\right)$$

$$= \tan^{-1}\left(\frac{12-1}{2}\right)$$

$$= \tan^{-1}\left(\frac{11}{2}\right)$$

Also, $2\tan^{-1}\left(\frac{1}{5}\right)$

$$= \tan^{-1}\left(\frac{2\cdot\frac{1}{5}}{1 - \left(\frac{1}{5}\right)^2}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2}{5}}{\frac{25-1}{25}}\right)$$

$$= \tan^{-1}\left(\frac{2}{5}\times\frac{25}{24}\right)$$

$$= \tan^{-1}\left(\frac{5}{12}\right)$$

Also, $\sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)$

$$= \tan^{-1}\left(\frac{142}{31}\right)$$

Therefore,

$3\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)$

$$= \tan^{-1}\left(\frac{11}{2}\right) + \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{142}{31}\right)$$

$$= \tan^{-1}\left(\frac{\frac{11}{2} + \frac{5}{12}}{1 - \frac{11}{2}\cdot\frac{5}{12}}\right) + \tan^{-1}\left(\frac{142}{31}\right)$$

$$= \tan^{-1}\left(\frac{132+10}{24-55}\right) + \tan^{-1}\left(\frac{142}{31}\right)$$

$$= \tan^{-1}\left(-\frac{142}{31}\right) + \tan^{-1}\left(\frac{142}{31}\right)$$
$$= -\tan^{-1}\left(\frac{142}{31}\right) + \tan^{-1}\left(\frac{142}{31}\right)$$
$$= 0$$

48. Given, $\sin^{-1}\left(\frac{ax}{c}\right) + \sin^{-1}\left(\frac{bx}{c}\right) = \sin^{-1}x$

$\Rightarrow \quad \sin^{-1}\left(\frac{ax}{c}\right) = \sin^{-1}x - \sin^{-1}\left(\frac{bx}{c}\right)$

$\Rightarrow \quad \sin^{-1}\left(\frac{ax}{c}\right) = \sin^{-1}\left(x\sqrt{1-\frac{b^2x^2}{c^2}} - \frac{bx}{c}\sqrt{1-x^2}\right)$

$\Rightarrow \quad \left(\frac{ax}{c}\right) = \left(x\sqrt{1-\frac{b^2x^2}{c^2}} - \frac{bx}{c}\sqrt{1-x^2}\right)$

$\Rightarrow \quad \left(\frac{ax}{c}\right) = \left(\frac{x}{c}\sqrt{c^2-b^2x^2} - \frac{bx}{c}\sqrt{1-x^2}\right)$

$\Rightarrow \quad x((\sqrt{c^2-b^2x^2} - b\sqrt{1-x^2}) - a) = 0$

$\Rightarrow \quad x = 0, \sqrt{c^2-b^2x^2} = b\sqrt{1-x^2} + a$

Thus $x = 0$ and

$\sqrt{c^2-b^2x^2} = b\sqrt{1-x^2} + a$

$\Rightarrow \quad c^2 - b^2x^2 = 2ab\sqrt{1-x^2} + b^2(1-x^2) + a^2$

$\Rightarrow \quad c^2 - b^2x^2 = 2ab\sqrt{1-x^2} + b^2 - b^2x^2 + a^2$

$\Rightarrow \quad c^2 = 2ab\sqrt{1-x^2} + b^2 + a^2$

$\Rightarrow \quad c^2 = 2ab\sqrt{1-x^2} + c^2$

$\Rightarrow \quad 2ab\sqrt{1-x^2} = 0$

$\Rightarrow \quad (1-x^2) = 0$

$\Rightarrow \quad x = \pm 1$

Hence, the solution sets is $\{-1, 0, 1\}$

49. Given, $\cos^{-1}(x\sqrt{6}) + \cos^{-1}(3\sqrt{3}\,x^2) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}(\sqrt{1-6x^2}) + \cos^{-1}(3\sqrt{3}\,x^2) = \frac{\pi}{2}$

It is possible only when, $\sqrt{1-6x^2} = 3\sqrt{3}x^2$

$\Rightarrow \quad 1 - 6x^2 = 27x^4$

$\Rightarrow \quad 27x^4 + 6x^2 - 1 = 0$

$\Rightarrow \quad 27x^4 - 9x^3 + 9x^3 - 3x^2 + 9x^2 - 1 = 0$

$\Rightarrow \quad 9x^3(3x-1) + 3x^2(3x-1) + (3x-1)(3x+1) = 0$

$\Rightarrow \quad ((3x-1)(9x^3 + 3x^2 + (3x+1)) = 0$

$\Rightarrow \quad (3x-1)(3x^2(3x+1) + (3x+1) = 0$

$\Rightarrow \quad (3x-1)(3x+1)(3x^2+1) = 0$

$\Rightarrow \quad x = -\frac{1}{3}, \frac{1}{3}, \pm\frac{i}{\sqrt{3}}$

Hence, the solutions are $\left\{\pm\frac{1}{3}, \pm\frac{i}{\sqrt{3}}\right\}$

50. Given equation is

$\Rightarrow \tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{b}{x}\right) + \tan^{-1}\left(\frac{c}{x}\right) + \tan^{-1}\left(\frac{d}{x}\right) = \frac{\pi}{2}$

$\tan^{-1}\left(\frac{\frac{a}{x}+\frac{b}{x}}{1-\frac{a}{x}\cdot\frac{b}{x}}\right) + \tan^{-1}\left(\frac{\frac{c}{x}+\frac{d}{x}}{1-\frac{c}{x}\frac{d}{x}}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}\left(\frac{(a+b)x}{x^2-ab}\right) + \tan^{-1}\left(\frac{(c+d)x}{x^2-cd}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}\left(\frac{(a+b)x}{x^2-ab}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{(c+d)x}{x^2-cd}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{(a+b)x}{x^2-ab}\right) = \cot^{-1}\left(\frac{(c+d)x}{x^2-cd}\right)$

$\Rightarrow \quad \left(\frac{(a+b)x}{x^2-ab}\right) = \left(\frac{x^2-cd}{(c+d)x}\right)$

$\Rightarrow \quad (x^2-ab)(x^2-cd) = (a+b)(c+d)\,x^2$

$\Rightarrow \quad x^4 - (a+b+cd)x^2 + abcd = (a+b)(c+d)\,x^2$

$\Rightarrow \quad x^4 - (a+b+cd+(a+b)(c+d))^{\,2} + abcd = 0$

since $x_1, x_2, x_3, x_4$ are the values of the above equation, we have

$x_1 + x_2 + x_3 + x_4 = 0$

$\Sigma x_1x_2 = (ab + cd + (a+b)(c+d))$

$\Sigma x_1x_2x_3 = 0$

$\Sigma x_1x_2x_3x_4 = abcd$

(i) $\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4 = 0$

(ii) $\sum_{i=1}^{4}\left(\frac{1}{x_i}\right)$

$= \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}$

$= \frac{x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_1x_2x_3}{x_1x_2x_3x_4}$

$= \frac{0}{abcd} = 0$

(iii) $\prod_{i=1}^{4}(x_i)$

$= x_1x_2x_3x_4$

$= abcd$

(iv) $\Pi(x_1 + x_2 + x_3)$

$= (x_1 + x_2 + x_3)(x_1 + x_2 + x_4)(x_1 + x_3 + x_4)(x_2 + x_3 + x_4)$

$= (-x_4)(-x_3)(-x_2)(-x_1)$

$= x_1x_2x_3x_4$

$= abcd$

51. We have,

$\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$

$\cos^{-1}(2x) + \cos^{-1}(3x) = \pi - \cos^{-1}(x)$

$\cos^{-1}(2x) + \cos^{-1}(3x) = \cos^{-1}(-x)$

$\cos^{-1}(2x \cdot 3x - \sqrt{(1-4x^2)(1-9x^2)}) = \cos^{-1}(-x)$

$6x^2 - \sqrt{(1-4x^2)(1-9x^2)} = -x$

$(6x^2 + x) = \sqrt{(1-4x^2)(1-9x^2)}$

$(6x^2 + x)^2 = (1 - 4x^2)(1 - 9x^2)$

$36x^4 + 12x + x^2 = 1 - 13x^2 + 36x^4$

$12x^3 + 14x^2 - 1 = 0$

Thus $a = 12$, $b = 14$, $c = 0$

Hence, the value of $(a + b + c + 2)$

$= 28$

52. We have $x = \sin(2\tan^{-1} 2)$

$\Rightarrow \quad x = \sin(2\theta), \tan^{-1} 2 = \theta$

$\Rightarrow \quad x = \dfrac{2\tan\theta}{1+\tan^2\theta}, \tan\theta = 2$

$\Rightarrow \quad x = \dfrac{4}{5}$

Also, $y = \sin\left(\dfrac{1}{2}\tan^{-1}\left(\dfrac{4}{3}\right)\right)$

$\Rightarrow \quad y = \sin(\theta), \dfrac{1}{2}\tan^{-1}\left(\dfrac{4}{3}\right) = \theta$

$\Rightarrow \quad y = \sin(\theta), \tan(2\theta) = \dfrac{4}{3}$

$\Rightarrow \quad y = \sin(\theta), \tan(\theta) = \dfrac{1}{2}$

$\Rightarrow \quad y = \dfrac{1}{\sqrt{5}}$

Hence, $y^2 = \dfrac{1}{5} = 1 - \dfrac{4}{5} = 1 - x$

## Level IV

1. We have

$\sin^{-1}(\cos(\sin^{-1} x)) + \cos^{-1}(\sin(\cos^{-1} x))$

$= \sin^{-1}(\sqrt{1-x^2}) + \cos^{-1}(\sqrt{1-x^2})$

$= \cos^{-1}(x) + \sin^{-1}(x)$

$= \dfrac{\pi}{2}$

2. We have

$\tan^{-1}\{\mathrm{cosec}(\tan^{-1} x) - \tan(\cot^{-1} x)\}$

$= \tan^{-1}\left(\dfrac{\sqrt{1+x^2} - 1}{x}\right)$

$= \tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right), x = \tan\theta$

$= \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\left(\dfrac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right)$

$= \tan^{-1}\left(\tan\left(\dfrac{\theta}{2}\right)\right)$

$= \dfrac{1}{2}\tan^{-1} x$

3. We have

$(\tan^{-1} x + \tan^{-1} y + \tan^{-1} z)$

$= \tan\left(\tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)\right)$

$= \left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$

Also, $\cot(\cot^{-1} x \cot^{-1} y + \cot^{-1} z)$

$= \cot\left(\tan^{-1}\left(\dfrac{1}{x}\right) + \tan^{-1}\left(\dfrac{1}{y}\right) + \tan^{-1}\left(\dfrac{1}{z}\right)\right)$

$= \cot\left(\tan^{-1}\left(\dfrac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{xyz}}{1-\left(\frac{1}{xy}+\frac{1}{xz}+\frac{1}{yz}\right)}\right)\right)$

$= \cot\left(\tan^{-1}\left(\dfrac{xy+yz+zx-1}{xyz-(x+y+z)}\right)\right)$

$= \cot\left(\cot^{-1}\left(\dfrac{xyz-(x+y+z)}{xy+yz+zx-1}\right)\right)$

$= \cot\left(\cot^{-1}\left(\dfrac{(x+y+z)-xyz}{1-(xy+yz+zx)}\right)\right)$

$= \left(\dfrac{(x+y+z)-xyz}{1-(xy+yz+zx)}\right)$

Hence, the result.

4. Given expression is

$\sin(\cot^{-1}(\tan(\cot^{-1} x)))$

$= \sin(\cot^{-1}(\tan\theta)), \theta = \cos^{-1} x$

$= \sin(\cot^{-1}(\tan\theta)), \cos\theta = x$

$= \sin\left(\cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)\right)$

$= \sin\varphi$, where $\cot\varphi = \dfrac{\sqrt{1-x^2}}{x}$

$= x$

5. Given expression is

$\sin(\mathrm{cosec}^{-1}(\cot(\tan^{-1} x)))$

$= \sin(\mathrm{cosec}^{-1}(\cot\theta))$, where $\tan\theta = x$

$= \sin\left(\mathrm{cosec}^{-1}\left(\dfrac{1}{x}\right)\right)$

$= \sin\varphi$, where $\operatorname{cosec}\varphi = \dfrac{1}{x}$

$= x$

6. Given expression is
$\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) + \tan^{-1}(\tan(-6)) + \cot^{-1}(\cot(-10))$

$= (5 - 2\pi) + (4\pi - 10)\ (6 - 2\pi) + \pi - (10\ 3\pi)$

$= (5 - 2\pi) + (4\pi - 10) + (2\pi - 6) + \pi + (3\pi - 10)$

$= 8\pi - 21$

7. We have

$$\cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3 - 3x^2}}{2}\right), \forall\, x \in \left(\frac{1}{2}, 1\right)$$

$$= \cos^{-1}(x) + \cos^{-1}\left(x \cdot \frac{1}{2} + \sqrt{1 - \frac{1}{4}}\sqrt{1 - x^2}\right)$$

$$= \cos^{-1}(x) + \cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(x)$$

$$= \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3}$$

8. We have

$$\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{\sqrt{5 - 2\sqrt{6}}}{1 + \sqrt{6}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{\sqrt{(\sqrt{3} - \sqrt{2})^2}}{1 + \sqrt{6}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{(\sqrt{3} - \sqrt{2})}{1 + \sqrt{3}\sqrt{2}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - (\tan^{-1}(\sqrt{3}) - \tan^{-1}(\sqrt{2}))$$

$$= \cot^{-1}(\sqrt{2}) + \tan^{-1}(\sqrt{2}) - \tan^{-1}(\sqrt{3})$$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

9. We have

$m = \sin^{-1}(a^6 + 1) + \cos^{-1}(a^4 + 1) - \tan^{-1}(a^2 + 1)$

since $\sin^{-1}(\ ) + \cos^{-1}(\ )$ is defined for $[-1, 1]$

Put $a = 0$, then

$m = \sin^{-1}(1) + \cos^{-1}(1) - \tan^{-1}(1)$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Hence, the image of the line $x + y = \dfrac{\pi}{4}$ w.r.t. to the $y$-axis is $x - y + \dfrac{\pi}{4} = 0$

10. Given equation is

$$(\sin^{-1}x)^3 + (\sin^{-1}y)^3 + (\sin^{-1}z)^3 = \frac{(3\pi)^3}{8}$$

It is possible only when

$$\sin^{-1}x = \frac{\pi}{2} = \sin^{-1}y = \sin^{-1}z$$

So, $x = 1, y = 1$ and $z = 1$

Hence, the value of $(3x + 4y - 5z + 2)$

$= 3 + 4 - 5 + 2$

$= 4$

11. We have

$$S = \sum_{r=1}^{n} \cot^{-1}\left(2^{r+1} + \frac{1}{2^r}\right)$$

$$= \sum_{r=1}^{n} \cot^{-1}\left(\frac{2^{2r+1} + 1}{2^r}\right)$$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^r}{1 + 2^{2r+1}}\right)$$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r+1} - 2^r}{1 + 2^{r+1} \cdot 2^r}\right)$$

$$= \sum_{r=1}^{n} [\tan^{-1}(2^{r+1}) - \tan^{-1}(2^r)]$$

$$= [\tan^{-1}(2^{n+1}) - \tan^{-1}(2)]$$

Thus, $\lim\limits_{n\to\infty}(S)$

$$= \operatorname*{Lim}_{n\to\infty}[\tan^{-1}(2^{n+1}) - \tan^{-1}(2)]$$

$$= [\tan^{-1}(\infty) - \tan^{-1}(2)]$$

$$= \frac{\pi}{2} - \tan^{-1}(2)$$

$$= \cot^{-1}(2)$$

12. We have

$$\lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n} \tan^{-1}\left(\frac{4}{4r^2 + 3}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{r^2 + \frac{3}{4}}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1 + \left(r^2 - \frac{1}{4}\right)}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n} \tan^{-1}\left(\frac{\left(r + \frac{1}{2}\right) - \left(r - \frac{1}{2}\right)}{1 + \left(r + \frac{1}{2}\right)\left(r - \frac{1}{2}\right)}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n} \tan^{-1}\left(r + \frac{1}{2}\right) - \tan^{-1}\left(r - \frac{1}{2}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\tan^{-1}\left(n + \frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\tan^{-1}\left(\frac{n+\frac{1}{2}-\frac{1}{2}}{1+\frac{1}{2}\left(n+\frac{1}{2}\right)}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{1+\frac{1}{2}\left(n+\frac{1}{2}\right)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{\frac{1}{n}+\frac{1}{2}\left(1+\frac{1}{2n}\right)}\right)$$

$$= 2$$

13. Given equation is

$$2\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi x^3$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \frac{\pi x^3}{2}$$

$$2\tan^{-1}x = \frac{\pi x^3}{2}$$

$$\tan^{-1}x = \frac{\pi x^3}{4}$$

Clearly, there are 3 solutions at $x = -1, 0, 1$.

14. Given, $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$

$$\Rightarrow \quad \cos^{-1}\left(\frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right) = \alpha$$

$$\Rightarrow \quad \left(\frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right) = \cos\alpha$$

$$\Rightarrow \quad \left(\frac{xy}{ab} - \cos\alpha\right)^2 = \left(\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right)^2$$

$$\Rightarrow \quad \left(\frac{xy}{ab}\right)^2 - 2\left(\frac{xy}{ab}\right)\cos\alpha + \cos^2\alpha$$

$$= 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \left(\frac{xy}{ab}\right)^2$$

$$\Rightarrow \quad \frac{x^2}{a^2} - 2\left(\frac{xy}{ab}\right)\cos\alpha + \frac{y^2}{b^2} = 1 - \cos^2\alpha$$

$$\Rightarrow \quad \frac{x^2}{a^2} - 2\left(\frac{xy}{ab}\right)\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$$

15. Put $A = \sin^{-1} x$, $B = \sin^{-1} y$, $C = \sin^{-1} z$

$\Rightarrow x = \sin A$, $y = \sin B$, $z = \sin C$

$\Rightarrow$ $\cos A = \sqrt{1-x^2}$, $\cos B = \sqrt{1-y^2}$, $\cos C = \sqrt{1-z^2}$

Thus, $A + B + C = \pi$

Now, LHS

$$= x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$$

$$= \sin A \cos A + \sin B \cos B + \sin C \cos C$$

$$= \frac{1}{2}[\sin 2A + \sin 2B + \sin 2C]$$

$$= \frac{1}{2}(4\sin A \sin B \sin C)$$

$$= 2\sin A \sin B \sin C$$

$$= 2xyz$$

Hence, the result.

16. Given,

$$f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$$

$$= (\sin^{-1} x + \cos^{-1} x)\{(\sin^{-1} x)^2 + (\cos^{-1} x)^2 - \sin^{-1} x \cos^{-1} x\}$$

$$= \frac{\pi}{2}\left\{\left(\frac{\pi}{2}\right)^2 - 3\sin^{-1}x\cos^{-1}x\right\}$$

$$= \frac{\pi}{2}\left\{\frac{\pi^2}{4} - 3\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)\right\}$$

$$= \frac{\pi}{2}\left\{\frac{\pi^2}{4} - 3a\left(\frac{\pi}{2} - a\right)\right\} \text{ where } a = \sin^{-1}x$$

$$= \frac{\pi}{2}\left\{3a^2 - \frac{3a\pi}{2} + \frac{\pi^2}{4}\right\}$$

$$= \frac{\pi}{8}\{12\alpha^2 - 6\pi a + \pi^2\}$$

$$= \frac{12\pi}{8}\left\{\alpha^2 - \frac{1}{2}\pi a + \frac{\pi^2}{12}\right\}$$

$$= \frac{12\pi}{8}\left\{\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{48}\right\}$$

Minimum value $= \frac{\pi^3}{32}$ at $x = \frac{1}{\sqrt{2}}$

and maximum value $= \frac{7\pi^3}{8}$ at $x = -1$

17. Given equation is

$$\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}(2x) = \frac{\pi}{2} - \sin^{-1}x = \cos^{-1}x$$

$$\Rightarrow \quad \sin^{-1}(2x) = \sin^{-1}(\sqrt{1-x^2})$$

$$\Rightarrow \quad 2x = \sqrt{1-x^2}$$

$$\Rightarrow \quad 4x^2 = 1 - x^2$$

$$\Rightarrow \quad 5x^2 = 1$$

$$\Rightarrow \quad x^2 = \frac{1}{5}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{1}{5}}$$

18. Given equation is

$$\tan^{-1}\left(\frac{1}{1+2x}\right)+\tan^{-1}\left(\frac{1}{1+4x}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{1+2x}+\frac{1}{1+4x}}{1-\frac{1}{1+2x}\times\frac{1}{1+4x}}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \frac{1+4x+1+2x}{1+6x+8x^2-1}=\frac{2}{x^2}$$

$$\Rightarrow \quad \frac{2+6x}{6x+8x^2}=\frac{2}{x^2}$$

$$\Rightarrow \quad \frac{1+3x}{3x+4x^2}=\frac{2}{x^2}$$

$\Rightarrow \quad 3x^3 + x^2 + 8x^2 + 6x = 0$

$\Rightarrow \quad 3x^3 + 9x^2 + 6x = 0$

$\Rightarrow \quad x^3 + 3x^2 + 2x = 0$

$\Rightarrow \quad x(x^2 + 3x = 2) = 0$

$\Rightarrow \quad x(x + 1)(x + 2) = 0$

$\Rightarrow \quad x = 0, -1, -2$

19. Given equation is

$\tan^{-1}(x-1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}(3x)$

$\Rightarrow \quad \tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}(3x) - \tan^{-1}(x)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x-1+x+1}{1-(x^2-1)}\right)=\tan^{-1}\left(\frac{3x-x}{1+3x\cdot x}\right)$$

$$\Rightarrow \quad \left(-\frac{2x}{x^2}\right)=\left(\frac{2x}{1+3x^2}\right)$$

$\Rightarrow \quad 2x(1 + 3x^2 + x^2) = 0$

$\Rightarrow \quad 2x = 0, (1 + 4x^2) = 0$

$\Rightarrow \quad x = 0$

Hence, the solution is $x = 0$

20. Given equation is

$$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cos^{-1}x=\frac{\pi}{4}$$

$$\Rightarrow \quad \cos^{-1}\left(\sqrt{1-\frac{1}{5}}\right)+\cos^{-1}x=\frac{\pi}{4}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)+\cos^{-1}x=\frac{p}{4}$$

$$\Rightarrow \quad \cos^{-1}x=\frac{\pi}{4}-\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \quad x=\cos\left(\frac{\pi}{4}-\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

$$\Rightarrow \quad x=\cos\left(\frac{\pi}{4}\right)\cos\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)+\sin\left(\frac{\pi}{4}\right)\sin\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

$$\Rightarrow \quad x=\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{5}}=\frac{3}{\sqrt{10}}$$

21. Given equation is

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \cos^{-1}\left(-\frac{1-x^2}{x^2+1}\right)+\tan^{-1}\left(-\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi-\cos^{-1}\left(\frac{1-x^2}{x^2+1}\right)-\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi-2\tan^{-1}(x)-2\tan^{-1}(x)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi-4\tan^{-1}(x)=\frac{2\pi}{3}$$

$$\Rightarrow \quad 4\tan^{-1}(x)=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}(x)=\frac{\pi}{12}$$

$$\Rightarrow \quad x=\tan\left(\frac{\pi}{12}\right)=(2-\sqrt{3})$$

22. Given equation is

$$2\tan^{-1}x=\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)-\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)$$

$\Rightarrow \quad 2\tan^{-1}x = 2\tan^{-1}(a) - 2\tan^{-1}(b)$

$\Rightarrow \quad \tan^{-1}x = \tan^{-1}(a) - \tan^{-1}(b)$

$$\Rightarrow \quad \tan^{-1}x=\tan^{-1}\left(\frac{a-b}{1+ab}\right)$$

$$\Rightarrow \quad x=\left(\frac{a-b}{1+ab}\right)$$

23. Given equation is

$\cot^{-1}x + \cot^{-1}(n^2 - x + 1) = \cot^{-1}(n-1)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right)+\tan^{-1}\left(\frac{1}{n^2-x+1}\right)=\tan^{-1}\left(\frac{1}{n-1}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1}{n^2-x+1}\right)=\tan^{-1}\left(\frac{1}{n-1}\right)-\tan^{-1}\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1}{n^2-x+1}\right)=\tan^{-1}\left(\frac{\frac{1}{n-1}-\frac{1}{x}}{1+\frac{1}{x(n-1)}}\right)$$

$$\Rightarrow \quad \left(\frac{1}{n^2-x+1}\right)=\frac{x-n+1}{nx-x+1}$$

$\Rightarrow \quad n^2x - x^2 + x - n^3 + nx - n + n^2 - x + 1 = nx - x + 1$

$\Rightarrow \quad n^2x - x^2 + x - n^3 - n + n^2 = 0$

$\Rightarrow \quad (n^2 + 1)x - (n^2 + 1)n - x^2 + n^2 = 0$

$\Rightarrow \quad (n^2 + 1)(x - n) - (x^2 - n^2) = 0$

$\Rightarrow \quad (x - n)(n^2 + 1 - x - n) = 0$

$\Rightarrow \quad (x - n) = 0, (n^2 + 1 - x - n) = 0$

$\Rightarrow \quad x = n, n^2 - n + 1$

Hence, the solutions are

$x = n, n^2 - n + 1$

24. Given equation is

$$\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1 - \left(\frac{x-1}{x+1} \cdot \frac{2x-1}{2x+1}\right)}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$

$$\Rightarrow \quad \frac{2x^2 - x - 1 + 2x^2 + x - 1}{2x^2 + 3x + 1 - 2x^2 + 3x - 1} = \frac{23}{26}$$

$$\Rightarrow \quad \frac{4x^2 - 2}{6x} = \frac{23}{36}$$

$$\Rightarrow \quad \frac{2x^2 - 1}{x} = \frac{23}{12}$$

$\Rightarrow \quad 24x^2 - 23x - 12 = 0$

$\Rightarrow \quad 24x^2 - 32x + 9x - 12 = 0$

$\Rightarrow \quad 8x(3x - 4) + 3(3x - 4) = 0$

$\Rightarrow \quad (8x + 3)(3x - 4) = 0$

$$\Rightarrow \quad x = \frac{4}{3}, -\frac{3}{8}$$

Hence, the solutions are $x = \frac{4}{3}, -\frac{3}{8}$

25. Given equation is

$$\sec^{-1}\left(\frac{x}{a}\right) - \sec^{-1}\left(\frac{x}{b}\right) = \sec^{-1}b - \sec^{-1}a$$

$$\Rightarrow \quad \sec^{-1}\left(\frac{x}{a}\right) + \sec^{-1}(a) = \sec^{-1}\left(\frac{x}{b}\right) + \sec^{-1}(b)$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{a}{x}\right) + \cos^{-1}\left(\frac{1}{a}\right) = \cos^{-1}\left(\frac{b}{x}\right) + \cos^{-1}\left(\frac{1}{b}\right)$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{a}{x} \cdot \frac{1}{a} - \sqrt{1 - \left(\frac{a}{x}\right)^2}\sqrt{1 - \frac{1}{a^2}}\right)$$

$$= \cos^{-1}\left(\frac{b}{x} \cdot \frac{1}{b} - \sqrt{1 - \left(\frac{b}{x}\right)^2}\sqrt{1 - \frac{1}{b^2}}\right)$$

$$\Rightarrow \quad \frac{1}{x} - \frac{\sqrt{(x^2 - a^2)(a^2 - 1)}}{ax} = \frac{1}{x} - \frac{\sqrt{(x^2 - b^2)(b^2 - 1)}}{bx}$$

$$\Rightarrow \quad \frac{\sqrt{(x^2 - a^2)(a^2 - 1)}}{a} = \frac{\sqrt{(x^2 - b^2)(b^2 - 1)}}{b}$$

$$\Rightarrow \quad \frac{(x^2 - a^2)(a^2 - 1)}{a^2} = \frac{(x^2 - b^2)(b^2 - 1)}{b^2}$$

$\Rightarrow \quad (x^2 - a^2)(a^2 - 1)\, b^2 = (x^2 - b^2)(b^2 - 1)\, n^2$

$\Rightarrow \quad x^2((a^2 - 1)b^2 - a^2(b^2 - 1)) = a^2b^2(a^2 - 1) - a^2b^2(b^2 - 1)$

$\Rightarrow \quad x^2[a^2 - b^2] = a^2b^2[(a^2 - 1) - (b^2 - 1)]$

$\Rightarrow \quad x^2[(a^2 - b^2)] = a^2b^2[(a^2 - b^2)]$

$\Rightarrow \quad x^2 = a^2b^2$

$\Rightarrow \quad x = ab$

26. We have

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{8n}{n^4 - 2n^2 + 5}\right)$$

$$= \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2\{(n+1)^2 - (n-1)^2\}}{4 + (n-1)^2}\right)$$

$$= \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\left\{\left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2\right\}}{1 + \left(\frac{n+1}{2}\right)^2 \cdot \left(\frac{n-1}{2}\right)^2}\right)$$

$$= \sum_{n=1}^{\infty} \tan^{-1}\left\{\left(\frac{n+1}{2}\right)^2\right\} - \tan^{-1}\left\{\left(\frac{n+1}{2}\right)^2\right\}$$

$$= \left[\tan^{-1}(1^2) - 0 + \tan^{-1}\left(\frac{3}{2}\right)^2 - \tan^{-1}(1^2)\right.$$

$$\left. + \tan^{-1}(2^2) - \tan^{-1}\left(\frac{3}{2}\right)^2 + \ldots\ldots + \right.$$

$$\left. = \tan^{-1}\left(\frac{n+1}{2}\right)^2 - \tan^{-1}\left(\frac{n-1}{2}\right)^2\right], n \to \infty$$

$$= \lim_{n\to\infty}\left\{\tan^{-1}\left(n + \frac{1}{2}\right)^2 - 0\right\}$$

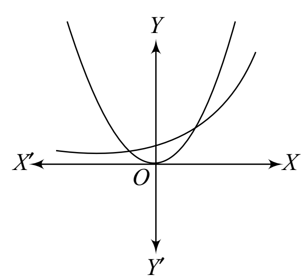
$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

27. Given equation is

$$\sin^{-1}(e^x) + \cos^{-1}(x^2) = \frac{\pi}{2}$$

It solutions exist only when

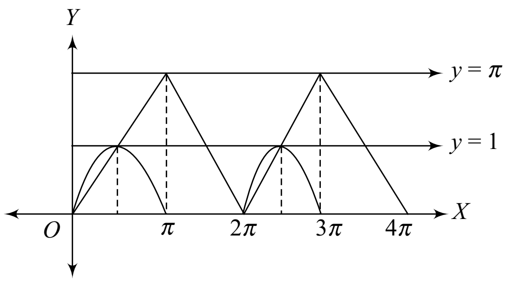
$e^x = x^2$




Hence, the number of solution is 2

28. Given equation is

$$\sqrt{\sin(x)} = \cos^{-1}(\cos x) \text{ in } (0, 2\pi)$$




From the graph, it is clear that, the number of real solution is 1 at $x = \frac{\pi}{2}$

29. Given equation is

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{3}\cdot\frac{1}{4}}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{7}{11}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{7}{11}+\frac{1}{5}}{1-\frac{7}{11}\cdot\frac{1}{5}}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{23}{24}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{23}{24}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{n}\right) = \tan^{-1}(1) - \tan^{-1}\left(\frac{23}{24}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{n}\right) = \tan^{-1}\left(\frac{1-\frac{23}{24}}{1+\frac{23}{24}}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{n}\right) = \tan^{-1}\left(\frac{1}{47}\right)$$

$\Rightarrow n = 47$

30. Given equation is

$x^3 + bx^2 + cx + 1 = 0$

Let $f(x) = x^3 + bx^2 + cx + 1$

Now, $f(0) = 1 > 0, f(-1) = b - c < 0$

So, the function $f(x)$ has a root in $-1 < \alpha < 0$

Now, $\tan^{-1}(\alpha) + \tan^{-1}\left(\frac{1}{\alpha}\right)$

$= \tan^{-1}\alpha - \pi + \cot^{-1}\alpha$

$= -\pi + (\tan^{-1}\alpha + \cot^{-1}\alpha)$

$= -\pi + \frac{\pi}{2} = -\frac{\pi}{2}$

31. Given equation is

$x^3 + bx^2 + cx + 1 = 0$

Let $f(x) = x^3 + bx^2 + cx + 1$

Now, $f(0) = 1 > 0, f(-1) = b - c < 0$

So, the function $f(x)$ has a root in $-1 < \alpha < 0$

Now, $2\tan^{-1}(\operatorname{cosec}\alpha) + \tan^{-1}(2\sin\alpha\sec^2\alpha)$

$$= 2\tan^{-1}\left(\frac{1}{\sin\alpha}\right) + \tan^{-1}\left(\frac{2\sin\alpha}{\cos^2\alpha}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{\sin\alpha}\right) + \tan^{-1}\left(\frac{2\sin\alpha}{1-\sin^2\alpha}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{\sin\alpha}\right) + 2\tan^{-1}(\sin\alpha)$$

$$= 2\left[\tan^{-1}\left(\frac{1}{\sin\alpha}\right) + \tan^{-1}(\sin\alpha)\right]$$

$$= 2\left(-\frac{\pi}{2}\right), \text{ as } \sin\alpha < 0$$

$= \pi$

32. Given, $\sin^{-1}x > \cos^{-1}x$

$$\Rightarrow 2\sin^{-1}x > \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}x > \frac{\pi}{4}$$

$$\Rightarrow x > \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Hence, the solution is $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$

33. Given, $\cos^{-1}x > \sin^{-1}x$

$\Rightarrow \sin^{-1}x + \cos^{-1}x > 2\sin^{-1}x$

$$\Rightarrow 2\sin^{-1}x < \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}x < \frac{\pi}{4}$$

$$\Rightarrow x < \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Hence, the solution set is $x \in \left[-1, \frac{1}{\sqrt{2}}\right)$

34. Let $a = \cot^{-1}x$

The given in-equation reduces to

$a^2 - 5a + 6 > 0$

$\Rightarrow (a-2)(a-3) > 0$

$\Rightarrow a < 2$ and $a > 3$

$\Rightarrow \cot^{-1}x < 2$ and $\cot^{-1}x > 3$

$\Rightarrow x > \cot(2)$ and $x > \cot(3)$

Hence, the solution set is

$(-\infty, \cot(2)) \cup (\cot 3, \infty)$

35. $$\tan^2\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) > 1$$

$$\Rightarrow \left\{\tan\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)\right\}^2 > 1$$

$$\Rightarrow \quad \frac{x^2}{(1-x^2)} > 1$$

$$\Rightarrow \quad \frac{x^2}{(1-x^2)} - 1 > 0$$

$$\Rightarrow \quad \frac{x^2 - 1 + x^2}{(1-x^2)} > 0$$

$$\Rightarrow \quad \frac{2x^2 - 1}{(x^2-1)} < 0$$

$$\Rightarrow \quad \frac{(\sqrt{2}x+1)(\sqrt{2}x-1)}{(x+1)(x-1)} < 0$$

$$\Rightarrow \quad x \in \left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(1, \frac{1}{\sqrt{2}}\right)$$

36. Given, $4(\tan^{-1} x)^2 - 8(\tan^{-1} x) + 3 < 0$

$\Rightarrow \quad 4a^2 - 8a + 3, a = \tan^{-1} x$

$\Rightarrow \quad 4a^2 - 6a - 2a = 3 < 0$

$\Rightarrow \quad 2a(2a-3) - 1(2a-3) < 0$

$\Rightarrow \quad (2a-1)(2a-3) < 0$

$$\Rightarrow \quad \frac{1}{2} < a < \frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{2} < \tan^{-1} x < \frac{3}{2}$$

$$\Rightarrow \quad \tan\left(\frac{1}{2}\right) < x < \tan\left(\frac{3}{2}\right)$$

37. Given, $4 \cot^{-1} x - (\cot^{-1} x)^2 - 3 \geq 0$

$\Rightarrow \quad 4a - a^2 - 3 \geq 0, a = \cot^{-1} x$

$\Rightarrow \quad a^2 - 4a + 3 \leq 0$

$\Rightarrow \quad (a-1)(a-3) \leq 0$

$\Rightarrow \quad 1 \leq a \leq 3$

$\Rightarrow \quad 1 \leq \cot^{-1} \leq x \leq 3$

$\Rightarrow \quad \cot(3) \leq x \leq \cot(1)$

38. Given in equation is

$$\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 2$$

$$\Rightarrow \quad \sin^{-1}\left(\sin\left(\frac{2(x^2+1)+2}{x^2+1}\right)\right) < \pi - 2$$

$$\Rightarrow \quad \sin^{-1}\left(\sin\left(2 + \frac{2}{x^2+1}\right)\right) < \pi - 2$$

$$\Rightarrow \quad \sin^{-1}\left(\sin\left(\pi - \frac{2x^2+4}{x^2+1}\right)\right) < \pi - 2$$

$$\Rightarrow \quad \left(\pi - \frac{2x^2+4}{x^2+1}\right) < \pi - 2$$

$$\Rightarrow \quad \left(\frac{2x^2+4}{x^2+1}\right) > 2$$

$$\Rightarrow \quad \left(\frac{x^2+2}{x^2+1}\right) > 1$$

which is a true statement.
Hence, $x \in R$

39. We have

$f(x) = \{\sin^{-1}(\sin x)\}^2 - \sin^{-1}(\sin x)$

$$= \left\{\sin^{-1}(\sin x) - \frac{1}{2}\right\}^2 - \frac{1}{4}$$

$$= \left\{\frac{\pi}{2} + \frac{1}{2}\right\}^2 - \frac{1}{4}$$

$= \dfrac{\pi}{4}(\pi + 2)$ ), since the maximum value of $\sin^{-1}(\sin x)$ is $-\dfrac{\pi}{2}$

40. Clearly, both terms are positive.
Applying, AM ≥ GM, we get,

$$\Rightarrow \quad \frac{8^{\sin^{-1}x} + 8^{\cos^{-1}x}}{2} \geq \sqrt{8^{\sin^{1-}x} \cdot 8^{\cos^{-1}x}}$$

$$\Rightarrow \quad \frac{f(x)}{2} \geq \sqrt{8^{\sin^{-1}x + \cos^{-1}x}}$$

$$\Rightarrow \quad \frac{f(x)}{2} \geq \sqrt{8^{\frac{p}{2}}}$$

$$\Rightarrow \quad f(x) \geq 2\sqrt{8^{\frac{\pi}{2}}} = 2.8^{\frac{\pi}{4}} = 2.2^{\frac{3\pi}{4}} = 2^{1+\frac{3\pi}{4}}$$

Hence, the minimum value of $2^{1+\frac{3\pi}{4}}$

41. Given in equation is

$x^2 - kx + \sin^{-1}(\sin 4) > 0$

$\Rightarrow \quad x^2 - kx + \sin^{-1}(\sin(\pi - 4)) > 0$

$\Rightarrow \quad x^2 - kx + (\pi - 4) > 0$

$\Rightarrow$ For, all $x$ in $R$, $D \geq 0$

$\Rightarrow \quad k^2 - 4(\pi - 4) \geq 0$

$\Rightarrow \quad k^2 \geq 4(\pi - 4)$

So, no real values of k satisfies the above in equation.
Hence, the solution is $k = \varphi$

42. Given,

$$A = 2\tan^{-1}(2\sqrt{2} - 1)$$

$\Rightarrow \quad A = 2\tan^{-1}(2.8 - 1) = 2\tan^{-1}(1.4)$

$$\Rightarrow \quad A > \frac{2\pi}{3}$$

and $\quad B = 3\sin^{-1}\left(\dfrac{1}{3}\right) + \sin^{-1}\left(\dfrac{3}{5}\right)$

$$\Rightarrow \quad B = \sin^{-1}\left(\frac{3}{3} - \frac{4}{27}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$\Rightarrow \quad B = \sin^{-1}\left(\frac{23}{27}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$\Rightarrow \quad B < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$\Rightarrow \quad B < \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

Hence, $A > B$

43. We have

$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right)+\cos^{-1}\left(\frac{\sqrt{12}}{4}\right)+\sec^{-1}(\sqrt{2})\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}(\sqrt{2})\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sin\left(\frac{\pi}{12}\right)\right)+\frac{\pi}{6}+\frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\frac{\pi}{12}+\frac{\pi}{6}+\frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left(\frac{6\pi}{12}\right)\right\}$$

$$= \sin^{-1}(0) = 0$$

44. Given,

$f(x) = \sin^{-1}(\cos^{-1} x + \tan^{-1} x + \cot^{-1} x)$

$= \sin^{-1}\left(\frac{\pi}{2} + \cos^{-1}x\right)$

Thus, $-1 \le \left(\frac{\pi}{2} + \cos^{-1}x\right) \le 1$

$\Rightarrow \quad -1 - \frac{\pi}{2} \le \cos^{-1}x \le 1 - \frac{\pi}{2}$

But the ranges of $\cos^{-1} x$ is $[0, \pi]$

So it has no solution

Therefore, $D_f = \varphi$

45. Clearly, $0 \le x \le 4$

We have

$$\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)+\sin^{-1}\left(\sqrt{1-\frac{x}{4}}\right)+\tan^{-1}y = \frac{2p}{3}$$

$\Rightarrow \quad \sin^{-1}\left(\frac{\sqrt{x}}{2}\right)+\cos^{-1}\left(\frac{\sqrt{x}}{2}\right)+\tan^{-1}(y) = \frac{2\pi}{3}$

$\Rightarrow \quad \frac{\pi}{2} + \tan^{-1}(y) = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}(y) = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$

$\Rightarrow \quad y = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

Hence, the maximum value of $(x^2 + y^2 + 1)$ is

$$= \left(16 + \frac{1}{3} + 1\right) = \frac{52}{3}$$

46. Given, $\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{1}{10}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{x}+\frac{1}{y}}{1-\frac{1}{x}\cdot\frac{1}{y}}\right) = \tan^{-1}\left(\frac{1}{10}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{xy-1}\right) = \tan^{-1}\left(\frac{1}{10}\right)$

$\Rightarrow \quad \left(\frac{x+y}{xy-1}\right) = \left(\frac{1}{10}\right)$

$\Rightarrow \quad 10(x + y) = xy - 1$

$\Rightarrow \quad y(10 - x) = -1 - 10x$

$\Rightarrow \quad y = \left(\frac{1+10x}{10-x}\right)$

Thus, there are four ordered pairs

(11, 111), (111, 11), (9, –91), (–91, 9)

satisfying the above equations.

47. Given, $\left[\cot\left(\sum_{k=1}^{10}\cot^{-1}(k^2+k+1)\right)\right] = \frac{a}{b}$

$\Rightarrow \quad \frac{a}{b} = \left[\cot\left(\sum_{k=1}^{10}\cot^{-1}(k^2+k+1)\right)\right]$

$$= \left[\cot\left(\sum_{k=1}^{10}\tan^{-1}\left(\frac{1}{1+k+k^2}\right)\right)\right]$$

$$= \left[\cot\left(\sum_{k=1}^{10}\tan^{-1}\left(\frac{(k+1)-k}{1+(k+1)k}\right)\right)\right]$$

$$= \left[\cot\left(\sum_{k=1}^{10}[\tan^{-1}(k+1) - \tan^{-1}(k)]\right)\right]$$

$$= [\cot(\tan^{-1}(11) - \tan^{-1}(1))]$$

$$= \left[\cot\left(\tan^{-1}\left(\frac{11-1}{1+11}\right)\right)\right]$$

$$= \left[\cot\left(\tan^{-1}\left(\frac{5}{6}\right)\right)\right]$$

$$= \left[\cot\left(\cot^{-1}\left(\frac{6}{5}\right)\right)\right]$$

$$= \frac{6}{5}$$

Hence, the value of $(a + b + 10)$ is 21.

48. We have

$$\tan^{-1}\left(\frac{p-q}{1+pq}\right) + \tan^{-1}\left(\frac{q-r}{1+qr}\right) + \tan^{-1}\left(\frac{r-p}{1+rp}\right)$$

$$= (\tan^{-1} p - \tan^{-1} q) + (\tan^{-1} q - \tan^{-1} r) + p + (\tan^{-1} r - \tan^{-1} p)$$

$$= \pi$$

49. Given equation is

$$(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = \pi a^3$$

$$\Rightarrow \quad \frac{\pi}{2}[(\sin^{-1}x)^2 + (\cos^{-1}x)^2 - 2\sin^{-1}x\cos^{-1}x] = \pi a^3$$

$$\Rightarrow \quad \left[\frac{\pi^2}{4} - 3\sin^{-1}x\cos^{-1}x\right] = 2a\pi^2$$

$$\Rightarrow \quad \left[\pi^2 - 12b\left(\frac{\pi}{2} - b\right)\right] = 8a\pi^2$$

$$\Rightarrow \quad [\pi^2 - 6b\pi + 12b^2] = 8a\pi^2$$

$$\Rightarrow \quad \left(b^2 - \frac{b\pi}{2} + \frac{\pi^2}{12}\right) = \frac{8a\pi^2}{12} = \frac{2a\pi^2}{3}$$

$$\Rightarrow \quad \left(b - \frac{\pi}{4}\right)^2 = \frac{p^2}{12}(8a-1) + \frac{\pi^2}{16}$$

$$\Rightarrow \quad \left(b - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{48}(32a-1)$$

$$\Rightarrow \quad \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{48}(32a-1)$$

As we know that,

$$-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{\pi}{2} - \frac{\pi}{4} \le \sin^{-1}x - \frac{\pi}{4} \le \frac{\pi}{2} - \frac{\pi}{4}$$

$$\Rightarrow \quad -\frac{3\pi}{4} \le \left(\sin^{-1}x - \frac{\pi}{4}\right) \le \frac{\pi}{4}$$

$$\Rightarrow \quad 0 \le \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 \le \frac{9\pi^2}{16}$$

$$\Rightarrow \quad 0 \le \frac{\pi^2}{48}(32a-1) \le \frac{9\pi^2}{16}$$

$$\Rightarrow \quad 0 \le (32a-1) \le 27$$

$$\Rightarrow \quad 1 \le 32a \le 28$$

$$\Rightarrow \quad \frac{1}{32} \le a \le \frac{7}{8}$$

50. Let $g(x) = \dfrac{x^2}{x^2+1} = 1 - \dfrac{1}{1+x^2}$

Clearly, $R_g = [0, 1)$

Now, $R_f = (f(1), f(0)] = (\cot^{-1}(1), \cot^{-1}(1)]$

$$= \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$$

Hence, the value of $\left(\dfrac{b}{a} + 2\right)$

$$= \left(\frac{\frac{\pi}{2}}{\frac{\pi}{4}} + 2\right) = 2 + 2 = 4$$

51. Given $\tan^{-1} y = 4\tan^{-1} x$

$$\Rightarrow \quad \tan^{-1}y = \tan^{-1}\left(\frac{4x - 4x^3}{1 - 6x^2 + x^4}\right)$$

$$\Rightarrow \quad y = \left(\frac{4x - 4x^3}{1 - 6x^2 + x^4}\right)$$

$$\Rightarrow \quad \frac{1}{y} = \left(\frac{1 - 6x^2 + x^4}{4x - 4x^3}\right)$$

$$\Rightarrow \quad \frac{1}{y} = \cot(4\theta), \text{ where, } x = \tan\theta$$

$\Rightarrow$ Clearly, $\dfrac{1}{y} = 0$ is zero only when $\theta = \dfrac{\pi}{8}$

Hence, $x^4 - 6x^2 + 1 = 0$

52. Let $x = \sqrt{\dfrac{a(a+b+c)}{bc}}$, $y = \sqrt{\dfrac{b(a+b+c)}{ac}}$

and $z = \sqrt{\dfrac{c(a+b+c)}{ab}}$

Now, $x + y + z - xyz$

$$= \sqrt{(a+b+c)}\left(\sqrt{\frac{a}{bc}} + \sqrt{\frac{b}{ca}} + \sqrt{\frac{c}{ab}}\right) - \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$$

$$= \sqrt{(a+b+c)}\left(\frac{a+b+c}{\sqrt{abc}}\right) - \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$$

$$= \frac{(a+b+c)^{3/2}}{\sqrt{abc}} - \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$$

$$= 0$$

Now, $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$

$$= \tan^{-1}\left(\frac{x + y + z - xyz}{1 - xy - yz - zx}\right) = \tan^{-1}(0) = \pi$$

Hence, the result.

53. We have

$$\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{6\tan\theta}{9 + \tan^2\theta}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{2\left(\frac{\tan\theta}{3}\right)}{1 + \frac{1}{3}\tan^2\theta}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2} \times 2\tan^{-1}\left(\frac{\tan\theta}{3}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{\tan\theta}{3}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2\tan^2\theta - \frac{\tan\theta}{3}}{1 + 2\tan^2\theta \cdot \frac{\tan\theta}{3}}\right)$$

$\Rightarrow \quad \tan(\theta) = \dfrac{6\tan^2\theta - \tan\theta}{3 + 2\tan^3\theta}$

$\Rightarrow \quad 3\tan(\theta) + 2\tan^4\theta - 6\tan^2\theta = 2\tan\theta = 0$

$\Rightarrow \quad 2\tan^4\theta - 6\tan^2\theta + 5\tan\theta = 0$

$\Rightarrow \quad \tan\theta(2\tan^3\theta - 6\tan\theta + 5) = 0$

$\Rightarrow \quad \tan\theta = 0, (2\tan^2\theta - 6\tan\theta + 5) = 0$

$\Rightarrow \quad \theta = n\pi, n \in I$

54. We have

$$\tan^{-1}\left(\frac{x\cos\theta}{1 - x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{x\cos\theta}{1 - x\sin\theta}\right) - \tan^{-1}\left(\frac{x - \sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{\dfrac{x\cos\theta}{1 - x\sin\theta} - \dfrac{x - \sin\theta}{\cos\theta}}{1 + \dfrac{x\cos\theta}{1 - x\sin\theta} \times \dfrac{x - \sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{x\cos^2\theta - x + x^2\sin\theta + \sin\theta - x\sin^2\theta}{\cos\theta(1 - x\sin\theta) + x\cos\theta(x - \sin\theta)}\right)$$

$$= \tan^{-1}\left(\frac{(x^2 + 1)\sin\theta - 2x\sin^2\theta}{(x^2 + 1)\cos\theta - x\sin 2\theta}\right)$$

$$= \tan^{-1}\left(\frac{(x^2 - 2x\sin\theta + 1)\sin\theta}{(x^2 - 2x\sin\theta + 1)\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta) = \theta$$

55. Given equation is

$$\cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right) + \sin^{-1}\left(\frac{2x}{x^2 + 1}\right) + \tan^{-1}\left(\frac{2x}{x^2 - 1}\right) = \frac{2\pi}{3}$$

$\Rightarrow \quad \pi - 2\tan^{-1}x + 2\tan^{-1}x - 2\tan^{-1}x = \dfrac{2\pi}{3}$

$\Rightarrow \quad \pi - 2\tan^{-1}x = \dfrac{2\pi}{3}$

$\Rightarrow \quad 2\tan^{-1}x = \dfrac{\pi}{3}$

$\Rightarrow \quad \tan^{-1}x = \dfrac{\pi}{6}$

$\Rightarrow \quad x = \tan\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{3}}$

Hence, the solution is $x = \tan\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{3}}$

56. We have

$$\tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{zx}{yr}\right) + \tan^{-1}\left(\frac{xy}{zr}\right)$$

$$= \tan^{-1}\left(\frac{\dfrac{yz}{xr} + \dfrac{zx}{yr} + \dfrac{xy}{zr} - \dfrac{xyz}{r^3}}{1 - \left(\dfrac{z^2}{r^2} + \dfrac{x^2}{r^2} + \dfrac{y^2}{r^2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\dfrac{yz}{xr} + \dfrac{zx}{yr} + \dfrac{xy}{zr} - \dfrac{xyz}{r^3}}{1 - \left(\dfrac{x^2 + y^2 + z^2}{r^2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\dfrac{yz}{xr} + \dfrac{zx}{yr} + \dfrac{xy}{zr} - \dfrac{xyz}{r^3}}{1 - 1}\right)$$

$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

57. Given, $\displaystyle\sum_{r=1}^{10}\tan^{-1}\left(\frac{3}{9r^2 + 3r - 1}\right)$

$$= \sum_{r=1}^{10}\tan^{-1}\left(\frac{3}{1 + (3r + 2)(3r - 1)}\right)$$

$$= \sum_{r=1}^{10}\tan^{-1}\left(\frac{(3r + 2) - (3r - 1)}{1 + (3r + 2)(3r - 1)}\right)$$

$$= \sum_{r=1}^{10}[\tan^{-1}(3r + 2) - \tan^{-1}(3r - 1)]$$

$$= \tan^{-1}(32) - \tan^{-1}(2)$$

$$= \tan^{-1}\left(\frac{32 - 2}{1 + 32.2}\right)$$

$$= \tan^{-1}\left(\frac{30}{65}\right) = \tan^{-1}\left(\frac{6}{13}\right)$$

$$= \cot^{-1}\left(\frac{13}{6}\right)$$

Hence, the value of $(2m + n + 4)$

$= 26 + 6 + 4$

$= 36$

58. We have

$$S = \sum_{b=1}^{10}\sum_{a=1}^{10}\tan^{-1}\left(\frac{a}{b}\right)$$

$$= \sum_{b=1}^{10}\left[\tan^{-1}\left(\frac{1}{b}\right) + \tan^{-1}\left(\frac{2}{b}\right) + \tan^{-1}\left(\frac{3}{b}\right) + \ldots + \tan^{-1}\left(\frac{10}{b}\right)\right]$$

$$= \tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \ldots + \tan^{-1}\left(\frac{1}{10}\right)$$

$$+ \tan^{-1}\left(\frac{2}{1}\right) + \tan^{-1}\left(\frac{2}{2}\right) + \tan^{-1}\left(\frac{2}{3}\right) + \ldots + \tan^{-1}\left(\frac{2}{10}\right)$$

$$+ \tan^{-1}\left(\frac{3}{1}\right) + \tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{3}{3}\right) + \ldots + \tan^{-1}\left(\frac{3}{10}\right)$$

$\vdots$

$$+ \tan^{-1}\left(\frac{10}{1}\right) + \tan^{-1}\left(\frac{10}{2}\right) + \tan^{-1}\left(\frac{10}{3}\right) + \ldots + \tan^{-1}\left(\frac{10}{10}\right)$$

$$= 10 \times \frac{\pi}{4} + 45 \times \frac{\pi}{2}$$

$$= 25\pi$$

Hence, the value of $(m + 4)$ is 29.

59. We have

$$f(x)=\frac{1}{\pi}(\sin^{-1}x+\cos^{-1}x+\tan^{-1}x)+\frac{(x+1)}{x^2+2x+10}$$

It will provide us the max value at $x = 1$

$$f(1)=\frac{1}{\pi}\left(\frac{\pi}{2}+\tan^{-1}(1)\right)+\frac{2}{13}$$

$$=\frac{1}{ð}\times\frac{3\pi}{4}+\frac{2}{13}$$

$$=\frac{3}{4}+\frac{2}{13}=\frac{39+8}{52}=\frac{47}{52}$$

Hence, the value of $(104m - 90)$ is 4.

60. We have

$\sin(2x) + \cos(2x) + \cos x + 1 = 0$

$\sin 2x + (1 + \cos 2x) + \cos x = 0$

Each term of the above equation is positive in $\left(0, \frac{\pi}{2}\right)$.

So it has no solution

Thus, $m = 0$

Also, $n=\sin\left[\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)+\cos^{-1}\left(\cos\left(\frac{7\pi}{3}\right)\right)\right]$

$$=\sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

Hence, the value of

$(m^2 + n^2 + m + n + 4)$

$= 0 + 1 + 0 + 1 + 4$

$= 6$

61. We have

$$f(n)=\sum_{k=-n}^{n}\left(\cot^{-1}\left(\frac{1}{k}\right)-\tan^{-1}(k)\right)$$

$$=\sum_{k=-n}^{-1}\left(\cot^{-1}\left(\frac{1}{k}\right)-\tan^{-1}(k)\right)+\sum_{k=1}^{n}\left(\cot^{-1}\left(\frac{1}{k}\right)-\tan^{-1}(k)\right)$$

$$=\sum_{k=-n}^{-1}(\tan^{-1}(k)+\pi-\tan^{-1}(k))+\sum_{k=1}^{n}(\tan^{-1}(k)-\tan^{-1}(k))$$

$$=\sum_{k=-n}^{-1}(\pi)+0$$

$$=n\pi$$

Now, $\sum_{n=2}^{10}(f(n)+f(n-1))$

$$=\sum_{n=2}^{10}(n\pi+(n-1)\pi)$$

$$=\sum_{n=2}^{10}((2n-1)\pi)$$

$= (3 + 5 + 7 + 9 + \ldots + 19)\pi$

$= (1 + 3 + 5 + 7 + 9 + \ldots + 19)\pi - \pi$

$= (10^2)\pi - \pi$

$= 99\pi$

Hence, the value of $(a + 1)$ is 100.

## Integer Type Questions

1. Given, $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3$

$$\Rightarrow \frac{2x^2+4}{x^2+1}>3$$

$$\Rightarrow \frac{2x^2+4}{x^2+1}-3>0$$

$$\Rightarrow \frac{2x^2+4-3x^2-3}{x^2+1}>0$$

$$\Rightarrow \frac{1-x^2}{x^2+1}>0$$

$$\Rightarrow \frac{x^2-1}{x^2+1}<0$$

$\Rightarrow x \in (-1, 1)$

Hence, the value of $(b - a + 5) = 1 + 1 + 5 = 7$.

2. Given, $a\sin^{-1}x - b\cos^{-1}x = c$

$$\Rightarrow a\sin^{-1}x-b\left(\frac{\pi}{2}-\sin^{-1}x\right)=c$$

$$\Rightarrow (a+b)\sin^{-1}x=c+\frac{b\pi}{2}$$

$$\Rightarrow \sin^{-1}x=\frac{2c+b\pi}{2(a+b)}$$

Now, $\cos^{-1}x=\frac{\pi}{2}-\frac{2c+b\pi}{2(a+b)}$

$$\Rightarrow \cos^{-1}x=\frac{(a+b)\pi-2c-b\pi}{2(a+b)}$$

$$\Rightarrow \cos^{-1}x=\frac{a\pi-2c}{2(a+b)}$$

Now, $a\sin^{-1}x - b\cos^{-1}x$

$$=a\left(\frac{2c+b\pi}{2(a+b)}\right)+b\left(\frac{a\pi-2c}{2(a+b)}\right)$$

$$=\left(\frac{2ac+ab\pi}{2(a+b)}\right)+\left(\frac{ab\pi-2bc}{2(a+b)}\right)$$

$$=\left(\frac{2ac+ab\pi+ab\pi-2bc}{2(a+b)}\right)$$

$$=\left(\frac{ab\pi+c(a-b)}{(a+b)}\right)$$

Clearly, $m = 1$

Hence, the value of $(m^2 + m + 2)$ is 4

3. Since sum of the roots is negative and product of the roots is positive.
So, both roots are negative
Thus $m$ is a negative root
Now, $\tan^{-1}(m) + \tan^{-1}\left(\frac{1}{m}\right)$
$= \tan^{-1}(m) - \pi + \cot^{-1}(m)$
$= -\pi + (\tan^{-1}(m) + \cot^{-1}(m))$
$= -\pi + \frac{\pi}{2} = -\frac{\pi}{2}$
Clearly, $k = -1$
Hence, the value of $(k + 4)$ is 3.

4. Given equation is
$$\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$$
It is true only when
$(x^2 - 2x + 1) = x^2 - x$
$\Rightarrow \quad x = 1$
Thus, the number of solutions is 1.

5. Given, $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$
$\Rightarrow \quad \cos^{-1}(3x) + \cos^{-1}(2x) = \pi - \cos^{-1}(x)$
$\Rightarrow \quad \cos^{-1}(3x \cdot 2x - \sqrt{1-9x^2}\sqrt{1-4x^2}) = \cos^{-1}(-x)$
$\Rightarrow \quad (3x \cdot 2x - \sqrt{1-9x^2}\sqrt{1-4x^2}) = (-x)$
$\Rightarrow \quad (6x^2 + x)^2 = (-\sqrt{1-9x^2}\sqrt{1-4x^2})^2$
$\Rightarrow \quad 36x^4 + 12x^3 + x^2 = 1 - 13x^2 + 36x^4$
$\Rightarrow \quad 12x^3 + 14x^2 - 1 = 0$
Thus, $a = 12$, $b = 14$, $c = 0$, $d = -1$
Hence, the value of $(b + c) - (a + d)$
$= 14 - 11 = 3.$

6. Given equation is
$x^3 - x^2 - 3x + 4 = 0$
$\Rightarrow \quad \alpha + \beta + \gamma = 1, \alpha\beta + \beta\gamma + \gamma\alpha = -3, \alpha\beta\gamma = -4$
It is given that,
$\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = 0$
$\Rightarrow \quad \tan^{-1}\left(\frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-(\alpha\beta+\beta\gamma+\gamma\alpha)}\right) = \theta$
$\Rightarrow \quad \tan^{-1}\left(\frac{1+4}{1+3}\right) = \theta$
$\Rightarrow \quad \tan\theta = \frac{5}{4}$
Hence, the value of $(p + q)$ is 9.

7. Given equation is
$\cos^{-1}x + \cos^{-1}(2x) + \pi = 0$
$\Rightarrow \quad \cos^{-1}(2x) + \cos^{-1}(x) = -\pi$
$\Rightarrow \quad \cos^{-1}(2x \cdot x - \sqrt{1-x^2}\sqrt{1-4x^2}) = -\pi$
$\Rightarrow \quad (2x^2 - \sqrt{1-x^2}\sqrt{1-4x^2}) = \cos(-\pi) = -1$
$\Rightarrow \quad (2x^2 + 1)^2 = (1 - x^2)(1 - 4x^2)$
$\Rightarrow \quad 4x^4 + 4x^2 + 1 = 1 - 5x^2 + 4x^4$
$\Rightarrow \quad 9x^2 = 0$
$\Rightarrow \quad x = 0$
But $x = 0$ does not satisfy the equation.
So it has no solution.
Therefore $M = 0$
Again, $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$,
$\Rightarrow \quad \sin^{-1}\left(\frac{5}{x}\right) + \cos^{-1}\left(\sqrt{1-\frac{144}{x^2}}\right) = \frac{\pi}{2}$
It is true only when,
$\left(\frac{5}{x}\right) = \sqrt{1-\frac{144}{x^2}}$
$\Rightarrow \quad \frac{25}{x^2} = 1 - \frac{144}{x^2}$
$\Rightarrow \quad \frac{169}{x^2} = 1$
$\Rightarrow \quad x^2 = 169$
$\Rightarrow \quad x = \pm 13$
Clearly, $x = 13$ only satisfies the equation. Thus, $N = 1$.
Hence, the value of $M + N + 4 = 5$.

8. We have
$$4\cos\left[\cos^{-1}\left(\frac{1}{4}(\sqrt{6}-\sqrt{2})\right) - \cos^{-1}\left(\frac{1}{4}(\sqrt{6}+\sqrt{2})\right)\right]$$
$= 4\cos\left[\cos^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) - \cos^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\right]$
$= 4\cos[\cos^{-1}(\cos(75°)) - \cos^{-1}(\cos(15°))]$
$= 4\cos(75° - 15°)$
$= 4\cos(60°)$
$= 2$

9. We have
$$5\cot\left(\sum_{k=1}^{5}\cot^{-1}(k^2 + k + 1)\right)$$
$= 5\cot\left(\sum_{k=1}^{5}\tan^{-1}\left(\frac{1}{1+k+k^2}\right)\right)$
$= 5\cot\left(\sum_{k=1}^{5}\tan^{-1}\left(\frac{(k+1)-\mathrm{k}}{1+(k+1)k}\right)\right)$
$= 5\cot\left(\sum_{k=1}^{5}[\tan^{-1}(k+1) - \tan^{-1}k]\right)$
$= 5\cot\left(\tan^{-1}(6) - \tan^{-1}(1)\right)$
$= 5\cot\left(\tan^{-1}\left(\frac{6-1}{1+6.1}\right)\right)$
$= 5\cot\left(\tan^{-1}\left(\frac{5}{7}\right)\right)$

$$= 5\cot\left(\cot^{-1}\left(\frac{7}{5}\right)\right)$$
$$= 7$$

10. $a = \sin^{-1}(\log_2 x)$ and $b = \cos^{-1}(\log_2 x)$
The given equations reduces to
$$\begin{cases} 3a + b = \dfrac{\pi}{2} \\ a + 2b = \dfrac{11\pi}{6} \end{cases}$$
On solving, we get,
$$a = -\frac{\pi}{6} \text{ and } b = \pi$$
$$\Rightarrow \sin^{-1}(\log_2 x) = -\frac{\pi}{6} \text{ and } \cos^{-1}(\log_2 y) = \pi$$
$$\Rightarrow (\log_2 x) = -\frac{1}{2} \text{ and } (\log_2 y) = -1$$
$$\Rightarrow x = 2^{-\frac{1}{2}} \text{ and } y = 2^{-1}$$
$$\Rightarrow \frac{1}{x} = \sqrt{2} \text{ and } \frac{1}{y} = 2$$
$$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} + 2 = 2 + 4 + 2 = 8$$

11. Here, both roots are negative
Now, $\cot(\cot^{-1}\alpha + \cot^{-1}\beta)$
$$= \cot\left(\pi + \tan^{-1}\left(\frac{1}{\alpha}\right) + \pi + \tan^{-1}\left(\frac{1}{\beta}\right)\right)$$
$$= \cot\left(2\pi + \tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\left(\frac{1}{\beta}\right)\right)$$
$$= \cot\left(\tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\left(\frac{1}{\beta}\right)\right)$$
$$= \cot\left(\tan^{-1}\left(\frac{\alpha+\beta}{\alpha\beta - 1}\right)\right)$$
$$= \cot\left(\cot^{-1}\left(\frac{\alpha\beta - 1}{\alpha+\beta}\right)\right)$$
$$= \cot\left(\cot^{-1}\left(\frac{\alpha\beta - 1}{\alpha+\beta}\right)\right)$$
$$= \left(\frac{44+1}{5}\right) = 9$$

12. Given equation is
$$\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{1}{7}\right)$$
$$\Rightarrow \tan^{-1}\left(\frac{x+y}{xy-1}\right) = \tan^{-1}\left(\frac{1}{7}\right)$$
$$\Rightarrow \left(\frac{x+y}{xy-1}\right) = \left(\frac{1}{7}\right)$$
$$\Rightarrow 7x + 7y = xy - 1$$
$$\Rightarrow y = \left(\frac{7x+1}{x-7}\right)$$
Hence, the possible ordered pairs are
(8, 57), (9, 32), (12, 17), (17, 12), (32, 9), (57, 8)
Thus, the number of ordered pairs is 6.

## Previous Years' JEE-Advanced Examinations

1. Now,
$$\tan^{-1}\left(\sqrt{\frac{a(a+b+c)}{bc}}\right) + \tan^{-1}\left(\sqrt{\frac{b(a+\ +c)}{ca}}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{\frac{a(a+b+c)}{bc}} + \sqrt{\frac{b(a+\ +c)}{ca}}}{1 - \sqrt{\frac{a(a+b+c)}{bc}} \cdot \sqrt{\frac{b(a+\ +c)}{ca}}}\right)$$
$$= \tan^{-1}\left(\frac{\left(\frac{(a+b+c)}{\sqrt{c}}\right) + \left(\frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}}\right)}{1 - \sqrt{\frac{(a+b+c)}{c}} \cdot \sqrt{\frac{(a+c)}{c}}}\right)$$
$$= \tan^{-1}\left(\frac{\left(\frac{(a+b+c)}{\sqrt{c}}\right)\left(\frac{a+b}{\sqrt{ab}}\right)}{\left(\frac{c-a-b-c}{\sqrt{c}}\right)}\right)$$
$$= \tan^{-1}\left(\frac{\left(\frac{a+b+c}{\sqrt{c}}\right)\left(\frac{a+b}{\sqrt{ab}}\right)}{\left(-\frac{a+b}{\sqrt{c}}\right)}\right)$$
$$= \tan^{-1}\left(-\sqrt{\frac{c(a+b+c)}{ab}}\right)$$
$$= -\tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right)$$
Therefore, $\theta$
$$= -\tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right) + \tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{a}}\right)$$
$$= 0$$

2. Given, $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$
$$= \tan\left(\tan^{-1}\left(\frac{\frac{2}{5}}{1 - \left(\frac{1}{5}\right)^2}\right) - \frac{\pi}{4}\right)$$
$$= \tan\left(\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}(1)\right)$$
$$= \tan\left(\tan^{-1}\left(\frac{\frac{5}{12} - 1}{1 + \frac{5}{12}\cdot 1}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(-\frac{7}{17}\right)\right)$$

$$= \left(-\frac{7}{17}\right)$$

3. Now, $\cos(2\cos^{-1}x + \sin^{-1}x)$

$$= \cos(\cos^{-1}x + \sin^{-1}x + \cos^{-1}x)$$

$$= \cos\left(\frac{\pi}{2} + \cos^{1}x\right)$$

$$= -\sin(\cos^{-1}x)$$

$$= -\sin(\sin^{-1}\sqrt{1-x^2})$$

$$= -\sqrt{1-x^2}$$

When $x = 1/5$, then the value of the given expression is

$$-\left(\sqrt{1-\frac{1}{25}}\right) = -\frac{2\sqrt{4}}{5}$$

4. Given, $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$

$$= \tan\left(\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right)$$

$$= \left(\frac{9+8}{12-6}\right)$$

$$= -\frac{17}{6}$$

5. No questions asked in between 1984 and 1985

6. Given, $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

$$= \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$$

$$= \left(\frac{\pi}{3}\right)$$

7. No questions asked in between 1987 and 1988

8. We have

$$A = 2\tan^{-1}(2\sqrt{2}-1)$$

$$> 2\tan^{-1}(\sqrt{3}) = \frac{2\pi}{3}$$

and $B = 3\tan^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$$= \sin^{-1}\left(3\left(\frac{1}{3}\right) - 4\left(\frac{1}{3}\right)^3\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\left(\frac{23}{27}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$< \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2\pi}{3}$$

Thus, $A > B$

10. Ans. (c)

Given,

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right) + \sin^{-1}(\sqrt{x^2+x+1}) = \frac{\pi}{2}$$

Thus, $\left(\frac{1}{\sqrt{x^2+x+1}}\right) = (\sqrt{x^2+x+1})$

$$\Rightarrow (\sqrt{x^2+x+1})^2 = 1$$

$$\Rightarrow x^2 + x + 1 = 1$$

$$\Rightarrow x^2 + x = 0$$

$$\Rightarrow x(x+1) = 0$$

$$\Rightarrow x = 0, -1$$

11. No questions asked in 2000.

12. We have

$$\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\right) = \left(x - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\right)$$

$$\Rightarrow x\left(1 - \frac{x}{2} + \frac{x^2}{4} - \ldots\right) = x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{4} - \ldots\right)$$

$$\Rightarrow x\left(\frac{1}{1-\left(-\frac{x}{2}\right)}\right) = x^2\left(\frac{1}{1-\left(-\frac{x^2}{2}\right)}\right)$$

$$\Rightarrow x\left(\frac{1}{1+\left(\frac{x}{2}\right)}\right) = x^2\left(\frac{1}{1+\left(\frac{x^2}{2}\right)}\right)$$

$$\Rightarrow x\left(1+\frac{x^2}{2}\right) = x^2\left(1+\frac{x}{2}\right)$$

$$\Rightarrow x(2+x^2) = x^2(2+x)$$

$$\Rightarrow (2x + x^3) = (2x^2 + x^3)$$

$$\Rightarrow x^2 = x$$

$$\Rightarrow x(x-1) = 0$$

$$\Rightarrow x = 0, 1$$

Hence, the solution set is $\{0, 1\}$

13. We have

$\cos(\tan^{-1}(\sin(\cot^{-1}x)))$

$$= \cos\left(\tan^{-1}\left(\sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)\right)$$

$$= \cos\left(\tan^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)$$

$$= \cos\left(\cos^{-1}\left(\sqrt{\frac{x^2+1}{x^2+2}}\right)\right)$$

$$= \left(\sqrt{\frac{x^2+1}{x^2+2}}\right)$$

14. Ans. (d)

Given, $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$

It is defined for, $\sin^{-1}(2x) + \frac{\pi}{6} \geq 0$

$$\sin^{-1}(2x) \geq -\frac{\pi}{6}$$

$$(2x) \geq \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$x \geq -\frac{1}{4}$$

Also, $-1 \leq 2x \leq 1$

$$-\frac{1}{2} \leq x \leq \frac{1}{2}$$

Thus, the domain of the given function $= \left[-\frac{1}{4}, \frac{1}{2}\right]$

15. Given, $\sin(\cot^{-1}(x+1) = \cos(\tan^{-1} x)$

$\Rightarrow \quad \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+2x+2}}\right)\right)$

$= \cos\left(\cos^{-1}\frac{1}{\sqrt{x^2+1}}\right)$

$\Rightarrow \quad \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{x^2+1}}$

$\Rightarrow \quad \sqrt{x^2+2x+2} = \sqrt{x^2+1}$

$\Rightarrow \quad x^2 + 2x + 2 = x^2 + 1$

$\Rightarrow \quad 2x + 2 = 1$

$\Rightarrow \quad x = -\frac{1}{2}$

16. No questions asked in between 2005 and 2006.

17. Given, $\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$

(A) when $a = 1$, $b = 0$, then

$\sin^{-1}(x) + \cos^{-1}(y) + \cos^{-1}(0) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}(x) + \cos^{-1}(y) + \frac{\pi}{2} = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}(x) + \cos^{-1}(y) = 0$

$\Rightarrow \quad \sin^{-1}(x) = -\cos^{-1}(y) = -\sin^{-1}\left(\sqrt{1-y^2}\right)$

$\Rightarrow \quad x = -\sqrt{1-y^2}$

$\Rightarrow \quad x^2\ 1 - y^2$

$\Rightarrow \quad x^{23} + y^2 = 1$

Ans. (P)

(B) When $a = 1$, $b = 1$, then

$\sin^{-1} x + \cos^{-1} y + \cos^{-1}(xy) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \cos^{-1} x + \cos^{-1} y + \cos^{-1}(xy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1} x = \cos^{-1} y + \cos^{-1}(xy)$

$\Rightarrow \quad \cos^{-1} x - \cos^{-1} y = \cos^{-1}(xy)$

$\Rightarrow \quad \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos^{-1}(xy)$

$\Rightarrow \quad (xy - \sqrt{1-x^2}\sqrt{1-y^2}) = (xy)$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} = 0$

$\Rightarrow \quad (1-x^2)(1-y^2) = 0$

Ans. (Q)

(C) When $a = 1$, $b = 2$, then

$\Rightarrow \quad \sin^{-1} x + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \cos^{-1} x + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1} x = \cos^{-1} y + \cos^{-1}(2xy)$

$\Rightarrow \quad \mathrm{c}\cos^{-1} x - \cos^{-1} y = \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos^{-1}(2xy)$

$\Rightarrow \quad (xy - \sqrt{1-x^2}\sqrt{1-y^2}) = (2xy)$

$\Rightarrow \quad (\sqrt{1-x^2}\sqrt{1-y^2})^2 = (-xy)^2$

$\Rightarrow \quad (1-x^2)(1-y^2) = x^2y^2$

$\Rightarrow \quad 1 - x^2 - y^2 + x^2y^2 = x^2y^2$

$\Rightarrow \quad x^2 + y^2 = 1$

Ans. (P)

(D) When $a = 2$, $b = 2$, then

$\Rightarrow \quad \sin^{-1}(2x) + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \cos^{-1}(2x) + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}(2x) = \cos^{-1} y + \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{-1}(2x) - \cos^{-1} y = \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{-1}(2x \cdot y - \sqrt{1-4x^2}\sqrt{1-y^2}) = \cos^{-1}(2xy)$

$\Rightarrow \quad (2x \cdot y - \sqrt{1-4x^2}\sqrt{1-y^2}) = (2xy)$

$\Rightarrow \quad (\sqrt{1-4x^2}\sqrt{1-y^2}) = 0$

$\Rightarrow \quad (1-4x^2)(1-y^2) = 0$

Ans. (S)

18. We have

$\sqrt{1+x^2} \times [\{x\cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 - 1]^{1/2}$

$$= \sqrt{1+x^2}\left[\left\{x\cos\left(\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)\right.\right.$$

$$\left.\left.+\sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+}}\right)\right)\right\}^2-1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left(\frac{x^2}{\sqrt{x^2+1}}+\frac{1}{\sqrt{x^2+1}}\right)^2-\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left(\frac{x^2+1}{\sqrt{x^2+1}}\right)^2-1\right]^{1/2}$$

$$= \sqrt{1+x^2}\,[(\sqrt{x^2+1})^2-]^{1/2}$$

$$= \sqrt{1+x^2}\,[(x^2+1)-1]^{1/2}$$

$$= x\sqrt{1+x^2}$$

Ans. (c)

19. No questions asked in between 2009 and 2010.

20. Given, $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

$$= \sin\left(\sin^{-1}\left(\frac{\sin\theta}{\sqrt{\sin^2\theta+\cos 2\theta}}\right)\right)$$

$$= \left(\frac{\sin\theta}{\sqrt{\sin^2\theta+\cos^2\theta-\sin^2\theta}}\right)$$

$$= \left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$$

$$= \left(\frac{\sin\theta}{\cos\theta}\right) = \tan\theta$$

Now, $\frac{d}{d(\tan\theta)}$ $(\tan\theta) = 1$

21. No questions asked in 2012

22. $\sum_{k=1}^{n}(2k) = 2(1+2+3+\ldots+n)$

$$= 2\left(\frac{n(n+1)}{2}\right)$$

$$= (n^2+n)$$

Therefore, $\sum_{n=1}^{23}\cot^{-1}(1+n+n^2)$

$$= \sum_{n=1}^{23}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)$$

$$= \sum_{n=1}^{23}\tan^{-1}\left(\frac{1}{1+(n+1)n}\right)$$

$$= \sum_{n=1}^{23}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)$$

$$= \sum_{n=1}^{23}(\tan^{-1}(n+1)-\tan^{-1}(n))$$

$$= (\tan^{-1}(2)-\tan^{-1}(1)) + (\tan^{-1}(3)-\tan^{-1}(2)) + \ldots + (\tan^{-1}(24)-\tan^{-1}(23))$$

$$= \tan^{-1}(24)-\tan^{-1}(1)$$

$$= \tan^{-1}\left(\frac{24-1}{1+24}\right) = \tan^{-1}\left(\frac{23}{25}\right)$$

Thus the given expression reduces to

$$= \cot\left(\tan^{-1}\left(\frac{23}{25}\right)\right)$$

$$= \cot\left(\cot^{-1}\left(\frac{25}{23}\right)\right)$$

$$= \frac{25}{23}$$

23. We have

$$\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)$$

$$= \left(\frac{\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{y^2+1}}\right)\right)+y\sin\left(\sin^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)\right)}{\cot\left(\cot^{-1}\left(\frac{\sqrt{1-y^2}}{y}\right)\right)+\tan\left(\tan^{-1}\left(\frac{y}{\sqrt{1-y^2}}\right)\right)}\right)$$

$$= \left(\frac{\left(\frac{1}{\sqrt{y^2-1}}\right)+y\left(\frac{y}{\sqrt{1+y^2}}\right)}{\left(\frac{\sqrt{1-y^2}}{y}\right)+\left(\frac{y}{\sqrt{1-y^2}}\right)}\right)$$

$$= \left(\frac{\left(\frac{1+y^2}{\sqrt{y^2+1}}\right)}{\left(\frac{1-y^2+y^2}{y-y^2}\right)}\right)$$

$$= y(\sqrt{1-y^4})$$

Thus, the given expression reduces to

$$= \left(\frac{1}{y^2}(y\sqrt{1-y^4})^2+y^4\right)^{1/2}$$

$$= \left(\frac{y^2(1-y^4)}{y^2}+y^4\right)^{1/2}$$

$$= ((1-y^4)+y^4)^{1/2}$$

$$= 1$$

24. Given, $\cos(\sin^{-1}\sqrt{1-x^2}) = \sin(\tan^{-1}(x\sqrt{6}))$

$$\Rightarrow \quad \cos\left(\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) = \sin\left(\sin^{-1}\left(\frac{x\sqrt{6}}{\sqrt{6x^2+1}}\right)\right)$$

$$\Rightarrow \quad \left(\frac{x}{\sqrt{1-x^2}}\right) = \left(\frac{x\sqrt{6}}{\sqrt{6x^2+1}}\right)$$

$$\Rightarrow \quad \left(\frac{1}{\sqrt{1-x^2}}\right) = \left(\frac{\sqrt{6}}{\sqrt{6x^2+1}}\right)$$

$$\Rightarrow \quad 6(1-x^2) = (6x^2+1)$$

$$\Rightarrow \quad 6 - 6x^2 = 6x^2 + 1$$

$$\Rightarrow \quad 12x^2 = 5$$

$$\Rightarrow \quad x^2 = \frac{5}{12}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{12}{5}}$$

CHAPTER

# 5 Properties of Triangles

## PROPERTIES OF TRIANGLES

### CONCEPT BOOSTER

### 5.1 INTRODUCTION

In any $\Delta$, the three sides and the three angles are generally called the elements of the triangle.

A triangle which does not contain a right angle is called an oblique triangle.

In any $\Delta ABC$, the measures of the angles $\angle BAC$, $\angle CBA$ and $\angle ACB$ are denoted by the letters $A$, $B$ and $C$ respectively and the sides $BC$, $CA$ and $AB$ opposite to the angles $A$, $B$ and $C$ are respectively denoted by $a$, $b$ and $c$. These six elements of a triangle are not independent and are connected by the relations.

(i) $A + B + C = \pi$

(ii) $a + b > c$; $b + c > a$; $c + a > b$

### 5.2 SINE RULE

#### Statement

The sides of a triangle are proportion to the sines of the angles opposite to them

i.e. In a $\Delta ABC$, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

The above rule may also be expressed as $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

### 5.3 COSINE RULE

#### Statement

In any $\Delta ABC$

(i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(ii) $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

(iii) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

### 5.4 PROJECTION FORMULAE

#### Statement

In any $\Delta ABC$

(i) $a = b \cos C + c \cos B$

(ii) $b = c \cos A + a \cos C$

(iii) $c = a \cos B + b \cos A$

### 5.5 NAPIER'S ANALOGY (LAW OF TANGENTS)

#### Statement

In any $\Delta ABC$,

(i) $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\left(\dfrac{A}{2}\right)$

(ii) $\tan\left(\dfrac{C-A}{2}\right) = \left(\dfrac{c-a}{c+a}\right)\cot\dfrac{B}{2}$

(iii) $\tan\left(\dfrac{A-B}{2}\right) = \left(\dfrac{a-b}{a+b}\right)\cot\dfrac{C}{2}$

### 5.6 HALF-ANGLED FORMULAE

In this section, we shall desire to the formulas for the sine, cosine and tangent to the half of the angles of any triangle in terms of its sides. The perimeter of $\Delta ABC$ will be denoted by $2s$

i.e., $a + b + c = 2s$

and its area is denoted by $\Delta$.

i.e., $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

1. Formulae for $\sin\left(\frac{A}{2}\right), \sin\left(\frac{B}{2}\right), \sin\left(\frac{C}{2}\right)$

In any $\Delta ABC$,

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

(ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

(iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

2. Formulae for $\cos\left(\frac{A}{2}\right), \cos\left(\frac{B}{2}\right), \cos\left(\frac{C}{2}\right)$:

In any $\Delta ABC$,

(i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

(iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

3. Formulae for $\tan\left(\frac{A}{2}\right), \tan\left(\frac{B}{2}\right), \tan\left(\frac{C}{2}\right)$:

In any $\Delta ABC$,

(i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

(iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

## 5.7 Area of Triangle

### Statement

Prove that the area of $\Delta ABC$ is given by

(i) $\Delta = \frac{1}{2}bc\sin A$

(ii) $\Delta = \frac{1}{2}ca\sin B$

(iii) $\Delta = \frac{1}{2}ab\sin C$

1. Area of a Triangle (Hero's Formula)

In any $\Delta ABC : \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

## 5.8 *m-n* Theorem

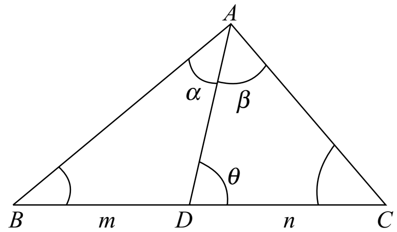
Statement: If $D$ is a point on the side $BC$ such that $BD : DC = m : n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$, then

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) $(m+n)\cot\theta = n\cot B - m\cot C$

*Solution*

(i) Given, $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$




So, $\angle ABD = 180° - (\alpha + 180° - \theta) = (\theta - \alpha)$
and $\angle ACD = 180° - (\theta + \beta)$

From, $\Delta ABD$, $\frac{BD}{\sin\alpha} = \frac{AD}{\sin(\theta-\alpha)}$ …(i)

From $\Delta ADC$, $\frac{DC}{\sin\beta} = \frac{AD}{\sin(180° - (\theta+\beta))}$

$\Rightarrow \frac{DC}{\sin\beta} = \frac{AD}{\sin(\theta+\beta)}$ …(ii)

Dividing (i) and (ii), we get

$$\frac{BD\sin\beta}{DC\sin\alpha} = \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)}$$

$$\Rightarrow \frac{m\sin\beta}{n\sin\alpha} = \frac{\sin\theta\cos\beta + \cos\theta\sin\beta}{\sin\theta\cos\alpha - \cos\theta\sin\alpha}$$

$$\Rightarrow \frac{m}{n} = \frac{(\sin\theta\cos\beta + \cos\theta\sin\beta)\sin\alpha}{(\sin\theta\cos\alpha - \cos\theta\sin\alpha)\sin\beta}$$

Dividing the R.H.S. by $\sin\alpha\sin\beta\sin\theta$, we get,

$$\frac{m}{n} = \frac{\cot\beta + \cot\theta}{\cot\alpha - \cot\theta}$$

$\Rightarrow n(\cot\beta + \cot\theta) = m(\cot\alpha - \cot\theta)$

$\Rightarrow (m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) Given, $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$

Thus, $\angle ADB = 180° - \theta$

Here, $\angle ABD = B$ and $\angle ACD = C$

So, $\angle BAD = 180° - (180° - \theta + B) = \theta - B$ and $\angle DAC = (180° - (\theta + C))$

Now, from $\Delta ABD$, $\frac{BD}{\sin(\theta - B)} = \frac{AD}{\sin B}$ …(i)

and from $\Delta ADC$

$$\frac{DC}{\sin(180° - (\theta + C))} = \frac{AD}{\sin C}$$

$\Rightarrow \frac{DC}{\sin(\theta + C)} = \frac{AD}{\sin C}$ …(ii)

Dividing (i) by (ii), we get

$$\frac{BD}{DC}\cdot\frac{\sin(\theta+C)}{\sin(\theta-B)} = \frac{\sin C}{\sin B}$$

$$\Rightarrow \frac{m}{n}\cdot\frac{\sin(\theta+C)}{\sin(\theta-B)} = \frac{\sin C}{\sin B}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{\sin C \cdot \sin(\theta - B)}{\sin B \cdot \sin(\theta + C)}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{\sin C(\sin\theta\cos B - \cos\theta\sin B)}{\sin B(\sin\theta\cos C + \cos\theta\sin C)}$$

Dividing the numerator and denominator on the right hand side by $\sin B \sin C \sin\theta$, we get,

$$\frac{m}{n} = \frac{\cot B - \cot\theta}{\cot C + \cot\theta}$$

$$\Rightarrow \quad (m + n)\cot\theta = n\cot B - m\cot C$$

Hence, the result.

This completes the proof of the statement.

## 5.9 Radii of Circle Connected with a Triangle

### 5.9.1 Circumcircle of a Triangle and its Radius

The circle which passes through the angular points of a $\Delta$ is called the circumcircle. The centre of this circle is the point of intersection of perpendicular bisectors of the sides and is called the circumcentre and its radius is always denoted by $R$.

The circumcentre may lie within outside or upon one of the sides of the $\Delta$. In a right angled triangle the circumcentre is the vertex whose right angle is formed.

### 5.9.2 Circum Radius

#### Statement

The circum radius $R$ of a $\Delta ABC$ is given by

(i) $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C}$

(ii) $R = \dfrac{abc}{4\Delta}$

## 5.10 Inscribed Circle and its Radius

On this topic, we shall discuss various circles connected with a triangle and the formula for the circle which can be inscribed within a $\Delta$ and touch each of the sides is called its inscribed circle or incircle.

The centre of this circle is the point of intersection of the bisecter of the angle of the triangle. The radius of this circle is a always denoted by '$r$' and is equal to the length of the perpendicular from its centre to any of the sides of the $\Delta$.

### 5.10.1 In-radius

#### Statement

The in-radius $r$ of the inscribed circle of a $\Delta ABC$ is given by

(i) $r = \dfrac{\Delta}{s}$.

(ii) $r = (s - a)\dfrac{A}{2}, r = (s - b)\dfrac{B}{2}, r = (s - c)\dfrac{C}{2}$;

(iii) $r = \dfrac{a\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$;

$r = \dfrac{b\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{B}{2}\right)}$;

$r = \dfrac{c\sin\left(\frac{B}{2}\right)\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{C}{2}\right)}$

(iv) $r = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

## 5.11 Escribed Circle of a Triangle and Their Radii

### 5.11.1

In circle which touches the side $BC$ and two sides $AB$ and $AC$ produced of a $\Delta ABC$ is called the Escribed opposite to the angle $A$.

Its radius is denoted by $r_1$. Similarly, $r_2$ and $r_3$ denote the radii of escribed circle opposite to the angle $B$ and $C$ respectively.

The centres of the escribed circle are called the ex-centres. The centre of the escribed circle opposite to the angle $A$ is the point of intersection of the external bisectors of angles $B$ and $C$.

The internal bisector of angle $A$ also passes through the same point.

The centre is generally denoted by $I_1$.

### 5.11.2 Radii of Ex-circles

#### Statement

In any $\Delta ABC$, the ex-radii are given by

(i) $r_1 = \dfrac{\Delta}{s - a}; r_2 = \dfrac{\Delta}{s - b}; r_3 = \dfrac{\Delta}{s - c}$

(ii) $r_1 = s\tan\dfrac{A}{2}; r_2 = s\tan\dfrac{B}{2}; r_3 = s\tan\dfrac{C}{2}$

(iii) $r_1 = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$;

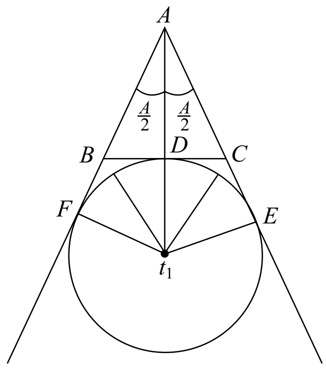
$r_2 = \dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$;

and $r_3 = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) $r_1 = 4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{B}{2}$

$r_2 = 4R \sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2}$

$r_3 = 4R \sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$




## 5.12 Regular Polygon

If a polygon has all its sides equal in length and also all its angles equal then the polygon is called a regular polygon.

The circle inscribed in the regular polygon and touching all the sides of the regular polygon is called inscribed circle.

The circle which pass through all the vertices of the regular polygon is called its **circum scribed circle**.

If the polygon has $n$-sides, then the sum of the interior angles is $(n-2)\times\pi$ and each angle is $\frac{(n-2)\times\pi}{n}$.

### 5.12.1 Statement

In a regular polygon of $A_1 A_2 A_3 \ldots. A_n$ of $n$-sides of each length $a$ is given by

(i) $R = \frac{a}{2}\operatorname{cosec}\left(\frac{a}{n}\right)$

where $R$ = circum-radius.

(ii) $r = \frac{a}{2}\cot\left(\frac{a}{n}\right)$, where $r$ = in-radius.

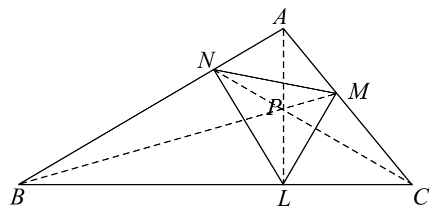
(iii) $\Delta = \frac{a^2 n}{4}\cot\left(\frac{\pi}{n}\right)$

where $\Delta$ = area of the regular polygon

(iv) $\Delta = \frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$

(v) $\Delta = nr^2\tan\left(\frac{2\pi}{n}\right)$

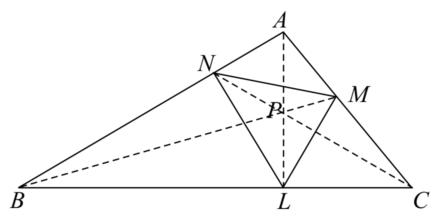
## 5.13 Orthocentre and Pedal Triangle of Any Triangle




Let $ABC$ be any triangle and let $AL$, $BM$ and $CN$ be the perpendiculars from $A$, $B$ and $C$ upon the opposite sides of the triangle. They meet at a point $P$. This point $P$ is called the orthocentre of the triangle

The $\Delta LMN$, which is formed by joining the feet of the perpendiculars, is called the pedal triangle.

### 5.13.1 Distances of the Orthocentre from the Angular Points of the Pedal Triangle




We have, $PL = LB\tan(PBL)$

$= LB\tan(90° - C)$

$= AB\cos B\cot C$

$= \frac{c}{\sin C}\times\cos B\cos C$

$= 2R\cos B\cos C$

Similarly,

$PM = 2R\cos A\cos C$,

$PN = 2R\cos A\cos B$

Again,

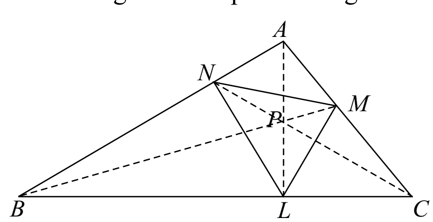
$AP = AM\sec(LAC) = c\cos A\operatorname{cosec} C$

$= \frac{c}{\sin C}\times\cos A = 2R\cos A$

Similarly, $BP = 2R\cos B$ and $CP = 2R\cos C$

### 5.13.2

The sides and the angles of the pedal triangle




Since the angles $PLC$, $PMC$ are right angles, so the points $P$, $L$, $C$ and $M$ lie on a circle.

Thus, $\angle PLM = \angle PCM = 90° - A$

Similarly, $P$, $L$, $B$ and $M$ lie on a circle and therefore

$\angle PLM = \angle PBN = 90° - A$

Hence, $\angle NLM = 180° - 2A$ = the supplement of $2A$.

So, $\angle LMN = 180° - 2B$ and $\angle MNL = 180° - 2C$.

Hence, it angles are the supplements of twice the angles of the triangle.

Again, from the triangle $AMN$, we have,

$$\frac{MN}{\sin A} = \frac{AM}{\sin(ANM)} = \frac{AB\cos A}{\cos(PNM)}$$

$$= \frac{c\cos A}{\cos(PAM)} = \frac{c\cos A}{\sin C}$$

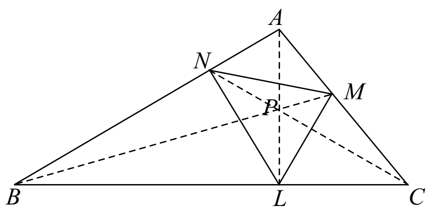
$$\Rightarrow \quad MN = \frac{c}{\sin C}\sin A\cos A = a\cos A$$

So, $NL = b\cos B$ and $LM = c\cos C$.

Hence, the sides of the pedal triangles are $a\cos A$, $b\cos B$ and $c\cos C$ respectively.

### 5.13.3

Area of a pedal $\Delta LMN$ of a $\Delta ABC$ is $2\Delta\cos A\cos B\cos C$




Here, $PL = 2R\cos B\cos C$,

$OM = 2R\cos C\cdot\cos A$,

$ON = 2R\cos A\cdot\cos B$

$ar(\Delta LMN)$

$$= \frac{1}{2}\times(R\sin 2A)\times(R\sin 2B)\times(\sin 2C)$$

$$= \frac{1}{2}\times R^2\times(\sin 2A\cdot\sin 2B\cdot\sin 2C)$$

$$= \frac{1}{2}\times R^2\times(8\sin A\cdot\sin B\cdot\sin C)\times(\cos A\cdot\cos B\cdot\cos C)$$

$$= \frac{1}{2}\times R^2\times\left(8\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}\right)\times(\cos A\cdot\cos B\cdot\cos C)$$

$$= \frac{1}{2}\times\frac{abc}{R}\times(\cos A\cdot\cos B\cdot\cos C)$$

$$= 2\Delta\times(\cos A\cdot\cos B\cdot\cos C)$$

### 5.13.4

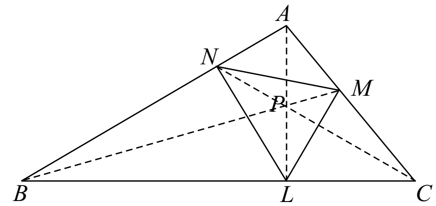
The circum-radius of a pedal $\Delta LMN$ of a $\Delta ABC$ is $\frac{R}{2}$

Circum-radius

$$= \frac{MN}{2\sin(MLN)} = \frac{R\sin 2A}{2\sin(180° - 2A)} = \frac{R\sin 2A}{2\sin 2A} = \frac{R}{2}$$

### 5.13.5

The in-radius of a pedal $\Delta LMN$ of a $\Delta ABC$ is $2R\cos A\cos B\cos C$.



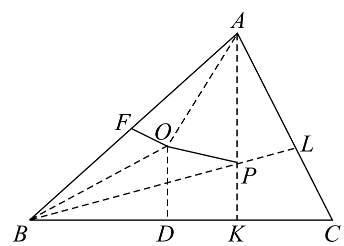

In-radius

$$= \frac{ar(\Delta LMN)}{\text{semi-perimeter }(\Delta LMN)}$$

$$= \frac{\frac{1}{2}R^2\cdot\sin 2A\cdot\sin 2B\cdot\sin 2C}{2R\cdot\sin A\cdot\sin B\cdot\sin C}$$

$$= 2R\cdot\cos A\cdot\cos B\cdot\cos C$$

## 5.14 Distance between the Circumcentre and Orthocentre




If $OF$ perpendicular to $AB$, we have,

$\angle OAF = 90° - \angle AOF = 90° - C$

Also, $\angle PAL = 90° - C$

Thus, $\angle OAP$

$= A - \angle OAF - \angle PAL$

$= A - 2(90° - C)$

$= A + 2C - 180°$

$= A + 2C - (A + B + C) = C - B$

Also, $OA = R$ and $PA = 2R\cos A$

Thus, $OP^2 = OA^2 + PA^2 - 2\cdot OA\cdot PA\cdot\cos(OAP)$

$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$

$= R^2 + 4R^2\cos A[\cos A - \cos(C - B)]$

$= R^2 - 4R^2\cos A[\cos(B + C) - \cos(B - C)]$

$= R^2 - 8R^2\cos A\cos B\cos C$

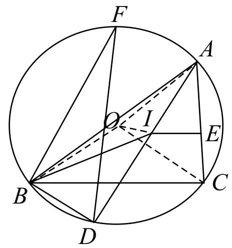
$= R^2(1 - 8\cos A\cos B\cos C)$

$$\Rightarrow \quad OP = R\sqrt{(1 - 8\cos A\cos B\cos C)}$$

## 5.15 Distance Between the Circumcentre and the Incentre

Let $O$ be the circumcentre and $OF$ be perpendicular to $AB$.

Let $I$ be the incentre and $IE$ perpendicular to $AC$.




Then $\angle OAF = 90° - C$

$$\angle OAI = \angle IAF - \angle OAF$$

$$= \frac{A}{2} - (90° - C)$$

$$= \frac{A}{2} + C - \frac{A+B+C}{2}$$

$$= \frac{C-B}{2}$$

Also, $AI = \dfrac{IE}{\sin\left(\dfrac{A}{2}\right)} = \dfrac{r}{\sin\left(\dfrac{A}{2}\right)}$

$$= 4R\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$= 1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{B}{2}+\frac{C}{2}\right)\right)$$

$$= 1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad OI = R\sqrt{1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)}$$

Also,

$$OI^2 = R^2 - 2R \times 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$= R^2 - 2Rr$$

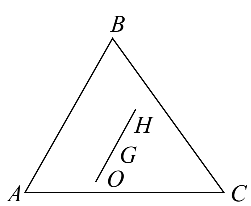
$$\Rightarrow \quad OI = \sqrt{R^2 - 2Rr}$$

Hence, the result

## 5.16 Distance between the Circumcentre and Centroid

i.e. $OG^2 = R^2 - \dfrac{1}{9}(a^2 + b^2 + c^2)$

*Solution*




As we know that, centroid divides the orthocentre and circumcentre in the ratio 2 : 1

Thus, $OG = \dfrac{1}{3} \cdot OH$

$$OG^2 = \frac{1}{9} \cdot OH^2$$

$$\Rightarrow \quad = \frac{1}{9}(R^2 - 8R^2\cos A \cdot \cos B \cdot \cos C)$$

$$\Rightarrow \quad OG^2 = \frac{R^2}{9} \times (1 - 4\{\cos(A+B) + \cos(A-B)\} \cdot \cos C)$$

$$\Rightarrow \quad OG^2 = \frac{R^2}{9} \times (1 + 4\cos^2 C + 4\cos(A-B) \cdot \cos(A+B))$$

$$\Rightarrow \quad OG^2 = \frac{R^2}{9}(1 + 4\cos^2 C + 4\cos^2 A - 4\sin^2 B)$$

$$\Rightarrow \quad OG^2 = \frac{R^2}{9}(1 + 4 - 4\sin^2 C + 4 - 4\sin^2 A - 4\sin^2 B)$$

$$\Rightarrow \quad OG^2 = \frac{R^2}{9}(9 - 4(\sin^2 A + \sin^2 B + \sin^2 C))$$

$$\Rightarrow \quad OG^2 = R^2 - \frac{1}{9} \times \{(2R\sin A)^2 + (2R\sin A)^2 + (2R\sin A)^2\}$$

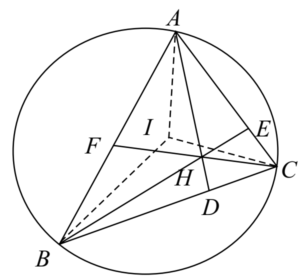
$$\Rightarrow \quad OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

## 5.17 Distance Between the Incentre and Orthocentre

If $OH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$

*Solution*

Let $ABC$ be a triangle, $H$ is the orthocentre and $I$ is the incentre.




Join $AH$, $AI$ and $IH$.

In $\Delta AIH$,

$$IH^2 = AH^2 + AI^2 - 2 \cdot AH \cdot AI \cdot \cos(\angle IAH)$$

$$\angle IAH = \frac{A}{2} - \angle HAC$$

$$= \frac{A}{2} - (90° - C) = \frac{1}{2}(C - B)$$

Thus, $4R^2\left[\cos^2 A + 4\sin^2\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right)\right.$

$$\left. - 4\cos A\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\cos\left(\frac{B}{2}\right)\right]$$

$$- \left[-16R^2\cos A \cdot \sin^2\left(\frac{B}{2}\right) \cdot \sin^2\left(\frac{C}{2}\right)\right]$$

$$= 4R^2[\cos^2 A + 4\sin^2\left(\frac{B}{2}\right)\sin^2\left(\frac{C}{2}\right)(1 - \cos A) - \cos A \cdot \cos B \cdot \sin C]$$

$$= 4R^2\left[\cos^2 A + 8\sin^2\left(\frac{A}{2}\right)\sin^2\left(\frac{B}{2}\right)\sin^2\left(\frac{C}{2}\right) - \cos A \cdot \cos B \cdot \sin C\right]$$

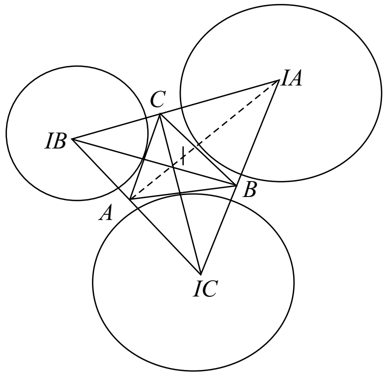
$$= 32R^2\sin^2\left(\frac{A}{2}\right)\sin^2\left(\frac{B}{2}\right)\sin^2\left(\frac{C}{2}\right) + 4R^2\cos A \cdot (\cos A - \sin B \cdot \sin C)$$

$$= 2\left[16R^2\sin^2\left(\frac{A}{2}\right)\sin^2\left(\frac{B}{2}\right)\sin^2\left(\frac{C}{2}\right)\right] - 4R^2\cos A[\cos(B+C) + \sin B \cdot \sin C]$$

$$= 2r^2 - 4R^2\cos A \cdot \cos B \cdot \cos C.$$

Hence, the result.

## 5.18 Excentral Triangle




The triangle formed by joining the three excentres $I_A I_B I_C$ of a $\Delta ABC$ is called an excentral of excentric triangle

(i) $\Delta ABC$ is the pedal triangle of $\Delta I_A I_B I_C$

(ii) Its angles are $\left(\frac{\pi}{2}-\frac{A}{2}\right), \left(\frac{\pi}{2}-\frac{B}{2}\right), \left(\frac{\pi}{2}-\frac{C}{2}\right)$

(iii) Its sides are

$$I_B I_C = 4R\cos\left(\frac{A}{2}\right), I_A I_C = 4R\cos\left(\frac{B}{2}\right),$$

and $I_A I_B = 4R\cos\left(\frac{C}{2}\right)$

(iv) $II_A = 4R\sin\left(\frac{A}{2}\right); II_B = 4R\sin\left(\frac{B}{2}\right)$

$$II_C = 4R\sin\left(\frac{C}{2}\right)$$

(v) Incentre $I$ of $\Delta ABC$ is the orthocentre of the excentric triangle $\Delta\, I_A I_B I_C$.

(vi) ar $(I_A I_B I_C)$

$$= \frac{1}{2}\times\left(4R\cos\left(\frac{B}{2}\right)\right)\times\left(4R\cos\left(\frac{B}{2}\right)\right)\times\sin\left(90° - \frac{A}{2}\right)$$

$$= 8\cdot R^2\cdot\cos\left(\frac{A}{2}\right)\cdot\cos\left(\frac{B}{2}\right)\cdot\cos\left(\frac{C}{2}\right)$$

(vii) Circum-radius $= \dfrac{I_2 I_3}{2\sin(\angle I_2 I_1 I_3)}$

$$= \frac{I_2 I_3}{2\sin\left(90° - \frac{A}{2}\right)}$$

$$= \frac{4R\cos\left(\frac{A}{2}\right)}{2\cos\left(\frac{A}{2}\right)} = 2R$$

*Solution*

We know that,

$$II_A = a\sec\left(\frac{A}{2}\right), II_B = a\sec\left(\frac{B}{2}\right), II_C = a\sec\left(\frac{C}{2}\right)$$

Also,

$$I_A I_B = c\,\text{cosec}\left(\frac{A}{2}\right), I_B I_C = c\,\text{cosec}\left(\frac{B}{2}\right), I_C I_A = c\,\text{cosec}\left(\frac{C}{2}\right)$$

Thus,

$$II_A\, II_B\, II_C = abc\times\sec\left(\frac{A}{2}\right)\sec\left(\frac{B}{2}\right)\sec\left(\frac{C}{2}\right) \quad \ldots\text{(i)}$$

Also, $a = 2R\sin A$, $b = 2R\sin B$ and $c = 2R\sin C$

From (i), we get,

$$II_A\, II_B\, II_C = (2R\sin A)(2R\sin B)(2R\sin C)$$
$$\times\sec\left(\frac{A}{2}\right)\sec\left(\frac{B}{2}\right)\sec\left(\frac{C}{2}\right)$$

$$= 8R^3\times\frac{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$= 64R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 16R^2\times 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 16R^2 r$$

Hence, the result.

(vi) If $I$ is the incentre and $I_A$, $I_B$, $I_C$ are the excentres of the triangle $\Delta ABC$, then prove that $II_A \,.\, II_B \,.\, II_C = 16r R^2$

**5.19**

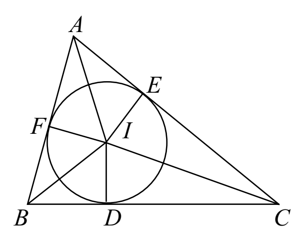
The distance between the incentre and the angular points of a $\Delta ABC$

$$IA = 4R\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right),$$

i.e. $IB = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right),$

and $IC = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)$

*Solution*




We have, $\sin\left(\frac{A}{2}\right) = \frac{IF}{IA}$

$\Rightarrow \quad r = IA\cdot\sin\left(\frac{A}{2}\right)$

$\Rightarrow \; 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

$$= IA\cdot\sin\left(\frac{A}{2}\right)$$

$\Rightarrow \; IA = 4R\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right),$

Similarly, $IB = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)$

and $\quad IC = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)$

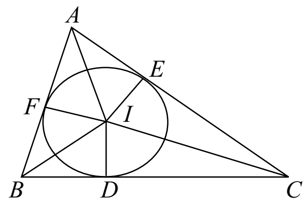
**5.20**

If $I$ be the incentre of a $\Delta ABC$, then

$$IA \cdot IB \cdot IC = abc \times \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)$$

*Solution*

From the above diagram,




$$IA = \frac{r}{\sin\left(\frac{A}{2}\right)}, IB = \frac{r}{\sin\left(\frac{B}{2}\right)},$$

$$IC = \frac{r}{\sin\left(\frac{C}{2}\right)}$$

Thus,

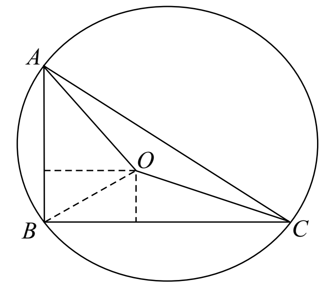
$$\begin{aligned}
IA \cdot IB \cdot IC &= \frac{r^3}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)} \\
&= \frac{r^3 \times 4R}{4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)} \\
&= \frac{r^3 \cdot 4R}{r} = 4r^2 \cdot R = 4 \cdot \frac{\Delta^2}{s^2} \cdot \frac{abc}{4\Delta} = abc \cdot \frac{\Delta}{s^2} \\
&= abc \times \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s^2} \\
&= abc \times \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^4}} \\
&= abc \times \sqrt{\frac{(s-a)}{s} \cdot \frac{(s-b)}{s} \cdot \frac{(s-c)}{s}} \\
&= abc \times \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \cdot \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\
&\quad \cdot \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \\
&= abc \times \tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) \\
&= abc \times \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)
\end{aligned}$$

Hence, the result.

**5.21**

$O$ is the circumcentre of a triangle $\Delta ABC$ and $R_1$ $R_2$ and $R_3$ are respectively the radii of the circumference of the $\Delta$'s $OBC$, $OCA$ and $OAB$ respectively, then $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$.

*Solution*




As we know that, $R = \frac{abc}{4\Delta}$

Let $\Delta OBC = \Delta_1$, $\Delta OCA = \Delta_2$, $\Delta OAB = \Delta_3$

In $\Delta OBC$, $R_1 = \frac{0B \cdot 0C \cdot BC}{4\Delta_1} = \frac{R \cdot R \cdot a}{4\Delta_1} = \frac{R^2 \cdot a}{4\Delta_1}$

$$\Rightarrow \quad \frac{a}{R_1} = \frac{4\Delta_1}{R^2}$$

Similarly, $\frac{b}{R_2} = \frac{4\Delta_2}{R^2}$ and $\frac{c}{R_3} = \frac{4\Delta_3}{R^2}$

Thus,

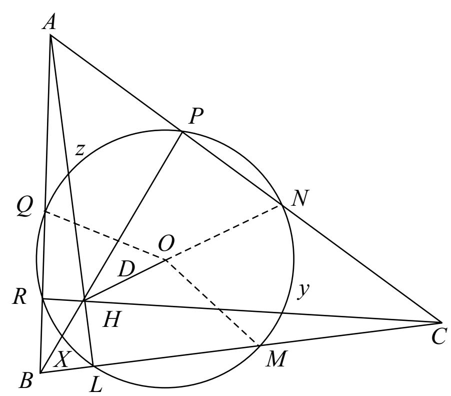
$$\begin{aligned}
\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} &= \frac{4}{R^2}(\Delta_1 + \Delta_2 + \Delta_3) \\
&= \frac{4}{R^2} \times \Delta = \frac{4\Delta}{R^2} = \frac{abc}{R^3}
\end{aligned}$$

Hence, the result.

**5.22**

The distance between the centre of the nine -point circle from the angle $A$ is $\frac{R}{2}\sqrt{(1 + 8\cos A \cdot \cos B \cdot \cos C)}$.

*Solution*




Let $ABC$ be a triangle, $H$ = orthocentre,

$O$ = circumcentre, $D$ = nine-point centre

As we know that, nine point centre is the mid-point of the orthocentre and circumcentre of a triangle.

Thus $AOH$ be a triangle, where, $AD$ is the median.

In $\Delta AOH$, $2(AD^2 + DO^2) = AH^2 + AO^2$

$$2AD^2 = AH^2 + AO^2 - \frac{1}{2}OH^2 \qquad \ldots\text{(i)}$$

Now from the $\Delta ABC$, we can write,

$$AH = 2R\cos A,\ OA = R,$$

$$OH = R\sqrt{(1 - 8\cos A\cdot\cos B\cdot\cos C)}$$

From (i), we get,

$$2AD^2 = R^2(4\cos^2 A + 1)\ \frac{R^2}{2}(1 - 8\cos A\cdot\cos B\cdot\cos C)$$

$$= \frac{R^2}{2}[1 + 8\cos A\cdot\{\cos(180° - B + C) + \cos B\cdot\cos C\}]$$

$$\Rightarrow \quad AD^2 = \frac{R^2}{4}\times[1 + 8\cos A\{-\cos B\cdot\cos C + \sin B\cdot\sin C + \cos B\cdot\cos C\}]$$

$$\Rightarrow \quad AD^2 = \frac{R^2}{4}(1 + 8\cos A\cdot\sin B\cdot\sin C)$$

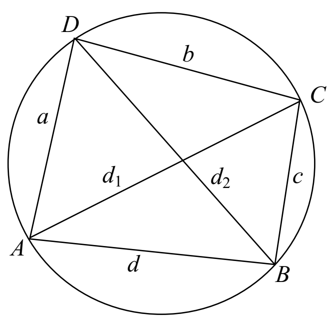
$$\Rightarrow \quad AD = \frac{R}{2}\sqrt{(1 + 8\cos A\cdot\sin B\cdot\sin C)}$$

Hence, the result.

## 5.23 Quadrilateral

### 5.23.1

Area of a quadrilateral, which is inscribed in a circle




Let $ABCD$ be a cyclic quadrilateral such that

$$AB = a,\ BC = b,\ CD = c \text{ and } AD = d.$$

$$\text{ar}(ABCD) = \text{ar}(\Delta ABC) + \text{ar}(\Delta ADC)$$

$$= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D$$

$$= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin(\pi - B)$$

$$= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin B$$

$$= \frac{1}{2}(ab + cd)\sin B$$

From $\Delta$'s, $BAC$ and $BCD$, we have

$$a^2 + b^2 - 2ab\cos B = c^2 + d^2 - 2cd\cos D$$

$$\Rightarrow \quad a^2 + b^2 - 2ab\cos B = c^2 + d^2 + 2cd\cos B$$

$$\Rightarrow \quad \cos B = \left(\frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}\right)$$

Now, $\sin^2 B = 1 - \cos^2 B$

$$= 1 - \frac{(a^2 + b^2 - c^2 - d^2)^2}{(2(ab + cd))^2}$$

$$= \frac{[\{2(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2\}]}{4(ab + cd)^2}$$

$$= \frac{1}{4(ab + cd)^2}\times[\{2(ab + cd) + (a^2 + b^2 - c^2 - d^2)^2\} \times \{2(ab + cd) + (a^2 + b^2 - c^2 - d^2)^2\}]$$

$$= \frac{1}{4(ab + cd)^2}[\{(a^2 + b^2 + 2ab) - (c^2 - 2cd + d^2)\} \times \{(c^2 + 2cd + d^2) - (a^2 + b^2 - 2ab)\}]$$

$$= \frac{\{(a + b)^2 - (c - d)^2\}\times\{(c + d)^2 - (a - b)^2\}}{4(ab + cd)^2}$$

$$= \frac{1}{4(ab + cd)^2}\times[\{(a + b + c - d)(a + b - c + d)\} \times (c + d + b - a)(c + d + a - b)]$$

Let $a + b + c + d = 2s$

Thus,

$$(a + b + c - d) = (a + b + c + d - 2d) = 2(s - d)$$

Similarly,

$$(a + b + d - c) = 2(s - c),$$
$$(a + c + d - b) = 2(s - b),$$
$$(b + c + d - a) = 2(s - a).$$

$$\Rightarrow \quad \sin^2 B = \frac{2(s - d)\times 2(s - c)\times 2(s - b)\times 2(s - a)}{4(ab + cd)^2}$$

$$\Rightarrow \quad \sin^2 B = \frac{16\times(s - a)\times(s - b)\times(s - c)\times(s - d)}{4(ab + cd)^2}$$

$$\Rightarrow \quad (ab + cd)\sin B = 2\times\sqrt{(s - a)(s - b)(s - c)(s - d)}$$

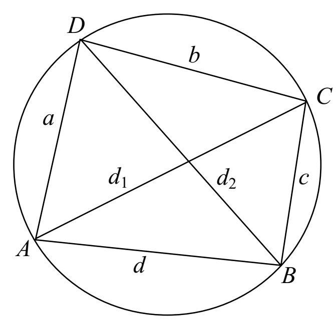
Hence, the area of the quadrilateral

$$= \frac{1}{2}(ab + cd)\sin B$$

$$= \sqrt{(s - a)(s - b)(s - c)(s - d)}$$

### 5.23.2

The radius of the circle circumscribing the quadrilateral $ABCD$.




We have $\cos B = \dfrac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$

and $\quad AC^2 = a^2 + b^2 - 2ab\cos B$

$$= a^2 + b^2 - 2ab \times \left(\frac{a^2 + b^2 - c^2 - d^2}{ab + cd}\right)$$

$$= \frac{(a^2 + b^2)cd + (c^2 + d^2)ab}{(ab + cd)}$$

$$= \frac{(ac + bd)(ad + bc)}{(ab + cd)}$$

In $\Delta ABC$,

$$R = \frac{1}{2} \times \frac{AC}{\sin B}$$

$$= \sqrt{\left(\frac{(ac + bd)(ad + bc)}{(ab + cd)}\right)}$$

$$\div 4\sqrt{\frac{(s-a)(s-b)(s-c)(s-d)}{(ab + cd)^2}}$$

$$= \frac{1}{2} \times \frac{AC}{\sin B} = \frac{1}{4}\sqrt{\left(\frac{(ab + cd)(ac + bd)(ad + bc)}{(s-a)(s-b)(s-c)(s-d)}\right)}$$

### 5.23.3

**Area of a quadrilateral *ABCD*, when it is not inscribed.**

$$\text{ar}(ABCD) = \text{ar}(\Delta ABC) + \text{ar}(\Delta ACD)$$

$$= \frac{1}{2}ab \sin B + \frac{1}{2}cd \sin D$$

$$\Rightarrow \quad 4\Delta = 2ab \sin B + 2cd \sin D \qquad \ldots\text{(i)}$$

Also,

$$a^2 + b^2 - 2ab \cos B = c^2 + d^2 - 2cd \cos D$$

$$2ab \cos B - 2cd \cos D = a^2 + b^2 - c^2 - d^2 \qquad \ldots\text{(ii)}$$

Squaring (i) and (ii) and adding, we get,

$$16\Delta^2 + (a^2 + b^2 - c^2 - d^2)^2$$

$$= 4a^2b^2 + 4c^2d^2 - 8\, abcd(\cos B \cos D - \sin B \sin D)$$

$$= 4(a^2b^2 + c^2d^2) - 8abcd \cos(B + D)$$

$$= 4(a^2b^2 + c^2d^2) - 8abcd \cos 2\alpha$$

$$= 4(a^2b^2 + c^2d^2) - 8abcd(2\cos^2\alpha - 1)$$

$$= 4(ab + cd)^2 - 16abcd \cos^2 \alpha$$

$$\Rightarrow \quad 16\Delta^2 = 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 - 16abcd \cos^2 \alpha \qquad \text{(iii)}$$

Thus,

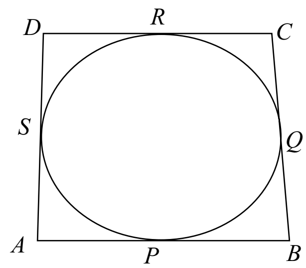
$$16\Delta^2 = 2(s - a) \cdot 2(s - b) \cdot 2(s - c) \cdot 2(s - d) - 16abcd \cos^2 \alpha$$

$$\Rightarrow \quad \Delta^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \alpha$$

$$\Rightarrow \quad \Delta = \sqrt{(s-a)\cdot(s-b)\cdot(s-c)\cdot(s-d) - abcd \cos^2\alpha}$$

### 5.23.4

Area of a quadrilateral which can have a circle inscribed in it.




We have

$$AP = AS,\ BP = BQ,\ CQ = CR \text{ and } DR = RS$$

$$AP + BP + CR + DR = AS + BQ + CQ + DS$$

$$\Rightarrow \quad AB + CD = BC + AD$$

$$\Rightarrow \quad a + c = b + d$$

Hence, $s = \dfrac{a + b + c + d}{2} = a + c = b + d$

Thus, $s - a = c$, $s - b = d$, $s - c = a$, $s - d = b$

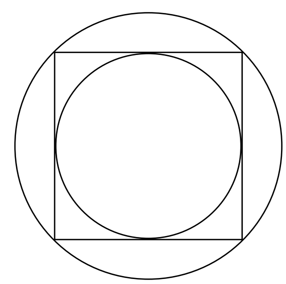
As we know that,

$$\Delta^2 = abcd - abcd \cos^2 \alpha = abcd \sin^2 \alpha$$

$$\Rightarrow \quad \Delta = \sqrt{abcd} \sin \alpha$$

### 5.23.5

The area of a quadrilateral, which can be both inscribed in a circle and circumscribed about another circle and the radius of the later circle is $\dfrac{2\sqrt{abcd}}{a + b + c + d}$



In a quadrilateral $ABCD$,

$$\angle B + \angle D = 180°$$

$$\Rightarrow \quad 2\alpha = 180°$$

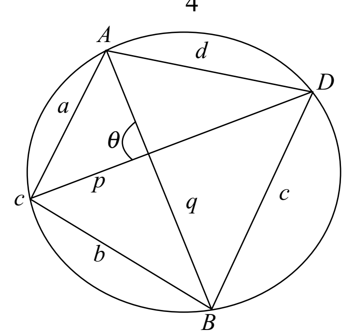
$$\Rightarrow \quad \alpha = 90°$$

Hence, the area of the quadrilateral, which is inscribed in a circle and circumscribed another circle is = $\sqrt{abcd}$ .

We have, $r = \dfrac{\Delta}{s} = \dfrac{2\Delta}{2s} = \dfrac{2\sqrt{abcd}}{a + b + c + d}$

### 5.23.6

$a$, $b$, $c$ and $d$ are the sides of a quadrilateral taken in order, and $\theta$ is the angle between the diagonals opposite to $b$ or $d$, then the area of the quadrilateral is $\dfrac{1}{4}(a^2 + c^2 - b^2 - d^2)\tan\theta$



Area of a quadrilateral

$$ABCD = \frac{1}{2} \times AC \times BD \times \sin\theta \qquad \ldots\text{(i)}$$

Now,

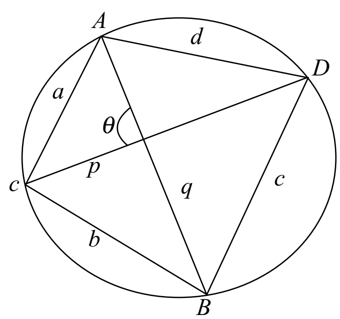
$$a^2 = OA^2 + OB^2 - 2 \cdot OA \cdot OB \cdot \cos(180° - \theta)$$
$$b^2 = OC^2 + OB^2 - 2 \cdot OC \cdot OB \cdot \cos\theta,$$
$$c^2 = OC^2 + OD^2 - 2 \cdot OC \cdot OD \cdot \cos(180° - \theta)$$
$$d^2 = OA^2 + OD^2 - 2 \cdot OA \cdot OD \cdot \cos\theta.$$

We have

$$a^2 - b^2 + c^2 - d^2 = 2\cos\theta(OA \cdot OB + OB \cdot OC + OC \cdot OD + OA \cdot OD)$$
$$= 2 \cdot \cos\theta \cdot AC \cdot BD$$

From (i), we get, area of the quadrilateral *ABCD*

$$= \frac{1}{2}(a^2 - b^2 + c^2 - d^2)\tan\theta$$




### 5.23.7

$a$, $b$, $c$ and $d$ are the sides of a quadrilateral and $p$ and $q$ be its diagonals, then its area is

$$\frac{1}{4} \times \sqrt{(4p^2q^2 - (a^2 + c^2 - b^2 - d^2)^2)}$$

As we know that,

$$a^2 - b^2 + c^2 - d^2 = 2pq\cos\theta$$

Area of a quadrilateral *ABCD*

$$= \frac{1}{2} pq \sin\theta$$

$$= \sqrt{\frac{1}{4} p^2 q^2 \sin^2\theta}$$

$$= \sqrt{\frac{1}{4} p^2 q^2 (1 - \cos^2\theta)}$$

$$= \sqrt{\frac{1}{4}(4p^2q^2 - (2pq\cos\theta)^2)}$$

$$= \frac{1}{2}\sqrt{(4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2)}$$

Hence, the result.

### 5.23.8

If a quadrilateral can be inscribed in a circle, then the angle between its diagonals is

$$\sin^{-1}\left(\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ac+bd)}\right)$$

*Solution*

Let *ABCD* be a quadrilateral, whose sides are $a$, $b$, $c$ and $d$ resp. and its diagonals are $p$ and $q$.

Let the angle between the diagonals be $\theta$.

Then, $pq = ac + bd$.

Area of a quadrilateral $ABCD = \frac{1}{2} \times pq \sin\theta$

$$\sin\theta = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ac+bd)}$$

$$\theta = \sin^{-1}\left(\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ac+bd)}\right)$$

Hence, the result.

### 5.23.9

If a quadrilateral can be inscribed in a circle as well as circumscribed about another circle, then the angle between its diagonal is $\cos^{-1}\left(\frac{ac-bd}{ac+bd}\right)$.

Since the quadrilateral be circumscribed, then we can write, $\frac{1}{2} pq \sin\theta = \sqrt{abcd}$

$$\Rightarrow \qquad \sin\theta = \left(\frac{2\sqrt{abcd}}{pq}\right)$$

Therefore, $\cos\theta = \sqrt{1 - \sin^2\theta}$

$$= \cos\theta = \sqrt{1 - \left(\frac{4\,abcd}{(ac+bd)^2}\right)}$$

$$= \left(\frac{ac-bd}{ac+bd}\right)$$

$$\Rightarrow \qquad \theta = \cos^{-1}\left(\frac{ac-bd}{ac+bd}\right)$$

Hence, the result.

## Exercises

### Level I

**(Problems Based on Fundamentals)**

**SINE RULE**

1. If in a triangle $ABC$, $\frac{a^2-b^2}{a^2+b^2}=\frac{\sin(A-B)}{\sin(A+B)}$, then the triangle is right angled or isosceles?
2. Prove that $\frac{\sin(B-C)}{\sin(B+C)}=\frac{b^2-c^2}{a^2}$
3. In a $\Delta ABC$ such that $\angle A = 45°$ and $\angle B = 75°$ then find $a+c\sqrt{2}$.
4. Prove that
$$\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0$$
5. Prove that
$$\frac{b^2-c^2}{\cos B+\cos C}+\frac{c^2-a^2}{\cos C+\cos A}+\frac{a^2-b^2}{\cos A+\cos B}=0$$
6. In a triangle $ABC$, if $a^2$, $b^2$, $c^2$ are in AP then prove that $\cot A$, $\cot B$, $\cot C$ are in AP.
7. If $\cot\frac{A}{2}=\frac{b+c}{a}$, then prove that $\Delta ABC$ is right angled.
8. Prove that $a^2$, $b^2$, $c^2$ are in A.P., if $\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}$
9. In any $\Delta ABC$, prove that
$$\prod\left(\frac{\sin^2 A+\sin A+1}{\sin A}\right)>27.$$
10. In a triangle $ABC$, prove that,
$$a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\left(\frac{A}{2}\right)$$
11. In a triangle $ABC$, prove that,
$$\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$$
12. Prove that $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\frac{a^2+b^2}{a^2+c^2}$
13. In a $\Delta ABC$, if $\cos A + 2\cos B + \cos C = 2$ then prove that the sides of a triangle are in AP
14. In a $\Delta ABC$, if
$\cos A\cos B + \sin A\sin B\sin C = 1$,
then prove that $a:b:c = 1:1:\sqrt{2}$.

**COSINE RULE**

15. In a $\Delta ABC$, prove that
$$a(b\cos C - c\cos B) = b^2 - c^2$$
16. Prove that
$$\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$$
17 Prove that
$$(a-b)^2\cos^2\left(\frac{C}{2}\right)+(a+b)^2\sin^2\left(\frac{C}{2}\right)=c^2$$
18. In a $\Delta ABC$, if
$(a + b + c)(a - b + c) = 3ac$, then find $\angle B$
19. In any $\Delta ABC$, if $2\cos B=\frac{a}{c}$
prove that the triangle is isosceles.
20. In a $\Delta ABC$, if $(a + b + c)(b + c - a) = \lambda bc$ then find the value of $\lambda$.
21. If the angles $A$, $B$, $C$ of a triangle are in AP and its sides $a$, $b$, $c$ are in GP, prove that $a^2$, $b^2$, $c^2$ are in AP.
22. If the line segment joining the points $P(a_1, b_1)$ and $Q(a_2, b_2)$ subtends an angle $\theta$ at the origin, prove that
$$\cos\theta=\left(\frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}\right)$$
23 In a triangle $ABC$, if $\cot A$, $\cot B$, $\cot C$ are in AP, prove that $a^2$, $b^2$, $c^2$ are in AP.
24. If the sides of a triangle are $a$, $b$ and $\sqrt{a^2+ab+b^2}$ then find its greatest angle.
25. In a triangle $ABC$, if $a\cos A = b\cos B$, then prove that triangle is right angled isosceles.
26. In a triangle $ABC$, the angles are in AP, then prove that,
$$2\cos\left(\frac{A-C}{2}\right)=\frac{a+c}{\sqrt{a^2-ac+c^2}}$$
27. In a triangle $ABC$, prove that
$$\left(\frac{b^2-c^2}{a^2}\right)\sin 2A+\left(\frac{c^2-a^2}{b^2}\right)\sin 2B+\left(\frac{a^2-b^2}{c^2}\right)\sin 2C=0$$
28. In a triangle $ABC$, if $\angle A = 60°$, then find the value of $\left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right)$.
29. If $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$, then find $\angle C$
30. If in a triangle $ABC$,
$$\frac{2\cos A}{a}+\frac{2\cos B}{b}+\frac{2\cos C}{c}=\frac{1}{bc}+\frac{b}{ca},$$
then find the angle $A$ in degrees.

## PROJECTION RULE

31. In any $\Delta ABC$, prove that
$$2\left(a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right)=a+c-b$$
32. In any $\Delta ABC$, prove that
$$2\left(b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}\right)=a+b+c$$
33. In any $\Delta ABC$, prove that
$$(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=(a+b+c)$$
34. In a $\Delta ABC$, prove that, $\dfrac{\sin B}{\sin C}=\dfrac{c-a\cos B}{b-a\cos C}$
35. In any $\Delta ABC$, prove that
$$2\left(a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right)=a+c-b$$
36. In any $\Delta ABC$, prove that
$$\frac{\cos A}{b\cos C+c\cos B}+\frac{\cos B}{c\cos A+a\cos C}+\frac{\cos C}{a\cos B+b\cos A}=\frac{a^2+b^2+c^2}{2abc}$$
37. In any $\Delta ABC$ prove that $2(bc\cos A+ca\cos B+ab\cos C)=(a^2+b^2+c^2)$

## NAPIER ANALOGY

38. In any $\Delta ABC$, $b=\sqrt{3}+1, c=\sqrt{3}-1$ and $\angle A=60°$ then find the value of $\tan\left(\dfrac{B-C}{2}\right)$.
39. In any $\Delta ABC$, $b=\sqrt{3}, c=1, B-C=90°$ then find $\angle A$.
40. If in a $\Delta ABC$, $a=6$, $b=3$ and $\cos(A-B)=\dfrac{4}{5}$, then find the angle $C$.
41. In a $\Delta ABC$, if
$$x=\tan\left(\frac{B-C}{2}\right)\tan\left(\frac{A}{2}\right),$$
$$y=\tan\left(\frac{C-A}{2}\right)\tan\left(\frac{B}{2}\right)\text{ and}$$
$$z=\tan\left(\frac{A-B}{2}\right)\tan\left(\frac{C}{2}\right),$$
then prove that $x+y+z+xyz=0$
42. In a $\Delta ABC$, if $a=5$, $b=4$ and $\cos(A-B)=\dfrac{31}{32}$, then prove that $c=6$.

## HALF ANGLED FORMULA

43. In a $\Delta ABC$, if $a=13$, $b=14$ and $c=15$, then find the value of
    (i) $\sin\dfrac{A}{2}$
    (ii) $\cos\dfrac{B}{2}$
    (iii) $\cos A$
44. In a $\Delta ABC$, if $\cos\left(\dfrac{A}{2}\right)=\sqrt{\dfrac{b+c}{2c}}$, prove that $\Delta ABC$ is right angled at $C$.
45. In a $\Delta ABC$, prove that,
$$b\cos^2\left(\frac{C}{2}\right)+c\cos^2\left(\frac{B}{2}\right)=s$$
46. In a $\Delta ABC$, prove that
$$bc\cos^2\left(\frac{A}{2}\right)+ca\cos^2\left(\frac{B}{2}\right)+ab\cos^2\left(\frac{C}{2}\right)=s^2$$
47. In a $\Delta ABC$, prove that
$$2ac\sin\left(\frac{A-B+C}{2}\right)=(a^2+c^2-b^2).$$
48. In a $\Delta ABC$, $3a=b+c$, then find the value of $\cot\left(\dfrac{B}{2}\right)\cot\left(\dfrac{C}{2}\right)$
49. In a $\Delta ABC$, prove that
$$1-\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)=\frac{2c}{(a+b+c)}$$
50. In a $\Delta ABC$, prove that:
$$\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)=\left(\frac{a+b+c}{b+c-a}\right)\cot\left(\frac{A}{2}\right)$$
51. In a $\Delta ABC$, if $\cot\left(\dfrac{A}{2}\right),\cot\left(\dfrac{B}{2}\right),\cot\left(\dfrac{C}{2}\right)$ are in AP, then prove that $a, b, c$ are in AP
52 In a $\Delta ABC$, $c(a+b)\cos\dfrac{B}{2}=b(a+c)\cos\dfrac{C}{2}$, then prove that the triangle is isosceles.
53. In a $\Delta ABC$, prove that
$$\frac{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}{\cot A+\cot B+\cot C}=\frac{(a+b+c)^2}{a^2+b^2+c^2}$$
54. In a $\Delta ABC$, if
$$c(a+b)\cos\left(\frac{B}{2}\right)=b(a+c)\cos\left(\frac{C}{2}\right)$$
then prove that the triangle $ABC$ is isosceles.

## AREA OF A TRIANGLE

55. In any $\Delta ABC$, if $a=\sqrt{2}, b=\sqrt{3}$ and $c=\sqrt{5}$, then find the area of the $\Delta ABC$.

56. In any $\Delta ABC$, prove that
$$\Delta = \frac{a^2 - b^2}{2} \times \frac{\sin A \sin B}{\sin (A - B)}$$

57. If the angles of triangle and 30° and 45° and the included side is ($\sqrt{3}$ + 1) cm., then prove that the area of the triangle is $\frac{1}{2}(\sqrt{3}+1)$ cm$^2$.

58. In a $\Delta ABC$, prove that
$$\frac{\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)}{\cot A + \cot B + \cot C} = \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

59. If in a $\Delta ABC$, prove that $\Delta < \frac{s^2}{4}$.

60. If $\alpha, \beta, \gamma$ are the lengths of the altitudes of a $\Delta ABC$, then prove that
$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{\Delta}(\cot A + \cot B + \cot C)$$

61. If $p_1, p_2, p_3$ are the altitudes of a triangle from the vertices $A$, $B$, $C$ and $\Delta$ be the area of the $\Delta ABC$, prove that
$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{2ab}{(a+b+c)\times\Delta} \times \cos^2\left(\frac{C}{2}\right).$$

62. If $a$, $b$, $c$ and $d$ are the sides of a quadrilateral, then find the minimum value of $\left(\frac{a^2+b^2+c^2}{d^2}\right)$

63. In a $\Delta ABC$, if $\cos A + \cos B + \cos C = \frac{3}{2}$, then the triangle is equilateral.

64. In a $\Delta PQR$, if $\sin P$, $\sin Q$, $\sin R$ are in AP then prove that its altitude are in HP

65. In a $\Delta ABC$, $\Delta = (6 + 2\sqrt{3})$ sq.u and $\angle B = 45°$, $a = 2(\sqrt{3}+1)$, then prove that the side $b$ is 4

66. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$, then prove that
$$\text{ar}(\Delta ABC) = \frac{1}{2}(\sqrt{3}+1)\text{sq.u}$$

67. The two adjacent sides of a cyclic quadrilateral are 2 and 3 and the angle between them is 60. If the area of the quadrilateral is $4\sqrt{3}$, then prove that the remaining two sides are 2 and 3 respectively.

### *M-N* THEOREM

68. The median $AD$ of a $\Delta ABC$ is perpendicular to AB. Prove that $\tan A + 2\tan B = 0$

69. If $D$ be the mid point of the side $BC$ of the triangle $ABC$ and $\Delta$ be its area, then prove that
$$\cot\theta = \frac{b^2 - c^2}{4\Delta}, \text{ where } \angle ADB = \theta$$

### CIRCUM-CIRCLE AND CIRCUM-RADIUS

70. In a $\Delta ABC$, if $a = 18$ cm, $b = 24$ cm and $c = 30$ cm, then find its circum-radius

71. In an equilateral triangle of side $2\sqrt{3}$ cm, then find the circum-radius.

72. If the length of the sides of a triangle are 3, 4 and 5 units, then find its circum-radius $R$.

73. If $8R^2 = a^2 + b^2 + c^2$, then prove that the triangle is right angled

74. In any $\Delta ABC$, prove that $a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C$

75. In any $\Delta ABC$, prove that
$$D = 2R^2 \sin A \sin B \sin C$$

76. In any $\Delta ABC$, prove that,
$$\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{3}{2R}$$

77. In any $\Delta ABC$, $a$, $b$, $c$ are in AP and $p_1$, $p_2$ and $p_3$ are the altitudes of the given triangle, then prove that, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \le \frac{3R}{\Delta}$.

78. If $p_1$, $p_2$ and $p_3$ are the altitudes of a $\Delta ABC$ from the vertices $A$, $B$ and $C$ respectively. then prove that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$

79. If $p_1$, $p_2$ and $p_3$ are the altitudes of a $\Delta ABC$ from the vertices $A$, $B$ and $C$ respectively. then prove that $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$

80. In an acute angled $\Delta ABC$, prove that
$$\frac{\cos C}{\sqrt{4R^2 - c^2}} = \frac{1}{2R}$$

81. If $p_1$, $p_2$ and $p_3$ are the altitudes of a $\Delta ABC$ from the vertices $A$, $B$ and $C$ respectively. then prove that $p_1 p_2 p_3 = \frac{a^2b^2c^2}{8R^3}$

82. If $p_1$, $p_2$ and $p_3$ are the altitudes of a $\Delta ABC$ from the vertices $A$, $B$ and $C$ respectively. then prove that
$$\frac{bp_1}{c} + \frac{cp_2}{a} + \frac{ap_3}{b} = \frac{a^2+b^2+c^2}{2R}$$

83. $O$ is the circum-centre of $\Delta ABC$ and $R_1$, $R_2$ and $R_3$ are respectively the radii of the circum-centre of the triangles $\Delta OBC$, $\Delta OCA$ and $\Delta OAB$, prove that
$$\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$$

84. In an acute angled $\Delta ABC$, prove that
$$\frac{a \sec A + b \sec B + c \sec C}{\tan A \tan B \tan C} = 2R$$
85. In any $\Delta ABC$, prove that
$(a \cos A + b \cos B + c \cos C) = 4R \sin A \sin B \sin C$

## IN-CIRCLE AND IN-RADIUS

86. In a $\Delta ABC$, if $a = 4$ cm, $b = 6$ cm and $c = 8$ cm, then find its in-radius.
87. If the sides of a triangle be 18, 24, 30 cm, then find its in-radius.
88. If the sides of a triangle are 3 : 7 : 8, then find $R : r$.
89. Two sides of a triangle are 2 and $\sqrt{3}$ and the included angle is 30°, then prove that its in-radius is $\frac{1}{2}(\sqrt{3} - 1)$.
90. In an equilateral triangle, prove that $R = 4r$
91. In a $\Delta ABC$, prove that
$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} + \frac{1}{2rR}$$
92. In a $\Delta ABC$, prove that
$$\cos A + \cos B + \cos C = \left(1 + \frac{r}{R}\right).$$
93. In a $\Delta ABC$, prove that
$$\sin A + \sin B + \sin C = \frac{s}{R} = \frac{\Delta}{Rr}$$
94. In any $\Delta ABC$, prove that $a \cot A + b \cot B + c \cot C = 2(r + R)$
95. In a $\Delta ABC$, prove that
$$\frac{a \sec A + b \sec B + c \sec C}{2 \tan A \cdot \tan B\, K \cdot \tan C} = R$$
96. In a $\Delta ABC$, prove that
$$(b + c) \tan\left(\frac{A}{2}\right) + (c + a) \tan\left(\frac{B}{2}\right) + (a + b) \tan\left(\frac{C}{2}\right)$$
$$= 4(r + R)$$
97. In a $\Delta ABC$, if $C = 90°$, prove that
$$\frac{1}{2}(a + b) = R + r$$
98. In any $\Delta ABC$, prove that
$$\cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right) = 2 + \frac{r}{2R}$$
99. If the distances of the sides of a triangle $ABC$ from a circum-center be $x$, $y$ and $z$ respectively, then prove that
$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}.$$
100. If in a $\Delta\Delta ABC$, $O$ is the circum center and $R$ is the circum-radius and $R_1$, $R_2$, $R_3$ are the circum radii of the traingles $\Delta OBC$, $\Delta OCA$ and $\Delta OAB$ respectively, then prove that
$$\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}.$$
101. If $p_1, p_2, p_3$ are respectively the perpendiculars from the vertices of a $\Delta$ to the opposite sides, then prove that
$$p_1 \cdot p_2 \cdot p_3 = \frac{(abc)^2}{8R^3}.$$
102. Find The bisectors of the angles of a $\Delta ABC$

## EXCIRCLE AND EX-RADII

103. In a $\Delta ABC$, if $a = 18$ cm, $b = 24$ cm, and $c = 30$ cm, then find the value of $r_1$, $r_2$ and $r_3$
104. In a triangle $\Delta ABC$, prove that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$, where $r$ is in radius and $R_1$, $R_2$, $R_3$ are exradii.
105. In a $\Delta ABC$, prove that
$$\frac{b - c}{r_1} + \frac{c - a}{r_2} + \frac{a - b}{r_3} = 0$$
106. In a $\Delta ABC$ if $\frac{s - c}{s - a} = \frac{b - c}{a - b}$, then prove that $a$, $b$, $c$ are in AP
107. In a triangle if $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, prove that the triangle is right angled.
108. In a triangle $\Delta ABC$, prove that $r_1 + r_2 + r_3 - r = 4R$
109. In a triangle $\Delta ABC$, prove that $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$
110. In a triangle $\Delta ABC$, prove that $r_1 + r_2 - r_3 + r = 4R \cos C$
111. If $r_1$, $r_2$, $r_3$ are in HP, then prove that $a$, $b$, $c$ are in AP.
112. In a triangle $ABC$, if $a$, $b$, $c$ are in AP as well as in GP then prove that the value of $\left(\frac{r_1}{r_2} - \frac{r_2}{r_3} + 10\right)$ is 10.
113. In a triangle $\Delta ABC$, prove that
$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} = \frac{a^2 + b^2 + c^2}{Ä^2}$$
114. In a triangle $\Delta ABC$, prove that $(r_1 - r)(r_2 - r)(r_3 - r) = 4r^2 R$
115. If $r_1 < r_2 < r_3$ and the ex-radii of a right angled triangle and $r_1 = 1$, $r_2 = 2$, then prove that $r_3 = \frac{3 + \sqrt{17}}{2}$.
116. Two sides of a triangle are the roots of $x^2 - 5x + 3 = 0$. If the angle between the sides is $\frac{\pi}{3}$. then prove that the value of $r$. $R$ is 2/3.
117. In an isosceles triangle of which one angle is 120°, circle of radius $\sqrt{3}$ is inscribed, then prove that the area of the triangle is $(12 + 7\sqrt{3})$ sq. u.
118. If in a triangle $r = r_1 - r_2 - r_3$, then prove that the triangle is right angled.
119. In a $\Delta ABC$, prove that $r \,.\, r_1 \,.\, r_2 \,.\, r_3 = \Delta^2$
120. Prove that $\frac{(r_1 + r_2)}{1 + \cos C} = \frac{(r_2 + r_3)}{1 + \cos A} = \frac{(r_3 + r_1)}{1 + \cos B}$.

121. Prove that $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=\frac{16R}{r^2(a+b+c)^2}$

122. Prove that $\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right)=\frac{64R^3}{(abc)^2}$

123. In a $\Delta ABC$, prove that
$r^2+r_1^2+r_2^2+r_3^2=16R^2-(a^2+b^2+c^2)$

124. In a $\Delta ABC$, prove that
$\frac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{(r_1r_2+r_2r_3+r_3r_1)}=4R$

**REGULAR POLYGON**

125. If $A_0, A_1, \dots A_5$ be the consecutive vertices of a regular hexagon inscribed in a unit circle. Then find the product of length of $A_0A_1$, $A_0A_2$ and $A_0A_4$.

126. If the Area of circle is $A_1$ and area of regular pentagon inscribed in the circle is $A_2$. Find the ratio of area of two.

127. Let $A_1, A_2, A_3, A_4$ and $A_5$ be the vertices of a regular pentagon inscribed in a unit circle taken in order. Show that $A_1A_2 \times A_1A_3 = \sqrt{5}$.

128. The sides of a regular do-decagon is 2 ft. Find the radius of the circumscribed circle.

129. A regular pentagon and a regular decagon have the same perimeter. Find the ratio of its area.

130. If $2a$ be the sides of a regular polygon of $n$-sides. $R$ and $r$ be the circum-radius and inradius, then prove that
$r+R=a\cot\left(\frac{\pi}{2n}\right)$.

131. A regular pentagon and a regular decagon have the same area, then find the ratio of their perimeter.

132. If the number of sides of two regular polygon having the same perimeter be $n$ and $2n$ respectively, prove that their areas are in the ratio
$$2\cos\left(\frac{\pi}{n}\right):\left(1+\cos\left(\frac{\pi}{n}\right)\right)$$

133. Let $A_1, A_2, A_3, \dots\dots\dots, A_n$ be the vertices of an $n$-sided regular polygon such that $\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$, then find the value of $n$.

134. If $A, A_1, A_2, A_3$ are the areas of incircle and the ex-circles of a triangle, then prove that $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{A}}$.

135. If the perimeter of the circle and the perimeter of the polygon of $n$-sides are same, then prove that the ratio of the area of the circle and the area of the polygon of $n$-sides is $\tan\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

136. Prove that the sum of the radii of the circle, which are respectively inscribed in and circum-scribed about a regular polygon of $n$-sides, is $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$

## LEVEL II

### (Mixed Problems)

1. In $\Delta ABC$, $a > b > c$, if
$\frac{a^3+b^3+c^3}{\sin^3 A+\sin^3 B+\sin^3 C}=8$, then the maximum value of $a$ is
(a) $\frac{1}{2}$ (b) 2 (c) 8 (d) 64

2. Sides of a $\Delta ABC$ are in AP. If $a < \min\{b, c\}$, then cos $A$ may be equal to
(a) $\frac{3c-4b}{2a}$ (b) $\frac{3c-4b}{2c}$
(c) $\frac{4c-3b}{2b}$ (d) $\frac{4c-3b}{2c}$

3. If a $\Delta ABC$, $a^4+b^4+c^4=2a^2b^2+b^2c^2+2c^2a^2$, then sin $A$ is
(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

4. In a $\Delta ABC$, $2a^2+4b^2+c^2=4ab+2ac$, then the numerical value of cos $B$ is
(a) 0 (b) $\frac{3}{8}$ (c) $\frac{5}{8}$ (d) $\frac{7}{8}$

5. If $a, b, c$ be the sides of $\Delta ABC$ and if roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ are equal then $\sin^2\left(\frac{A}{2}\right)\cdot\sin^2\left(\frac{B}{2}\right)\cdot\sin^2\left(\frac{C}{2}\right)$ are in
(a) AP (b) GP (c) HP (d) AGP

6. In a $\Delta ABC$, $(a+b+c)(b+c-a)=kbc$ if
(a) $k<0$ (b) $k>6$
(c) $0<k<4$ (d) $k>4$

7. $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)$ is equal
(a) $3abc$ (b) $(a+b+c)$
(c) $abc(a+b+c)$ (d) 0

8. If $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$ and $a=2$, then the area of the triangle is
(a) 1 (b) 2 (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$

9. If in a $\Delta ABC$, $\cos A+2\cos B+\cos C=2$, then $a, b, c$ are in
(a) AP (b) GP (c) HP (d) None

10. If in a $\Delta ABC$, $\sin^3 A+\sin^3 B+\sin^3 C=3\sin A\sin B\sin C$, then the value of
$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$
(a) 0
(b) $(a+b+c)^2$

(c) $(a + b + c)(ab + bc + ca)$
(d) None

11. If $b + c = 3a$, then the value of $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$ is
(a) 1 (b) 2 (c) $\sqrt{3}$ (d) $\sqrt{2}$

12. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. The product of length of the line segment $A_0A_1$, $A_0A_2$, $A_0A_4$ is
(a) $\frac{3}{4}$ (b) $3\sqrt{3}$ (c) 3 (d) $\frac{3\sqrt{3}}{2}$

13. In a $\Delta ABC$, the value of $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is
(a) $\frac{R}{r}$ (b) $\frac{R}{2r}$ (c) $\frac{r}{R}$ (d) $\frac{2r}{R}$

14. In a $\Delta ABC$, the sides $a, b, c$ are the roots of the equation $x^3 - 11x^2 + 38x - 40 = 0$.
Then $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ is
(a) 1 (b) $\frac{3}{4}$ (c) $\frac{9}{16}$ (d) None

15. The ex-radii of a $\Delta$ $r_1, r_2, r_3$ are in AP, then the sides $a$, $b$, $c$ are in
(a) AP (b) GP (c) HP (d) AGP

16. In any $\Delta ABC$, $\sum\frac{\sin^2 A + \sin A + 1}{\sin A}$ is always greater than
(a) 9 (b) 3 (c) 27 (d) 36

17. In a triangle $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right) = 2$, then the triangle is
(a) right angled (b) isosceles
(c) equilateral (d) None

18. In a $\Delta ABC$, $a = 2b$ and $|A - B| = \frac{\pi}{3}$, then $\angle C$ is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) None

19. If the median of $\Delta ABC$, through $A$ is perpendicular to $AB$, then
(a) $\tan A + \tan B = 0$ (b) $2\tan A + \tan B = 0$
(c) $\tan A + 2\tan B = 0$ (d) None

20. In a $\Delta ABC$, $\cos A + \cos B + \cos C = \frac{3}{2}$, then the $\Delta$
(a) isosceles (b) right angled
(c) equilateral (d) None

21. If $A_1 \ldots A_n$ be a regular polygon of $n$-sides and $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, then
(a) $n = 5$ (b) $n = 6$ (c) $n = 7$ (c) None

22. In a $\Delta ABC$, $\tan\left(\frac{A}{2}\right) = \frac{5}{6}$ and $\tan\left(\frac{C}{2}\right) = \frac{2}{5}$, then
(a) $a, c, b \in$ AP (b) $a, b, c \in$ AP
(c) $b, a, c \in$ AP (d) $a, b, c \in$ GP

23. If the angles of a triangle are in the ratio 1 : 2 : 3, then the corresponding sides are in the ratio
(a) 2 : 3 : 1 (b) $\sqrt{3}$ : 2 : 1
(c) 2 : $\sqrt{3}$ : 1 (d) 1 : $\sqrt{3}$ : 2

24. In a $\Delta ABC$, $a\cot A + b\cot B + c\cot C$ is
(a) $r + R$ (b) $r - R$
(c) $2(r + R)$ (d) $2(r - R)$

25. If $A$, $A_1$, $A_2$, $A_3$ are the areas of in circle and the ex circles of a triangle, then $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$ is
(a) $\frac{2}{\sqrt{A}}$ (b) $\frac{1}{\sqrt{A}}$ (c) $\frac{1}{2\sqrt{A}}$ (d) $\frac{3}{\sqrt{A}}$

26. In any $\Delta ABC$, $\Pi\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right)$ is always greater than
(a) 9 (b) 3 (c) 27 (d) 36

27. In an equilateral triangle, $R : r : r_2$ is
(a) 1 : 1 : 1 (b) 1 : 2 : 3
(c) 2 : 1 : 3 (d) 3 : 2 : 4

28. In a $\Delta ABC$, $\tan A \tan B \tan C = 9$. For such triangles, if $\tan^2 A + \tan^2 B + \tan^2 C = \lambda$, then
(a) $9.\sqrt[3]{3} < \lambda < 27$ (b) $\lambda \nleq 27$
(c) $\lambda < 9.\sqrt[3]{3}$ (d) $\lambda > 27$

29. In a $\Delta ABC$, $a^2\cos^2 A = b^2 + c^2$, then
(a) $A < \frac{\pi}{4}$ (b) $\frac{\pi}{4} < A < \frac{\pi}{2}$
(c) $A > \frac{\pi}{2}$ (d) $A = \frac{\pi}{2}$

30. In a $\Delta ABC$, $A : B : C = 3 : 5 : 4$, then $a + b + c\sqrt{2}$ is
(a) $2b$ (b) $2c$ (c) $3b$ (d) $3a$

31. If $A$, $B$, $C$ are angles of a triangle such that the angle $A$ is obtuse, then $\tan B \tan C <$
(a) 0 (b) 1 (c) 2 (d) 3

32. In a triangle, if $r_1 > r_2 > r_3$, then
(a) $a > b > c$ (b) $a < b < c$
(c) $a > b$ and $b < c$ (d) $a < b$ and $b > c$

33. $4rR\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ is
(a) $s$ (b) $s^2$ (c) $\Delta^2$ (d) $\Delta$

34. If $(a - b)(s - c) = (b - c)(s - a)$, then $r_1, r_2, r_3$ are in
(a) HP (b) GP (c) AP (d) AGP

35. If $c^2 = a^2 + b^2$, $2s = a + b + c$, then $4s(s - a)(s - b)(s - c)$ is
(a) $s^4$ (b) $b^2c^2$ (c) $c^2a^2$ (d) $a^2b^2$

36. $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2}$ is
(a) $\frac{a^2 + b^2 + c^2}{s^2}$ (b) $\frac{\Sigma a^2}{\Delta^2}$
(c) $4R$ (d) $4r$

37. $(r_1 - r)(r_2 - r)(r_3 - r)$ is
(a) $\frac{R}{r}$ (b) $4R^2 r$ (c) $4Rr^2$ (d) $4R$

38. If the sides be 13, 14, 15, then $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$ is
(a) 5 (b) 4 (c) 0 (d) 1

39. $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab}$ is
(a) $\frac{1}{2R} - \frac{1}{r}$ (b) $2R - r$
(c) $r - 2R$ (d) $\frac{1}{r} - \frac{1}{2R}$

40. $r_1 + r_2 =$
(a) $c \tan\left(\frac{C}{2}\right)$ (b) $c \cot\left(\frac{C}{2}\right)$
(c) $c \sin\left(\frac{C}{2}\right)$ (d) $c \cos\left(\frac{C}{2}\right)$

41. $16R^2 r\, r_1\, r_2\, r_3$ is
(a) $abc$ (b) $a^3b^3c^3$ (c) $a^2b^2c^2$ (d) $a^2b^3c^4$

42. If $\frac{r}{r_1} = \frac{r_2}{r_3}$, then
(a) $A = 90°$ (b) $B = 90°$
(c) $C = 90°$ (d) None

43. In a $\Delta ABC$, the value of $r\, r_1 r_2 r_3$ is
(a) $\Delta$ (b) $\Delta^2$ (c) $\Delta^3$ (d) $\Delta^4$

44. If $r_1 = r_2 + r_3 + r$, then the $\Delta$ is
(a) equilateral (b) isosceles
(c) right angled (d) none

45. $(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)$ is
(a) $Rs^2$ (b) $2Rs^2$ (c) $3Rs^2$ (d) $4Rs^2$

46. The diameter of the circum-circle of a triangle with sides 5 cm, 6 cm, 7 cm is
(a) $\frac{3\sqrt{6}}{2}$ cm (b) $2\sqrt{6}$ cm
(c) $\frac{35}{48}$ cm (d) $\frac{35}{2\sqrt{6}}$

47. In a $\Delta ABC$, the sides are in the ratio 4 : 5 : 6. The ratio of the circum-radius and the in-radius is
(a) 8 : 7 (b) 3 : 2 (c) 7: 3 (d) 16 : 7

48. If in a triangle, $R$ and $r$ are the circum radius and in-radius respectively, then the HM of the ex-radii of the triangle is
(a) $3r$ (b) $2R$ (c) $R + r$ (d) None

49. If $a$, $b$ and $c$ are the sides of a triangle $ABC$ and $3a = b + c$, the value of $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$ is
(a) 3 (b) 2 (c) 4 (d) 1

50. In a $\Delta ABC$, if $\cos A + \cos B = 4 \sin 2\left(\frac{C}{2}\right)$, then $a$, $b$ and $c$ are in
(a) AP (b) GP (c) HP (d) None

## Level III

### (Problems for JEE Advanced)

1. The sides of a triangle are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$, prove that the greatest angle is 120°.
2. In any $\Delta ABC$,
$$\cos\theta = \frac{a}{b+c},\ \cos\varphi = \frac{b}{a+c},\ \cos\psi = \frac{c}{a+b}$$
where $\varphi$, $\theta$ and $\phi$ lie between 0 and $\pi$, prove that
$$\tan^2\left(\frac{\theta}{2}\right) + \tan^2\left(\frac{\varphi}{2}\right) + \tan^2\left(\frac{\psi}{2}\right) = 1.$$
3. Given the product $p$ of sines of the angles of a triangle and the product $q$ of their cosines, find the cubic equation, whose co-efficients are functions of $p$ and $q$ and whose roots are the tangents of the angles of the triangle.
4. In a $\Delta ABC$, if
$\sin^3\theta = \sin(A - \theta)\sin(B - \theta)\sin(C - \theta)$
prove that $\cot\theta = \cot A + \cot B + \cot C$.
5. The base of a triangle is divided into three parts. If $t_1$, $t_2$, $t_3$ be the tangents of the angles subtended by these parts at the opposite vertex, prove that
$$\left(\frac{1}{t_1} + \frac{1}{t_2}\right)\left(\frac{1}{t_2} + \frac{1}{t_3}\right) = 4\left(1 + \frac{1}{t_2^2}\right).$$
6. In a $\Delta ABC$, prove that $\cos A + \cos B + \cos C = \left(1 + \frac{r}{R}\right)$.
7. In a $\Delta ABC$, prove that $\sin A + \sin B + \sin C = \frac{s}{R} = \frac{\Delta}{Rr}$.
8. In any $\Delta ABC$, prove that $a \cot A + b \cot B + c \cot C = 2(r + R)$.
9. In any $\Delta ABC$, prove that
$$\cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right) = 2 + \frac{r}{2R}$$
10. If $p_1$, $p_2$ and $p_3$ are the altitudes of a $\Delta ABC$ from the vertices $A$, $B$ and $C$ respectively. then prove that $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$.
11. If the distances of the sides of a $\Delta ABC$ from a circum-center be $x$, $y$ and $z$ respectively, then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$.
12. If in a $\Delta ABC$, $O$ is the circum-center and $R$ is the circum-radius and $R_1$, $R_2$, $R_3$ are the circum-radii of the triangles $\Delta OBC$, $\Delta OCA$ and $\Delta OAB$ respectively, then prove that $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$.
13. In any $\Delta ABC$, prove that
$$\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$$

14. In any $\Delta ABC$ prove that $\frac{bc}{r_1}+\frac{ca}{r_2}+\frac{ab}{r_3}$
$$= 2R\left[\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)-3\right]$$
15. In a $\Delta ABC$, prove that $(r + r_1)\tan\left(\frac{B-C}{2}\right)$
$+ (r + r_2)\tan\left(\frac{C-A}{2}\right) + (r + r_3)\tan\left(\frac{A-B}{2}\right) = 0.$
16. If a triangle of maximum area is inscribed within a circle of radius $R$, then prove that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{\sqrt{2}+1}{R}$.
17. Let $A_1, A_2, A_3, \ldots, A_n$ be the vertices of an $n$-sided regular polygon such that $\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$, then find the value of $n$.
18. If $A, A_1, A_2, A_3$ are the areas of incircle and the ex-circles of a triangle, then prove that
$$\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{A}}$$
19. The sides of a triangle are in AP and the greatest and the least angles are $\theta$ and $\varphi$, then prove that $4(1 + \cos\theta)(1 - \cos\varphi) = \cos\theta + \cos\varphi$.
20. If $\alpha, \beta, \gamma$ are the lengths of the altitudes of a $\Delta ABC$, then prove that
$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{1}{\Delta}(\cot A+\cot B+\cot C)$$
21. If $p_1, p_2, p_3$ are the altitudes of a triangle from the vertices $A, B, C$ and $\Delta$ be the area of the $\Delta ABC$, prove that
$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{2ab}{(a+b+c)\times\Delta}\times\cos^2\left(\frac{C}{2}\right)$$
22. Three circles whose radii are $a, b, c$ touch one another externally and the tangents at their points of contact meet in a point, prove that the distance of this point from either of their points of contact is
$$\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}$$
23. Two circles of radii $a$ and $b$ cut each other at an angle $\theta$, then prove that the length of the common chord is $\frac{2ab\sin\theta}{\sqrt{(a^2+b^2+2ab\cos\theta)}}$.
24. If the sides of a triangle are in AP and if the greatest angle exceeds the least angle by $\alpha$, then show that the sides are in the ratio $(1 - x) : x : (1 + x)$ where $x=\sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}$
25. The sides $a, b, c$ of a $\Delta ABC$ are the roots of $x^3 - px^2 + qx - r = 0$, then prove that its area is $\frac{1}{4}\sqrt{p(4pq-p^3-8r)}$.
26. Let $O$ be a point inside a $\Delta ABC$ such that $\angle OAB = \angle OBC = \angle OCA = \omega$ then prove that
(i) $\cot\omega = \cot A + \cot B + \cot C$
(ii) $\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$
27. $ABCD$ is a trapezium such that $AB$ is parallel to $CD$ and $CB$ is perpendicular to them. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then prove that $AB=\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$.
28. In an $\Delta ABC$, if $\theta$ be any angle, then prove that $b\cos\theta = c\cos(A - \theta) + a\cos(C + \theta)$.
29. If the median of a $\Delta ABC$ through $A$ is perpendicular to $AB$, prove that $\tan A + 2\tan B = 0$.
30. In a $\Delta ABC$, prove that
$$\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\le\frac{1}{8}$$
31. In a $\Delta ABC$, prove that
$$\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)+\tan^2\left(\frac{C}{2}\right)\ge 1$$
32. If the distances of the vertices of a triangle from the points of contact of the incircle with the sides be $\alpha, \beta, \gamma$, then prove that $r^2=\frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}$ where $r$ is in-radius.
33. If $t_1, t_2$ and $t_3$ are the lengths of the tangents drawn from the centre of the ex-circle to the circum-circle of $\Delta ABC$, prove that
$$\frac{1}{t_1^2}+\frac{1}{t_2^2}+\frac{1}{t_3^2}=\frac{abc}{a+b+c}$$
34. In $\Delta ABC$, find the min value of
$$\frac{\sum\cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right)}{\prod\cot^2\left(\frac{A}{2}\right)}$$
35. A triangle has base 10 cm long and the base angles are 50° and 70°. If the perimeter of the triangle is $x + y\cos(z°)$ where $z \in (0, 90°)$, then find the value of $(x + y + z)$.
36 In a $\Delta ABC$, find the value of
$$\frac{a\cos A+b\cos B+c\cos C}{a+b+c}.$$
37. In a right angled $\Delta ABC$, The hypotenuse $BC$ of length '$a$' is divided into $n$ equal parts ($n$ an odd positive integers) Let $\alpha$ be the acute angle subtending from $A$ by that segment which contains the mid-point of the hypotenuse of the triangle, prove that $\tan\alpha=\frac{4nh}{(n^2-1)a}$

**[Roorkee, 1983]**

***Note:*** No questions asked in 1984, 1985.

38. If $a$, $b$, $c$ are the sides of a $\Delta ABC$ and $3a = b + c$, then prove that
$$\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right) = 2$$
**[Roorkee, 1986]**

39. If in a triangle, $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A - B)}{\sin(A + B)}$. Prove that it is either a right angled triangle or an isosceles triangle. **[Roorkee, 1987]**

40. In any $\Delta ABC$, show that
$$\left(\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right)\right)\left(a\sin^2\left(\frac{B}{2}\right) + b\sin^2\left(\frac{A}{2}\right)\right) = c\cot\left(\frac{C}{2}\right)$$
**[Roorkee, 1988]**

***Note:*** No questions asked in 1989.

41. If $x$, $y$, $z$ are the perpendicular distances of the vertices of a $\Delta ABC$ from the opposite sides and $\Delta$ be the area of the triangle, then prove that
$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{\Delta}(\cot A + \cot B + \cot C)$$
**[Roorkee Main, 1990]**

42. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides. **[Roorkee, 1991]**

***Note:*** No questions asked in 1992.

43. In a $\Delta ABC$, $R$ is the circum-radius and $8R^2 = a^2 + b^2 + c^2$. The the triangle $ABC$ is
(a) acute angled (b) right angled
(c) obtuse angled (d) none of these

44. If the sides $a$, $b$, $c$ of a triangle are in AP, then find the value of $\tan\left(\frac{A}{2}\right) + \tan\left(\frac{C}{2}\right)$ in terms of $\cot\left(\frac{B}{2}\right)$ **[Roorkee, 1993]**

45. A cyclic quadrilateral $ABCD$ of area $\frac{3\sqrt{3}}{4}$ is inscribed in a unit circle. If one of its sides $AB = 1$ and the diagonal $BD = \sqrt{3}$. Find the length of the other sides. **[Roorkee, 1995]**

46. In a $\Delta ABC$, $\angle C = 60°$ and $\angle A = 75°$. If $D$ is a point on $AC$ such that the area of the $\Delta BAD$ is $\sqrt{3}$ times the area of the $\Delta BCD$, find the angle $\angle ABD$. **[Roorkee, 1996]**

47. If in a $\Delta ABC$, $a = 6$, $b = 3$ and $\cos(A - B) = \frac{4}{5}$, then find its area. **[Roorkee Main, 1997]**

48. Two sides of a triangle are of lengths $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30°. How many such triangles are possible? Find the length of their third side and area. **[Roorkee Main, 1998]**

49. The radii $r_1$, $r_2$, $r_3$ of escribed circles of a $\Delta ABC$ are in the harmonic progression. If its area is 24sq. cm. and its perimeter is 24 cm, then find the lengths of its sides. **[Roorkee Main, 1999]**

***Note:*** No questions asked in 2000, 2001.

50. In a $\Delta ABC$, let the sides $a$, $b$, $c$ are the roots of $x^3 - 11x^2 + 38x - 40 = 0$. If the value of $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{m}{n}$, where $m$ and $n$ are least +ve integers, then find $(m + n)$

## LEVEL IV

### (Tougher Problems for JEE Advanced)

1 If in a $\Delta ABC$,
$$\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C},$$
prove that the triangle $ABC$ is either isosceles or right angled.

2. In a $\Delta ABC$, if
$$a\tan A + b\tan B = (a + b)\tan\left(\frac{A + B}{2}\right),$$
prove that the triangle is isosceles.

3. In any $\Delta ABC$, prove that,
$$(r_2 + r_2)(r_3 + r_2)\sin \mathrm{C} = 2r_3\sqrt{r_2r_3 + r_1r_3 + r_1r_2}$$

4. In any $\Delta ABC$, prove that,
$$r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - (a^2 + b^2 + c^2)$$

5. In any triangle $ABC$, prove that,
$$\left((r + r_1)\tan\left(\frac{B - C}{2}\right)\right) + \left((r + r_2)\tan\left(\frac{C - A}{2}\right)\right) + \left((r + r_3)\tan\left(\frac{C - A}{2}\right)\right) = 0$$

6. In any $\Delta ABC$, prove that,
$$\frac{\tan\left(\frac{A}{2}\right)}{(a - b)(a - c)} + \frac{\tan\left(\frac{B}{2}\right)}{(b - a)(b - c)} + \frac{\tan\left(\frac{C}{2}\right)}{(c - a)(c - b)} = \frac{1}{\Delta}$$

7. If $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, prove that the triangle is right angled triangle.

8. In $\Delta ABC$, prove that,
$$\frac{\text{area of the incircle}}{\text{area of triangle } ABC} = \frac{\pi}{\cot\left(\frac{A}{2}\right)\cdot\cot\left(\frac{B}{2}\right)\cdot\cot\left(\frac{C}{2}\right)}$$

9. If $a$, $b$, $c$ are in AP, prove that,
$$\cos A \cdot \cot\left(\frac{A}{2}\right), \cos B \cdot \cot\left(\frac{B}{2}\right),$$
$$\cos C \cdot \cot\left(\frac{C}{2}\right) \text{ are in AP.}$$
10. If the circumference of the $\Delta ABC$ lies on its incircle, then prove that,
$$\cos A + \cos B + \cos C = \sqrt{2}$$
11. $ABCD$ is a trapezium such that $AB$, $DC$ are parallel and $BC$ is perpendicular to them. If angle $\angle ADB = \theta$, $BC = p$ and $CD = q$, prove that, $AB = \dfrac{(p^2+q^2)\sin\theta}{p\sin\theta + q\sin\theta}$
12. Let $O$ be the circumcenter and $H$ be the orthocenter of $\Delta ABC$. If $Q$ is the mid-point of $OH$, then show that
$$AQ = \frac{R}{2}\sqrt{1 + 8\cos A\cos B\cos C}$$
13. If $I_1$, $I_2$ and $I_3$ are the centres of escribed circles of $\Delta ABC$, prove that the area of $\Delta I_1I_2I_3 = \dfrac{abc}{2r}$
14. In $\Delta ABC$, prove that,
$$a^2(s-a) + b^2(s-b) + c^2(s-c)$$
$$= 4R\Delta\left(1 - 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)$$
15. Let $O$ be a point inside a $\Delta ABC$ such that $\angle OAB = \angle OBC = \angle OCA = \omega$, then prove that
(i) $\cot A + \cot B + \cot C = \cot\omega$
(ii) $\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C = \operatorname{cosec}^2\omega$
16. Find the distance between the circum-center and the mid-points of the sides of a triangle.
17. Find the distance between the in-center and the angular points of a triangle.
18. Prove that the distance between the circum-centre(O) and the in-center(I) is
$$OI = R \times \sqrt{\left(1 - 8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)}$$
19. Prove that the ratio of circum-radius and inradius of an equilateral triangle is 1/2.
20. Prove that the ratio of the area of the in-circle to the area of a triangle is
$$\pi : \tan\left(\frac{A}{2}\right)\cdot\tan\left(\frac{B}{2}\right)\cdot\tan\left(\frac{C}{2}\right)$$
21. Prove that the distance of the orthocenter from the sides and angular points of a triangle is $2R\cos A$, $2R\cos B$ and $2R\cos C$.
22. Prove that the distance between the circum-center and the orthocenter of a triangle is $OH =$
$$R\sqrt{1 - 8\cos A\cdot\cos B\cdot\cos C}$$
23. Prove that the area of an ex-central triangle is $8R^2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$.
24. Prove that the circum-radius of an ex-central triangle is $\dfrac{I_2I_3}{2\sin(I_1I_2I_3)} = 2R$.
25. Prove that the the distance between the in-center and the ex-centers are
$$II_1 = 4R\sin\left(\frac{A}{2}\right), II_2 = 4R\sin\left(\frac{B}{2}\right),$$
$$II_2 = 4R\sin\left(\frac{C}{2}\right),$$
26. If $a^2$, $b^2$, $c^2$ are in AP, then prove that $\cot A$, $\cot B$, $\cot C$ are in AP.
27. In any $\Delta ABC$, prove that
$a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-\mathrm{B}) = 3abc$
28. The sides of a triangle are in AP and the greatest and least angles are $\theta$ and $\phi$ respectively, then prove that
$$4(1-\cos\theta)(1-\cos\varphi) = \cos\theta + \cos\varphi$$
29. If in a triangle, the bisector of the side $c$ be perpendicular to the side $d$, prove that $2\tan A + \tan C = 0$.
30. In any triangle, if $\theta$ be any angle, then prove that, $b\cos\theta = c\cos(A-\theta) + a\cos(C+\theta)$.
31. If $AD$, $BE$ and $CF$ are the internal bisectors of the angles of a $\Delta ABC$, prove that
$$\frac{1}{AD}\cos\left(\frac{A}{2}\right) + \frac{1}{BE}\cos\left(\frac{B}{2}\right) + \frac{1}{CF}\cos\left(\frac{C}{2}\right)$$
$$= \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$
32. Prove that the triangle having sides $3x + 4y$, $4x + 3y$ and $5x + 5y$ units respectively, where $x, y > 0$, is obtuse angled.
33. If $\Delta$ be the area and '$s$' be the semi perimeter of a triangle, then prove that, $\Delta \le \dfrac{s^2}{3\sqrt{3}}$
34. Let $ABC$ be a triangle having altitudes $h_1$, $h_2$ and $h_3$ from the vertices $A$, $B$, $C$ respectively and $r$ be the inradius, prove that,
$$\frac{h_1+r}{h_1-r} + \frac{h_2+r}{h_2-r} + \frac{h_3+r}{h_3-r} \ge 6$$
35. Two circles of radii $a$ and $b$ cut each other at an angle $\theta$. Prove that the length of the common chord is $\dfrac{2ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}$
36. If $\alpha$, $\beta$, $\gamma$ are the distances of the vertices of a triangle from the corresponding points of contact with the incircle, prove that $r^2 = \dfrac{\alpha\beta\gamma}{\alpha+\beta+\gamma}$
37. Tangents are drawn to the in-circle of a triangle $ABC$ which are parallel to its sides. If $x$, $y$, $z$ be the lengths of

the tangents and $a$, $b$, $c$ be the sides of a triangle, then prove that, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

38. If $t_1$, $t_2$ and $t_3$ be the lengths of the tangents from the centres of escribed circles to the circum circles, prove that
$$\frac{1}{t_1^2}+\frac{1}{t_1^2}+\frac{1}{t_1^2}=\frac{2s}{abc}$$

39. If $x$, $y$, $z$ be the lengths of the perpendiculars from the circum-centre on the sides $BC$, $CA$, $AB$ of a $\Delta ABC$, prove that,
$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$$

40. If $p_1$, $p_2$, $p_3$ are the altitudes of the triangle $ABC$ from the vertices $A$, $B$ and $C$ respectively, prove that
$$\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}=\frac{1}{R}$$

41. The product of the sines of the angles of a triangle is $p$ and the product of their cosines is $q$. Prove that the tangents of the angles are the roots of $qx^3 - px^2 + (1 + q)x - p = 0$.

42. In a $\Delta ABC$, if $\cos A \cos B + \sin A \sin B \sin C = 1$, prove that the sides are in the ratio $1: 1 : \sqrt{2}$.

43. The base of a triangle is divided into three equal parts. If $t_1$, $t_2$ and $t_3$ be the tangents of the angles subtended by these parts at the opposite vertices, prove that,
$$\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right)=4\left(1+\frac{1}{t_2^2}\right)$$

44. The three medians of a $\Delta ABC$ make angle $\alpha$, $\beta$, $\gamma$ with each other, prove that,
$$\cot\alpha+\cot\beta+\cot\gamma$$
$$+\cot A+\cot B+\cot C=0$$

45. Perpendiculars are drawn from the angles $A$, $B$, $C$ of an acute angled triangle on the opposite sides and produced to meet the circumscribing circle. If these parts be $\alpha$, $\beta$, $\gamma$ respectively, then prove that
$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=2(\tan A+\tan B+\tan C)$$

46. In a $\Delta ABC$, the vertices $A$, $B$, $C$ are at distances of $p$, $q$, $r$ from the orthocentre respectively. Prove that $\left(\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)=\frac{abc}{pqr}$

47. The internal bisectors of the angles of a $\Delta ABC$ meet the sides in $D$, $E$ and $F$. Prove that the area of the $\Delta DEF$ is
$$\frac{2\Delta abc}{(a+b)(b+c)(c+a)}$$

48. In a $\Delta ABC$, the measures of the angles $A$, $B$, $C$ are $3\alpha$, $3\beta$ and $3\gamma$ respectively. $P$, $Q$, and $R$ are the points within the triangle such that $\angle BAR = \angle RAQ = \angle QAC = \alpha$, $\angle CBP = \angle PBR = \angle RBA = \beta$ and $\angle ACQ = \angle QCP = \angle PCB = \gamma$, then prove that $AR = 8R \sin\beta \sin\gamma \cos(30° - \beta)$

49. If in a $\Delta ABC$, the median $AD$ and the perpendicular $AE$ from the vertex $A$ to the side $BC$ divides the angle $A$ into three equal parts, show that $\cos\left(\frac{A}{3}\right)\cdot\sin^2\left(\frac{A}{3}\right)=\frac{3a^2}{32bc}$.

50. If the sides of a triangle are in AP and if its greatest angle exceeds the least angle by $\alpha$, show that the sides are in the ratio
$$(1-x):1:(1+x), \text{ where } x=\sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}$$

51. Let $A$, $B$, $C$ be three angles such that $A=\frac{\pi}{4}$ and $\tan B \tan C = p$. Find all the possible values of $p$ such that $A$, $B$, $C$ are the angles of a triangle.
Ans. $p < 0$ and $p \geq 3 + 2\sqrt{2}$.

## Integer Type Questions

1. In any right angled triangle, find the value of $\frac{a^2+b^2+c^2}{R^2}$

2. In any $\Delta ABC$, $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$ find the value of $\left(\frac{r_1+r_2+r_3}{r}\right)$

3. In any $\Delta ABC$, find the minimum value of $\left(\frac{r_1+r_2+r_3}{r}\right)$

4. In any $\Delta ABC$, find the minimum value of $\left(\frac{\sin^2 A+\sin A+1}{\sin A}\right)$

5. In a $\Delta ABC$, find the value of
$$\frac{1}{8R^2}(r^2+r_1^2+r_2^2+r_3^2+(a^2+b^2+c^2))$$
where $r$ = in-radius, $R$ = circum-radius and $r_1$, $r_2$, $r_3$ are ex-radii.

6. In a $\Delta ABC$, the median $AD=\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides the angle $A$ into angles 30° and 45°. Find the length of the side $BC$.

7. In a $\Delta ABC$, find the value of $\frac{(r_1+r_2)(r_2+r_3)(r_1+r_3)}{Rs^2}$, where $R$ = circum-radius, $r_1$, $r_2$, $r_3$ are ex-radii. and $s$ is the semi perimeter.

8. If in a $\Delta ABC$, $a = 6$, $b = 3$ and $\cos(A-B)=\frac{4}{5}$, then find its area.

9. A triangle has base 10 cm long and the base angles are 50° and 70°. If the perimeter of the triangle is $x + y \cos(z°)$ where $z \in (0, 90°)$, then find the value of $\left(\dfrac{x+y+z}{y}\right)$

10. In any $\Delta ABC$, find the value of
$$\frac{a \cot A + b \cot B + c \cot C}{(r+R)}$$

## Comprehensive Link Passages

In these questions, a passage (paragraph) has been given followed by questions based on each of the passage. You have to answer the questions based on the passage given.

**Passage I**

If $p_1, p_2, p_3$ are the altitudes of a $\Delta ABC$, from the vertices $A$, $B$, $C$ respectively and $\Delta$ is the area of the triangle and '$s$' is the semi perimeter of the triangle.

On the basis of the above information, answer the following questions.

1. If $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \dfrac{1}{p_3} = \dfrac{1}{2}$, then the least value of $p_1, p_2, p_3$ is
   (a) 8 (b) 27 (c) 125 (d) 216

2. The value of $\dfrac{\cos A}{p_1} + \dfrac{\cos B}{p_2} + \dfrac{\cos C}{p_3}$ is
   (a) $\dfrac{1}{r}$ (b) $\dfrac{1}{r}$
   (c) $\dfrac{a^2+b^2+c^2}{2R}$ (d) $\dfrac{1}{\Delta}$

3. The minimum value of $\dfrac{b^2 p_1}{c} + \dfrac{c^2 p_1}{a} + \dfrac{a^2 p_1}{b}$ is
   (a) $\Delta$ (b) $2\Delta$ (c) $3\Delta$ (d) $4\Delta$

4. The value of $\dfrac{1}{p_1^3} + \dfrac{1}{p_2^3} + \dfrac{1}{p_3^3}$ is
   (a) $\dfrac{(\Sigma a)^2}{4\Delta^2}$ (b) $\dfrac{(\prod a)^3}{8\Delta^3}$ (c) $\dfrac{\Sigma a^2}{4\Delta^2}$ (d) $\dfrac{\Pi a^2}{8\Delta^2}$

5. In the $\Delta ABC$, the altitudes are in AP, then
   (a) $a, b, c$, are in AP (b) $a, b, c$ are in HP
   (c) $a, b, c$ are in GP (d) angles $A, B, C$ are in AP

**Passage 2**

$ABCD$ be a cyclic quadrilateral inscribed in a circle of radius $R$.

Then $\cos B = \dfrac{(a^2+b^2-c^2-d^2)}{2(ab+cd)}$ and the area of the quadrilateral $= \dfrac{1}{2}(ab+cd)\sin B$

$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$,

where, $s = \dfrac{a+b+c+d}{2}$.

Also, $AC^2 \cdot BD^2 = (ac+bd)^2$

i.e., $AC \cdot BD = AB \cdot CD + BC \cdot AD$

and $R = \dfrac{AC}{2 \sin B}$

On the basis of the above information, answer the following questions.

1. The side of a quadrilateral which can be inscribed in a circle are 6, 6, 8 and 8 cm. Then the circum radius is
   (a) $\dfrac{5}{2}$ cm (b) $\dfrac{24}{7}$ cm (c) $\dfrac{11}{7}$ cm (d) None

2. The sides of a quadrilateral, which can be inscribed in a circle are 5, 5, 12 and 12 cm. Then the in-radius is
   (a) $\dfrac{15}{17}$ cm (b) $\dfrac{30}{17}$ cm (c) $\dfrac{60}{17}$ cm (d) None

3. If a quadrilateral with sides $a, b, c, d$ can be inscribed in one circle and circumscribed about an other circle, then its area is
   (a) $\sqrt{abcd}$ (b) $\sqrt{2(abcd)}$
   (c) $2 \times \sqrt{(abcd)}$ (d) None.

**Passage 3**

$G$ is the centroid of the $\Delta ABC$. Perpendiculars from vertices $A, B, C$ meet the sides $BC, CA, AB$ at $D, E, F$ respectively. $P, Q, R$ are the feet of perpendiculars from $G$ on sides $BC, CA, AB$ respectively, $L, M, N$ are the mid points of the sides $BC, CA, AB$ respectively.

On the basis of the above information, answer the following questions.

1. Length of the side $PG$ is
   (a) $\dfrac{1}{2} b \sin C$ (b) $\dfrac{1}{2} c \sin C$
   (c) $\dfrac{2}{3} b \sin C$ (d) $\dfrac{1}{3} c \sin B$

2. ar($\Delta GPL$) : ar($\Delta ALD$) is
   (a) $\dfrac{1}{3}$ (b) $\dfrac{1}{9}$ (c) $\dfrac{2}{3}$ (d) $\dfrac{4}{9}$

3. Area of $\Delta PQR$ is
   (a) $\dfrac{1}{9}(a^2+b^2+c^2) \sin A \cdot \sin B \cdot \sin C$
   (b) $\dfrac{1}{18}(a^2+b^2+c^2) \sin A \cdot \sin B \cdot \sin C$
   (c) $\dfrac{2}{9}(a^2+b^2+c^2) \sin A \cdot \sin B \cdot \sin C$
   (d) $\dfrac{1}{3}(a^2+b^2+c^2) \sin A \cdot \sin B \cdot \sin C$

**Passage 4**

In a $\Delta ABC$, $R$ be the circum radius such that $R = \dfrac{abc}{4\Delta}$ and $r_1$, $r_2$ and $r_3$ are the exradii, where

$r_1 = \dfrac{\Delta}{s-a}, r_2 = \dfrac{\Delta}{s-a}$ and $r_3 = \dfrac{\Delta}{s-a}$ and $r$ be in-radius such that $r = \dfrac{\Delta}{s}$.

On the basis of above information, answer the following questions.

1. If $r_1 = r + r_2 + r_3$, then the triangle is
   (a) equilateral (b) isosceles
   (c) right angled (d) None
2. The value of $\cos A + \cos B + \cos C$ is
   (a) $1+\dfrac{r}{R}$ (b) $2\left(1+\dfrac{r}{R}\right)$
   (c) $\dfrac{1}{2}\left(1+\dfrac{r}{R}\right)$ (d) $\dfrac{1}{4}\left(1+\dfrac{r}{R}\right)$
3. If $\left(1-\dfrac{r_1}{r_2}\right)\left(1-\dfrac{r_1}{r_3}\right) = 2$, then the triangle is
   (a) right angled (b) equilateral
   (c) isosceles (d) None
4. The value of $r_1 + r_2 + r_3 - 4R$ is
   (a) $2r$ (b) $3r$ (c) $r$ (d) $4r$
5. The value of $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}$ is
   (a) $\dfrac{1}{r}$ (b) $\dfrac{2}{r}$ (c) $\dfrac{1}{2r}$ (d) $\dfrac{3}{2r}$
6. The value of $\dfrac{r_1}{bc}+\dfrac{r_2}{ca}+\dfrac{r_3}{ab}$ is
   (a) $\dfrac{1}{r}-\dfrac{1}{R}$ (b) $\dfrac{1}{2r}-\dfrac{1}{R}$
   (c) $\dfrac{1}{r}-\dfrac{1}{2R}$ (d) $\dfrac{1}{r}-\dfrac{1}{3R}$

**Passage 5**

In any $\Delta ABC$,

$$\cos A = \frac{b^2+c^2-a^2}{2bc},$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

and $$\cos C = \frac{a^2+b^2-c^2}{2ab},$$

where $a = BC$, $b = CA$ and $c = AB$ respectively.

On the basis of the above information, answer the following questions.

1. If the angles $A, B, C$ are in AP, then $\dfrac{a+c}{\sqrt{a^2+c^2-ac}}$ is
   (a) $2\cos\left(\dfrac{A-C}{2}\right)$ (b) $2\sin\left(\dfrac{A-C}{2}\right)$
   (c) $\sin\left(\dfrac{A-C}{2}\right)$ (d) $\cos\left(\dfrac{A-C}{2}\right)$
2. If $\dfrac{1}{a+c}+\dfrac{1}{b+c}=\dfrac{1}{a+b+c}$, then
   (a) $\angle C = 75°$ (b) $\angle A = 75°$
   (c) $\angle A = 60°$ (d) $\angle C = 60°$
3. The value of $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}$ is
   (a) $\dfrac{a^2+b^2+c^2}{2abc}$ (b) $\dfrac{a^2+b^2+c^2}{abc}$
   (c) $\dfrac{a^2-b^2+c^2}{abc}$ (d) $\dfrac{a^2+b^2-c^2}{abc}$
4. If $c^4 - 2(a^2+b^2)c^2 + a^4 + a^2b^2 + b^4 = 0$, then the angle $C$ is
   (a) 60° (b) 30° (c) 75° (d) 45°
5. The value of $\dfrac{\cos 2A}{a^2}-\dfrac{\cos 2B}{b^2}$ is
   (a) $\dfrac{1}{a^2}+\dfrac{1}{b^2}$ (b) $\dfrac{1}{a^2}-\dfrac{1}{b^2}$
   (c) $\dfrac{2}{a^2}-\dfrac{2}{b^2}$ (d) $\dfrac{2}{a^2}+\dfrac{2}{b^2}$.

## Matrix Match
## (For JEE-Advanced Examination Only)

1. Match the following columns
   In any $\Delta ABC$, then

| | Column I | | Column II |
|---|---|---|---|
| (A) | $b\cos C + c\cos B$ | (P) | $c$ |
| (B) | $c\cos A + a\cos C$ | (Q) | $b$ |
| (C) | $a\cos B + b\cos A$ | (R) | $a$ |
| (D) | $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C$ | (S) | $a+b+c$ |

2. Match the following columns
   In any $\Delta ABC$, the value of

| | Column I | | Column II |
|---|---|---|---|
| (A) | $a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$ is | (P) | $3abc$ |
| (B) | $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B)$ is | (Q) | $abc$ |
| (C) | $\left(\dfrac{b^2-c^2}{a^2}\right)\sin 2A + \left(\dfrac{c^2-a^2}{b^2}\right)\sin 2B + \left(\dfrac{a^2-b^2}{c^2}\right)\sin 2C$ is | (R) | 0 |
| (D) | $a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B)$ is | (S) | $2abc$ |

3. Match the following columns
In any $\Delta ABC$, the value of

| | Column I | | Column II |
|---|---|---|---|
| (A) | $\tan B \cot C$ is | (P) | $(a^2 + b^2 + c^2)$ |
| (B) | $2(bc \cos A + ca \cos B + ab \cos C)$ is | (Q) | $\dfrac{(a^2+b^2-c^2)}{(a^2+b^2+c^2)}$ |
| (C) | $\dfrac{\cot A + \cot B + \cot C}{\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)}$ is | (R) | $\dfrac{a^2+b^2+c^2}{(a+b+c)^2}$ |
| (D) | $\dfrac{\cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)}{\cot\left(\frac{A}{2}\right)}$ is | (S) | $\left(\dfrac{2a}{b+c-a}\right)$ |

4. Match the following columns
In any $\Delta ABC$, if

| | Column I | | Column II |
|---|---|---|---|
| (A) | $\cot A, \cot B, \cot C$ are in AP, then | (P) | $a, b, c$ are in AP |
| (B) | $\cos A \cdot \cot\left(\frac{A}{2}\right)$, $\cos B \cdot \cot\left(\frac{B}{2}\right)$, $\cos C \cdot \cot\left(\frac{C}{2}\right)$ are in AP, then | (Q) | $a^2, b^2, c^2$ are in AP |
| (C) | $\sin^2\left(\frac{A}{2}\right), \sin^2\left(\frac{B}{2}\right)$, $\sin^2\left(\frac{C}{2}\right)$ are in AP, then | (R) | $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP |
| (D) | $\tan\left(\frac{A}{2}\right), \tan\left(\frac{B}{2}\right), \tan\left(\frac{C}{2}\right)$ are in AP, then | (S) | $a, b, c$ are in GP |

5. Match the following columns
In any $\Delta ABC$, if

| | Column I | | Column II |
|---|---|---|---|
| (A) | $\cot A + \cot B + \cot C = \sqrt{3}$, then $\Delta$ is | (P) | Isoceles |
| (B) | $(a^2 + b^2) \sin (A - B) = (a^2 - b^2) \sin (A + B)$, then $\Delta$ is | (Q) | Right angled |
| (C) | $2 \cos A = \dfrac{\sin B}{\sin C}$, then $\Delta$ is | (R) | Equilateral |
| (D) | $a \tan A + b \tan B$, $= (a+b) \tan\left(\dfrac{A+B}{2}\right)$ then $\Delta$ is | (S) | Acute angled |

6. Match the following columns
In any $\Delta ABC$, if

| | Column I | | Column II |
|---|---|---|---|
| (A) | $\sin A + \sin B + \sin C = \dfrac{3\sqrt{3}}{2}$, then $\Delta$ is | (P) | Isoceles |
| (B) | $\tan A + \tan B + \tan C = 3\sqrt{3}$, then $\Delta$ is | (Q) | Right angled |
| (C) | $8 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right) = 1$, then $\Delta$ is | (R) | Equilat-eral |
| (D) | $a^2 + b^2 + c^2 = 8R^2$, then $\Delta$ is | (S) | Obtuse angled |

7. Match the following columns
In any $\Delta ABC$, if $r$ be the inradius and $r_1, r_2$ and $r_3$ be the ex-radii of the given $\Delta ABC$, then the value of

| | Column I | | Column II |
|---|---|---|---|
| (A) | $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$ is | (P) | $r$ |
| (B) | $\dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} + \dfrac{1}{r^2}$ is | (Q) | $\dfrac{\Delta^2}{a^2+b^2+c^2}$ |
| (C) | $r_1 + r_2 + r_3 - 4R$ is | (R) | $\dfrac{1}{r}$ |
| (D) | $\dfrac{r_1}{bc} + \dfrac{r_2}{ac} + \dfrac{r_3}{ba} + \dfrac{1}{2R}$ is | (S) | $\dfrac{a^2+b^2+c^2}{\Delta^2}$ |

8. Match the following columns
In any $\Delta ABC$, if $r$ be the inradius and $R$ be the circum-radius of the given $\Delta ABC$, then the value of

| | Column I | | Column II |
|---|---|---|---|
| (A) | $\cos A + \cos B + \cos C$ is | (P) | $\dfrac{\Delta}{r \cdot R}$ |
| (B) | $a \cot A + b \cot B + c \cot C$ is | (Q) | $\left(1 + \dfrac{r}{R}\right)$ |
| (C) | $\sin A + \sin B + \sin C$ is | (R) | $2(r + R)$ |

| | | | |
|---|---|---|---|
| (D) | $\cos^2\left(\frac{A}{2}\right)+\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)$ | (S) | $\left(2+\frac{r}{2R}\right)$ |

9. Match the following columns
In any $\Delta ABC$, if $r$ be the inradius and $r_1$, $r_2$ and $r_3$ be the ex-radii of the given $\Delta ABC$, then the value of

| Column I | | Column II | |
|---|---|---|---|
| (A) | $r - r_1 + r_2 + r_3$ is | (P) | $4R \cos C$ |
| (B) | $r - r_2 + r_1 + r_3$ is | (Q) | $4R \cos B$ |
| (C) | $r - r_3 + r_1 + r_2$ is | (R) | $4R \cos A$ |
| (D) | $r_1 + r_2 + r_3 - r$ is | (S) | |

10. Match the following columns
In any $\Delta ABC$, the minimum value of

| Column I | | Column II | |
|---|---|---|---|
| (A) | $\cot^2 A + \cot^2 B + \cot^2 C$ is | (P) | 9 |
| (B) | $\tan^2 A + \tan^2 B + \tan^2 C$ is | (Q) | 1 |
| (C) | $\operatorname{cosec}\left(\frac{A}{2}\right)+\operatorname{cosec}\left(\frac{B}{2}\right)+\operatorname{cosec}\left(\frac{C}{2}\right)$ is | (R) | 6 |
| (D) | $\cos A + \cos B + \cos C$ is | (S) | 3. |

11. Match the following columns
In any $\Delta ABC$, then the maximum value of

| Column I | | Column II | |
|---|---|---|---|
| (A) | $\cos A + \cos B + \cos C$ is | (P) | 1 |
| (B) | $\cos A \,.\, \cos B \,.\, \cos C$ is | (Q) | $\frac{3}{2}$ |
| (C) | $\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)+\tan^2\left(\frac{C}{2}\right)$ is | (R) | $\frac{1}{4}$ |
| (D) | $\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)$ is | (S) | $\frac{1}{8}$ |

## Assertion and Reason

**Codes:**

(A) Both A and R are individually true and R is the correct explanation of A.
(B) Both A and R are individually true and R is not the correct explanation of A.
(C) A is true and R is false.
(D) A is false and R is true.

1. *Assertion (A):* If $\Delta$ be the area of a triangle and $s$ be the semi-perimeter, then $\Delta^2 \le \frac{s}{4}$
*Reason (R):* AM ≥ GM
(a) *A* (b) *B* (c) *C* (d) *D*

2. *Assertion (A):* In a $\Delta ABC$, if $\cos A + 2\cos B + \cos C = 2$, then $a$, $b$, $c$ are in AP
*Reason (R):* In a $\Delta ABC$, $\cos A + \cos B + \cos C = 1 + 4\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)$
(a) *A* (b) *B* (c) *C* (d) *D*

3. *Assertion (A):* In a right angled triangle, $a^2 + b^2 + c^2 = 8R^2$, where $R$ is the circum-radius
*Reason (R):* $a^2 = b^2 + c^2$
(a) *A* (b) *B* (c) *C* (d) *D*

4. *Assertion (A):* If $A$, $B$, $C$ and $D$ are the angles of a cyclic quadrilateral, then $\sin A + \sin B + \sin C + \sin D = 0$
*Reason (R):* If $A$, $B$, $C$ and $D$ are the angles of a cyclic quadrilateral, then $\cos A + \cos B + \cos C + \cos D = 0$
(a) *A* (b) *B* (c) *C* (d) *D*

5. *Assertion (A):* In any triangle, $a\cos A + b\cos B + c\cos C \le s$
*Reason(R):* In any triangle, $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \le \frac{1}{8}$
(a) *A* (b) *B* (c) *C* (d) *D*

6. *Assertion (A):* In any $\Delta ABC$, the minimum value of $\frac{r_1 + r_2 + r_3}{r}$ is 9
*Reason (R):* In a $\Delta ABC$, if $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then $\frac{r_1 + r_2 + r_3}{r} = 9$
(a) *A* (b) *B* (c) *C* (d) *D*

7. *Assertion (A):* In a $\Delta ABC$, the harmonic mean of the ex-radii is three times the in-radius.
*Reason (R):* In any $\Delta ABC$, $r_1 + r_2 + r_3 = 4R$
(a) *A* (b) *B* (c) *C* (d) *D*

8. *Assertion (A):* If $A$, $A_1$, $A_2$, $A_3$ are the areas of in-circle and ex-circles of a triangle, then
$$\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{A}}$$
*Reason (R):* In a triangle, $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$
(a) *A* (b) *B* (c) *C* (d) *D*

9. *Assertion (A):* If $x$, $y$ and $z$ are respectively the distances of the vertices of a $\Delta ABC$ from its orthocentre,
$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$$
*Reason (R):* In a $\Delta ABC$
$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$$
(a) *A* (b) *B* (c) *C* (d) *D*

10. *Assertion (A):* If $x$, $y$ and $z$ are respectively the distances of the vertices of a $\Delta ABC$ from its orthocentre, $x + y + z = 2(R + r)$
*Reason (R):* In a $\Delta ABC$,

$$r = 4R\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)$$

(a) A (b) B (c) C (d) D

11. *Assertion (A):* In a $\Delta ABC$, $r_1 + r_2 + r_3 - r = 4R$

*Reason (R):* In a $\Delta ABC$, $R = \frac{abc}{4\Delta}$

(a) A (b) B (c) C (d) D

12. *Assertion (A):* In any triangle $ABC$, $2\left(a\sin^2\left(\frac{C}{2}\right) + c\sin^2\left(\frac{A}{2}\right)\right) = a + c - b$

*Reason (R):* In any $\Delta ABC$, $b = c\cos A + a\cos C$.

(a) A (b) B (c) C (d) D

## Questions Asked In Previous Years' JEE-Advanced Examinations

1. If $p_1, p_2, p_3$ are the altitudes of a triangle from the vertices $A$, $B$, $C$ respectively and $\Delta$ be the area of a triangle, prove that
$$\frac{1}{p_1} + \frac{1}{p_1} + \frac{1}{p_1} = \frac{2ab}{(a+b+c)\Delta}\cos^2\left(\frac{C}{2}\right)$$
**[IIT-JEE, 1978]**

2. A quadrilateral $ABCD$ is inscribed in a circle $S$ and $A$, $B$, $C$, $D$ are the points of contacts with $S$ of an other quadrilateral which is circumscribed about $S$. If this quadrilateral is also cyclic, prove that $AB^2 + CD^2 = BC^2 + AD^2$
**[IIT-JEE, 1978]**

3. If two sides of a triangle and the included angle are given by $a = (\sqrt{3}+1)$ cm, $b = 2$ cm and $C = 60°$, find the other two angles and the third side.
**[IIT-JEE, 1979]**

4. If a circle is inscribed in a right angled triangle $ABC$ with right angled at $B$, show that the diameter of the circle is equal to $AB + BC - AC$. **[IIT-JEE, 1979]**

5. $ABC$ is a triangle, $D$ is the middle point of $BC$. If $AD$ is perpendicular to $AC$, prove that $\cos A\cos C = \frac{2(c^2 - a^2)}{3ac}$
**[IIT-JEE, 1980]**

6. Let the angles $A$, $B$, $C$ of a $\Delta ABC$ be in A.P and let $b:c = \sqrt{3}:\sqrt{2}$. Find the angle $A$. **[IIT-JEE, 1981]**

7. No questions asked in 1982.

8. The ex-radii $r_1, r_2, r_3$ of $\Delta ABC$ are in H.P Show that the sides $a$, $b$, $c$ are in AP. **[IIT-JEE, 1983]**

9 For a triangle $ABC$, it is given that $\cos A + \cos B + \cos C = \frac{3}{2}$, prove that the triangle is equilateral. **[IIT-JEE, 1984]**

10. With usual notation, if in a $\Delta ABC$
$\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, prove that
$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$ **[IIT-JEE, 1984]**

11. In a triangle $ABC$, the median to the side $BC$ is at length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides the angle $A$ into angles of 30° and 45°. Find the length of the side $BC$.
**[IIT-JEE, 1985]**

12. In a triangle $ABC$, if $\cot A$, $\cot B$, $\cot C$ are in AP, then prove that $a^2 + b^2 + c^2$ are in AP.
**[IIT-JEE, 1985]**

13. The set of all real numbers a such that $a^2 + 2a$, $2a + 3$, $a^2 + 3a + 8$ are the sides of a triangle. is...........
**[IIT-JEE, 1985]**

14. The sides of a triangle inscribed in a given circle subtends angle $\alpha$, $\beta$ and $\gamma$ at the centre. The minimum value of the arithmetic mean of $\cos\left(\alpha + \frac{\pi}{2}\right), \cos\left(\beta + \frac{\pi}{2}\right), \cos\left(\gamma + \frac{\pi}{2}\right)$ is ______
**[IIT-JEE, 1986]**

15 In a $\Delta ABC$,
$$\cos A\cos B + \sin A\sin B\sin C = 1,$$
show that $a:b:c = 1:1:\sqrt{2}$ **[IIT-JEE, 1986]**

16. There exists a $\Delta ABC$ satisfying the conditions

(i) $b\sin A = a,\ A < \frac{\pi}{2}$

(ii) $b\sin A > a,\ A > \frac{\pi}{2}$

(iii) $b\sin A > a,\ A < \frac{\pi}{2}$

(iv) $b\sin A < a,\ A < \frac{\pi}{2}, b > a$

**[IIT-JEE, 1986]**

17. In a triangle, the lengths of the two larger sides are 10 and 9 respectively. If the angles are in AP, then the length of the third side can be

(a) $5-\sqrt{6}$ (b) $3\sqrt{3}$ (c) 5 (d) $5+\sqrt{6}$
**[IIT-JEE, 1987]**

18. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$, then the area of the triangle is...... **[IIT-JEE, 1988]**

19. $ABC$ is an isosceles triangle inscribed in a circle of radius $r$. If $AB = AC$ and $h$ is the altitude from $A$ to $BC$, then the triangle $ABC$ has perimeter $P$ =...... and area $\Delta$ = ...... and also $\lim_{x\to 0}\left(\frac{\Delta}{p^3}\right)$ =...... **[IIT-JEE, 1989]**

20. In a triangle $ABC$, $A$ is greater than angle $B$. If the measures of angles $A$ and $B$ satisfy the equation $3\sin x - 4\sin^3 x - k = 0$, then the measure of angle $C$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$
**[IIT-JEE, 1990]**

21. $ABC$ is a triangle such that
$$\sin(2A+B) = \sin(C-A) = -\sin(B+C) = \frac{1}{2}.$$

If $A$, $B$ and $C$ are in AP, determine the values of $A$, $B$ and $C$.

**[IIT-JEE, 1990]**

22. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of a triangle.

**[IIT-JEE, 1991]**

23. In a triangle of base $a$, the ratio of the other two sides is $r$ (<1). Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$

**[IIT-JEE, 1991]**

24. Three circles touch one another externally. The tangents at their points of contact meet a point whose distance from a point of a contact is 4. Find the ratio of the product of the radii to the sum of the radii of the circles.

**[IIT-JEE, 1992]**

25. If in a $\Delta ABC$,

$$\frac{2\cos A}{a}+\frac{2\cos B}{b}+\frac{2\cos C}{c}=\frac{1}{bc}+\frac{b}{ca},$$ then find the angle $A$ in degrees.

**[IIT-JEE, 1993]**

26. Let $A_1, A_2, \ldots\ldots, A_n$ be the vertices of $n$ sided regular polgon such that $\frac{1}{A_1A_2}+\frac{1}{A_1A_3}=\frac{1}{A_1A_2}$, then find $n$.

**[IIT-JEE, 1994]**

27. Consider the following statement concerning a $\Delta ABC$
   (i) The sides $a$, $b$, $c$ and area of $\Delta$ are rational
   (ii) $a, \tan\left(\frac{B}{2}\right), \tan\left(\frac{C}{2}\right)$ are rational
   (iii) a, $\sin A$, $\sin B$, $\sin C$ are rational Then prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

**[IIT-JEE, 1994]**

28. In a $\Delta ABC$, $AD$ is an altitude from $A$, Given $b > c$, $\angle C = 23°$ and $AD=\frac{abc}{a^2-c^2}$, then find $\angle B$.

**[IIT-JEE, 1994]**

29. If the length of the sides of a triangle are 3, 5, 7, then the largest angle of the triangle is
   (a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{3\pi}{4}$

**[IIT-JEE, 1994]**

30. In a $\Delta ABC$, $\angle B=\frac{\pi}{3}$ and $\angle C=\frac{\pi}{4}$. Let $D$ divide $BC$ internally in the ratio 1:3, then $\frac{\sin(\angle BAD)}{\sin(\angle CAD)}$ is equal to
   (a) $\frac{1}{\sqrt{6}}$ (b) $\frac{1}{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{\frac{2}{3}}$

**[IIT-JEE, 1995]**

31. In a $\Delta ABC$, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circum-circle to that of in-circle is ______

**[IIT-JEE, 1996]**

32. Let $ABC$ be three angles such that $A=\frac{\pi}{4}$ and $\tan(B)\tan(C) = p$. Find all possible values of $p$ such that $A$, $B$, $C$ are the angles of a triangle.

**[IIT-JEE, 1997]**

33. Prove that a $\Delta ABC$ is equilateral if and only if $\tan(A)+\tan(B)+\tan(C)=3\sqrt{3}$

**[IIT-JEE, 1998]**

34. If in a $\Delta PQR$, $\sin P$, $\sin Q$, $\sin R$ are in AP then
   (a) the altitude are in AP
   (b) the altitude are in HP
   (c) the medians are in GP
   (d) the medians are in AP

**[IIT-JEE, 1998]**

35. Let $ABC$ be a triangle having $O$ and $I$ as its circum-centre and in-centre respectively. If $R$ and $r$ are the circum-radius and in-radius respectively, then prove that $(IO)^2 = R^2 - 2Rr$. Further show that the $\Delta BIO$ is a right angled triangle if and only if $b$ is the arithmetic mean of $a$ and $c$.

**[IIT-JEE, 1999]**

36. In any $\Delta ABC$, prove that
$$\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)=\cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$$

**[IIT-JEE, 2000]**

37. Let $ABC$ be a triangle with incentre $I$ and in-radius $r$. Let $D$, $E$, $F$ be the feet of the perpendicular from $I$ to the sides $BC$, $CA$ and $AB$ respectively, If $r_1$, $r_2$, $r_3$ are the radii of the circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEIF$ respectively, prove that,
$$\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1r_2r_3}{(r_1-r_2)(r_2-r_3)(r_3-r_1)}$$

**[IIT-JEE, 2000]**

38. In a $\Delta ABC$, $2ac\sin\left(\frac{A-B+C}{2}\right)=$
   (a) $a^2+b^2-c^2$ (b) $c^2+a^2-b^2$
   (c) $b^2-c^2-a^2$ (d) $c^2-a^2-b^2$

**[IIT-JEE, 2000]**

39. In a $\Delta ABC$, let $\angle C=\frac{\pi}{2}$. If $r$ is the in-radius and $R$ is the circum-radius of the triangle, then $2(R + r)$ is equal to
   (a) $a+b$ (b) $b+c$
   (c) $c+a$ (d) $a+b+c$

**[IIT-JEE, 2000]**

40. If $\Delta$ is the area of a triangle with side lengths $a$, $b$ and $c$, then show that $\Delta \le \frac{1}{4}\sqrt{(a+b+c)abc}$. Also show that the equality occurs in the above inequality if and only if $a = b = c$

**[IIT-JEE, 2001]**

41. Which of the following pieces of data does not uniquely determine an acute angled triangle $ABC$ ($R$ being the radius of the circum-circle)?

(a) $a$, $\sin A$, $\sin B$ (b) $a, b, c$
(c) $a$, $\sin B$, $R$ (d) $a$, $\sin A$, $R$

**[IIT-JEE, 2002]**

42. If the angles of a triangle are in the ratio 4:1:1, then ratio of the longest side to the perimeter is

(a) $\frac{\sqrt{3}}{(2+\sqrt{3})}$ (b) $\frac{1}{6}$

(c) $\frac{1}{(2+\sqrt{3})}$ (d) $\frac{2}{3}$

**[IIT-JEE, 2003]**

43 If $I_n$ is the area of $n$-sided regular polygon inscribed in a circle of unit radius and $O_n$ be the area of the polygon circumscribing the given circle, prove that

$$I_n = \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$$

**[IIT-JEE, 2003]**

44. The side of a triangle are in the ratio $1:\sqrt{3}:2$, then the angles of the triangle are in the ratio is

(a) 1:3:5 (b) 2:3:4 (c) 3:2:1 (d) 1:2::3

**[IIT-JEE, 2004]**

45. In an equilateral triangle, three coins of radii 1 unit each are kept so that they touch each other and also the sides of the triangle. Area of the triangle is

(a) $4+2\sqrt{3}$ (b) $6+4\sqrt{3}$

(c) $\left(12+\frac{7\sqrt{3}}{4}\right)$ (d) $\left(3+\frac{7\sqrt{3}}{4}\right)$

**[IIT-JEE, 2005]**

46. In a $\Delta ABC$, $a$, $b$, $c$ are the lengths of its sides and $A$, $B$, $C$ are the angles of a $\Delta ABC$. The correct relation is given by

(a) $(b-c)\sin\left(\frac{B-C}{2}\right) = a\cos\left(\frac{A}{2}\right)$

(b) $(b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B-C}{2}\right)$

(c) $(b+c)\sin\left(\frac{B+C}{2}\right) = a\cos\left(\frac{A}{2}\right)$

(d) $(b-c)\cos\left(\frac{A}{2}\right) = 2a\sin\left(\frac{B+C}{2}\right)$

**[IIT-JEE, 2005]**

47. One angle of an isosceles triangle is 120° and radius of its incircle is $\sqrt{3}$. Then the area of the triangle in sq. units is

(a) $(7+12\sqrt{3})$ (b) $(12-7\sqrt{3})$

(c) $(12+7\sqrt{3})$ (d) $4\pi$

**[IIT-JEE, 2006]**

48. In a $\Delta ABC$, internal angle bisector of $\angle A$ meets side $BC$ in $D$, $DE \perp AD$ meets $AC$ in $E$ and $AB$ in $F$. Then

(a) $AE$ is HM of $b$ and $c$

(b) $AD = \left(\frac{2bc}{b+c}\right)\cos\left(\frac{A}{2}\right)$

(c) $EF = \left(\frac{4bc}{b+c}\right)\sin\left(\frac{A}{2}\right)$

(d) $\Delta AEF$ is isosceles

**[IIT-JEE, 2006]**

49. Let $ABCD$ be a quadrilateral with area 18, with side $AB$ parallel to the side $CD$ and $AB = 2\,CD$. Let $AD$ be perpendicular to $AB$ and $CD$. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides,then its area is

(a) 3 (b) 2 (c) 3/2 (d) 1

**[IIT-JEE, 2007]**

50. A straight line through the vertex $P$ of a $\Delta PQR$ intersects the side $QR$ at the point $S$ and the cirum-circle of the triangle $PQR$ at the point $T$. If $S$ is not the centre of the circum-circle, then

(a) $\frac{1}{PS}+\frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$

(b) $\frac{1}{PS}+\frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

(c) $\frac{1}{PS}+\frac{1}{ST} < \frac{4}{QR}$

(d) $\frac{1}{PS}+\frac{1}{ST} > \frac{4}{QR}$

**[IIT-JEE, 2008]**

51. In a $\Delta ABC$ with fixed base $BC$, the vertex $A$ moves such that $\cos B + \cos C = 4\sin^2\left(\frac{A}{2}\right)$

If $a$, $b$ and $c$ denote the lengths of the sides of the triangle opposite to the angles $A$, $B$ and $C$ respectively, then

(a) $b + c = 4a$
(b) $b + c = 2a$
(c) locus of points $A$ is an ellipse
(d) locus of points $A$ is a pair of straight lines

**[IIT-JEE, 2009]**

52. Two parallel chords of a circle of radius 2 are at a distance $(\sqrt{3}+1)$ apart. If the chord subtends angles $\frac{\pi}{k}$ and $\frac{2\pi}{k}$ at the centre, where $k > 0$, then find the value of $[k]$., [,] = GIF

**[IIT-JEE, 2010]**

53. Consider a $\Delta ABC$ and let $a$, $b$, $c$ denote the lengths of the sides opposite to vertices $A$, $B$, and $C$ respectively. Suppose $a = 6$, $b = 10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if $r$ denotes the radius of the in-circle of the triangle, then find the value of $r^2$.

**[IIT-JEE, 2010]**

54. Let $ABC$ be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let $a$, $b$ and $c$ denote the lengths of the side opposite to $A$, $B$ and $C$ respectively. The values of $x$ for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

(a) $-(2+\sqrt{3})$ (b) $(1+\sqrt{3})$
(c) $(2+\sqrt{3})$ (d) $4\sqrt{3}$

**[IIT-JEE, 2010]**

55. No questions asked in 2011

56. Let $PQR$ be a triangle of area $\Delta$ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where $a$, $b$, $c$ are the lengths of the triangle opposite to the angles at $P$, $Q$, and $R$ respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

(a) $\frac{3}{4\Delta}$ (b) $\frac{45}{4\Delta}$ (c) $\left(\frac{3}{4\Delta}\right)^2$ (d) $\left(\frac{45}{4\Delta}\right)^2$

**[IIT-JEE, 2012]**

57. No questions asked in 2013.

58. In a triangle, the sum of two sides is $x$ and the product of the same two sides is $y$. If $x^2 - c^2 = y$ is (are), where $c$ is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is

(a) $\frac{3y}{2x(x+c)}$ (b) $\frac{3y}{2c(x+c)}$

(c) $\frac{3y}{4x(x+c)}$ (d) $\frac{3y}{4c(x+c)}$

**[IIT-JEE, 2014]**

59. No questions asked in 2015.

## Answers

### Level II

| | | | | |
|---|---|---|---|---|
| 1. (b) | 2. (d) | 3. (b) | 4. (d) | 5. (c) |
| 6. (c) | 7. (a) | 8. (d) | 9. (a) | 10. (a) |
| 11. (b) | 12. (c) | 13. (c) | 14. (c) | 15. (a) |
| 16. (a) | 17. (a) | 18. (b) | 19. (c) | 20. (c) |
| 21. (c) | 22. (b) | 23. (d) | 24. (c) | 25. (b) |
| 26. (c) | 27. (c) | 28. (b) | 29. (c) | 30. (c) |
| 31. (b) | 32. (a) | 33. (d) | 34. (c) | 35. (d) |
| 36. (b) | 37. (c) | 38. (c) | 39. (d) | 40. (b) |
| 41. (c) | 42. (c) | 43. (b) | 44. (c) | 45. (d) |
| 46. (d) | 47. (d) | 48. (a) | 49. (b) | 50. (a) |

#### INTEGER TYPE QUESTIONS

| | | | | |
|---|---|---|---|---|
| 1. 8 | 2. 1 | 3. 9 | 4. 3 | 5. 2 |
| 6. 5 | 7. 1 | 8. 9 | 9. 2 | 10. 2 |

#### COMPREHENSIVE LINK PASSAGES

Passage I: 1. (d) 2. (b) 3. (d) 4. (c) 5. (b)
Passage II: 1. (d) 2. (a) 3. (b)
Passage III:
Passage IV: 1. (c) 2. (a) 3. (a) 4. (c) 5. (a) 6. (c)
Passage V: 1. (a) 2. (d) 3. (a) (a) 5. (b)

#### MATRIX MATCH

1. (A) → (R); (B) →(Q); (C) →(P); (D) → (P)
2. (A)→ (R); (B)→ (R); (C) → (R); (D) → (P)
3. (A) → (Q); (B) → (P); (C) → (R); (D) → (S)
4. (A)→ (Q);(B)→(P);(C)→ (R); (D)→(P)
5. (A)→(R); (B)→(P, Q); (C)→(P);(D)→ (P, Q).
6. (A) → (R); (B) → (R); (C) → (R); (D) → (Q)
7. (A) →(R); (B)→(S); (C)→(P); (D)→ (R)
8. (A) → (Q; (B) → (R); (C) → (P); (D) → (S)
9. (A) → (R); (B) → (Q); (C) → (P); (D) → (S)
10. (A) → (Q); (B) → (P); (C) → (R); (D) → (Q)
11. (A) → (Q); (B) → (S); (C) → (P); (D) → (S)

#### ASSERTION AND REASON

| | | | | |
|---|---|---|---|---|
| 1. (a) | 2. (b) | 3. (a) | 4. (d) | 5. (a) |
| 6. (b) | 7. (c) | 8. (a) | 9. (a) | 10. (a) |
| 11. (a) | 12. (a) | | | |

## Hints and Solutions

### Level I

1. We have $\frac{a^2-b^2}{a^2+b^2} = \frac{\sin(A-B)}{\sin(A+B)}$

$$\Rightarrow \frac{a^2-b^2}{a^2+b^2} = \frac{\sin(A-B)\sin(A+B)}{\sin^2(A+B)}$$

$$\Rightarrow \frac{a^2-b^2}{a^2+b^2} = \frac{\sin^2 A - \sin^2 B}{\sin^2(\pi - C)}$$

$$\Rightarrow \quad \frac{a^2-b^2}{a^2+b^2} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$

$$\Rightarrow \quad \frac{a^2-b^2}{a^2+b^2} = \frac{k^2(a^2-b^2)}{k^2c^2}$$

$$\Rightarrow \quad \frac{a^2-b^2}{a^2+b^2} = \frac{(a^2-b^2)}{c^2}$$

$$\Rightarrow \quad (a^2-b^2)\left(\frac{1}{a^2+b^2} - \frac{1}{c^2}\right) = 0$$

$$\Rightarrow \quad (a^2-b^2) = 0, \left(\frac{1}{a^2+b^2} - \frac{1}{c^2}\right) = 0$$

$$\Rightarrow \quad a^2 = b^2, \frac{1}{a^2+b^2} = \frac{1}{c^2}$$

$\Rightarrow \quad a = b, a^2 + b^2 = c^2$

Thus, the triangle is isosceles or right angled.

2. We have

$$\frac{\sin(B-C)}{\sin(B+C)} = \frac{\sin(B-C)}{\sin(B+C)} \times \frac{\sin(B+C)}{\sin(B+C)}$$
$$= \frac{\sin^2(B) - \sin^2(C)}{\sin^2(B+C)}$$
$$= \frac{\sin^2(B) - \sin^2(C)}{\sin^2(\pi - A)}$$
$$= \frac{\sin^2(B) - \sin^2(C)}{\sin^2(A)}$$
$$= \frac{k^2b^2 - k^2c^2}{k^2a^2}$$
$$= \frac{b^2-c^2}{a^2}$$

3. We have

$\angle C = 180° - (A + B)$
$= 180° - (45° + 75°)$
$= 180° - 120°$
$= 60°$

From the sine rule, we can write

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{a}{\sin(45°)} = \frac{b}{\sin(75°)} = \frac{c}{\sin(60°)}$$

$$\Rightarrow \quad \frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{c}{\frac{\sqrt{3}}{2}} = k(\text{say})$$

Now, $a + c\sqrt{2}$

$$= \frac{k}{\sqrt{2}} + \left(\frac{k\sqrt{3}}{2}\right)\sqrt{2}$$
$$= \frac{k}{\sqrt{2}} + \left(\frac{k\sqrt{3}}{\sqrt{2}}\right)$$
$$= \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)k$$
$$= 2\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)k$$
$$= 2b$$

4. Now,

$$\frac{a^2\sin(B-C)}{\sin B + \sin C} = \frac{ak\sin A\sin(B-C)}{\sin B + \sin C}$$
$$= \frac{ak\sin(B+C)\sin(B-C)}{\sin B + \sin C}$$
$$= ak\left(\frac{\sin^2(B) - \sin^2(C)}{\sin B + \sin C}\right)$$
$$= ak(\sin B - \sin C)$$
$$= k\sin A(\sin B - \sin C)$$
$$= k(\sin A\sin B - \sin A\sin C)$$

Similarly,

$$\frac{b^2\sin(C-A)}{\sin C + \sin A} = k(\sin B\sin C - \sin A\sin B)$$

and $$\frac{c^2\sin(A-B)}{\sin A + \sin B} = k(\sin A\sin C - \sin C\sin B)$$

Thus, LHS

$= k[\sin A\sin B - \sin A\sin C + \sin B\sin C$
$- \sin A\sin B + \sin A\sin C - \sin C\sin B]$
$= 0$

5. Now,

$$\frac{b^2-c^2}{\cos B + \cos C} = \frac{k^2(\sin^2 B - \sin^2 C)}{\cos B + \cos C}$$
$$= \frac{k^2(1 - \cos^2 B - 1 + \cos^2 C)}{\cos B + \cos C}$$
$$= -\frac{k^2(\cos^2 B - \cos^2 C)}{\cos B + \cos C}$$
$$= -k^2(\cos B - \cos C)$$

Similarly,

$$\frac{c^2-a^2}{\cos C + \cos A} = -k^2(\cos C - \cos A)$$

and $$\frac{a^2-b^2}{\cos A + \cos B} = -k^2(\cos A - \cos B)$$

Thus, LHS

$= -k^2[\cos B - \cos C + \cos C - \cos A]$

$\cos A - \cos B$

$= 0$

6. Given,

$a^2, b^2, c^2$ are in AP

$\Rightarrow\ b^2 - a^2 = c^2 - b^2$

$\Rightarrow\ \sin^2 B - \sin^2 A = \sin^2 C - \sin^2 B$

$\Rightarrow\ \sin(B + A)\sin(B - A)$
$= \sin(C + B)\sin(C - B)$

$\Rightarrow\ \sin C(\sin B\cos A - \cos B\sin A)$
$= \sin A(\sin C\cos B - \cos C\sin B)$

Dividing both the sides by $\sin A \sin B \sin C$, we get,

$\Rightarrow\ \cot A - \cot B = \cot B - \cot C$

$\Rightarrow\ \cot A, \cot B, \cot C$ are in AP

7. We have

$$\cot\left(\frac{A}{2}\right) = \frac{b + c}{a}$$

$$\Rightarrow\ \cot\left(\frac{A}{2}\right) = \frac{\sin B + \sin C}{\sin A}$$

$$\Rightarrow\ \cot\left(\frac{A}{2}\right) = \frac{2\sin\left(\frac{B + C}{2}\right)\cos\left(\frac{B - C}{2}\right)}{\sin A}$$

$$\Rightarrow\ \cot\left(\frac{A}{2}\right) = \frac{2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)\cos\left(\frac{B - C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$\Rightarrow\ \cot\left(\frac{A}{2}\right) = \frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B - C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$\Rightarrow\ \cot\left(\frac{A}{2}\right) = \frac{\cos\left(\frac{B - C}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$\Rightarrow\ \frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)} = \frac{\cos\left(\frac{B - C}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$\Rightarrow\ \cos\left(\frac{A}{2}\right) = \cos\left(\frac{B - C}{2}\right)$$

$$\Rightarrow\ \frac{A}{2} = \frac{B - C}{2}$$

$\Rightarrow\ A + C = B$

$\Rightarrow\ 2B = A + B + C = 180°$

$\Rightarrow\ B = 90°$

Thus, the triangle is right angled.

8. Given,

$a^2, b^2, c^2$ are in AP.

$b^2 - a^2 = c^2 - b^2$

$\sin^2 B - \sin^2 C = \sin^2 C - \sin^2 B$

$\sin(B + A)\sin(B - A) = \sin(C + B)\sin(C - B)$

$\sin(C)\sin(B - A) = \sin(A)\sin(C - B)$

$$\frac{\sin(B - A)}{\sin(C - B)} = \frac{\sin(A)}{\sin(C)}$$

$$\frac{\sin(A)}{\sin(C)} = \frac{\sin(A - B)}{\sin(B - C)}$$

9. We have

$$\prod\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right) = \left(\sin A + \frac{1}{\sin A} + 1\right)\left(\sin B + \frac{1}{\sin B} + 1\right)\left(\sin C + \frac{1}{\sin C} + 1\right)$$

$> (2 + 1)(2 + 1)(2 + 1) = 27$

(applying AM $\geq$ GM)

10. Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$

We have,

$$\frac{b + c}{a} = \frac{k(\sin B + \sin C)}{k\sin A}$$

$$= \frac{(\sin B + \sin C)}{\sin A}$$

$$= \frac{2\sin\left(\frac{B + C}{2}\right)\cos\left(\frac{B - C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)\cos\left(\frac{B - C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B - C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{\cos\left(\frac{B - C}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$= \frac{\sin\left(\frac{A}{2} + B\right)}{\sin\left(\frac{A}{2}\right)}$$

11. Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$

We have, $\frac{a\sin(B - C)}{b^2 - c^2}$

$$= \frac{k \sin A \cdot \sin (C - A)}{k^2(\sin^2 C - \sin^2 A)}$$

$$= \frac{\sin (B + C) \cdot \sin (B - C)}{k(\sin^2 B - \sin^2 C)}$$

$$= \frac{(\sin^2 B - \sin^2 C)}{k(\sin^2 B - \sin^2 C)} = \frac{1}{k}\frac{b \sin (C - A)}{c^2 - a^2}$$

$$= \frac{k \sin (C + A) \cdot \sin (C - A)}{k^2(\sin^2 C - \sin^2 A)}$$

$$= \frac{\sin^2 C - \sin^2 A}{k(\sin^2 C - \sin^2 A)} = \frac{1}{k}$$

Also,

$$\frac{c \sin (A - B)}{a^2 - b^2} = \frac{k \sin C \sin (A - B)}{k^2(\sin^2 A - \sin^2 B)}$$

$$= \frac{\sin (A + B) \sin (A - B)}{k(\sin^2 A - \sin^2 B)}$$

$$= \frac{\sin^2 A - \sin^2 B}{k(\sin^2 A - \sin^2 B)} = \frac{1}{k}$$

Hence, the result.

12. Given,

$$\frac{1 + \cos (A - B) \cos C}{1 + \cos (A - C) \cos B}$$

$$= \frac{1 + \cos (A - B) \cos (\pi - (A + B))}{1 + \cos (A - C) \cos (\pi - (A + C))}$$

$$= \frac{1 - \cos (A - B) \cos (A + B)}{1 - \cos (A - C) \cos (A + C)}$$

$$= \frac{1 - \{\cos^2 A - \sin^2 B\}}{1 - \{\cos^2 A - \sin^2 C\}}$$

$$= \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C}$$

$$= \frac{a^2 + b^2}{a^2 + c^2}$$

13. Given,

$\cos A + 2\cos B + \cos C = 2$

$\Rightarrow \cos A + \cos C = 2(1 - \cos B)$

$$\Rightarrow 2 \cos\left(\frac{A + C}{2}\right)\cos\left(\frac{A - C}{2}\right) = 4 \sin^2\left(\frac{B}{2}\right)$$

$$\Rightarrow \cos\left(\frac{\pi}{2} - \frac{B}{2}\right)\cos\left(\frac{A - C}{2}\right) = 2 \sin^2\left(\frac{B}{2}\right)$$

$$\Rightarrow \sin\left(\frac{B}{2}\right)\cos\left(\frac{A - C}{2}\right) = 2 \sin^2\left(\frac{B}{2}\right)$$

$$\Rightarrow \cos\left(\frac{A - C}{2}\right) = 2 \sin\left(\frac{B}{2}\right)$$

Multiplying both sides by $2 \cos\left(\frac{B}{2}\right)$, we get,

$$\Rightarrow 2 \cos\left(\frac{B}{2}\right)\cos\left(\frac{A - C}{2}\right) = 2\left(2 \cos\left(\frac{B}{2}\right)\sin\left(\frac{B}{2}\right)\right)$$

$$\Rightarrow 2 \sin\left(\frac{A + C}{2}\right)\cos\left(\frac{A - C}{2}\right) = 2 \sin B$$

$\Rightarrow \sin A + \sin C = 2 \sin B$

$\Rightarrow a + c = 2b$

$\Rightarrow a, b, c$ are in AP

14 We have

$\cos A + \cos B + \sin A \sin B \sin C = 1$

$$\Rightarrow \frac{1 - \cos A \cos B}{\sin A \sin B} = \sin C$$

$$\Rightarrow \frac{1 - \cos A \cos B}{\sin A \sin B} = \sin C \le 1$$

$\Rightarrow 1 - \cos A \cos B \le \sin A \sin B$

$\Rightarrow 1 \le \cos A \cos B - \sin A \sin B$

$\Rightarrow \cos (A - B) \ge 1$

$\Rightarrow \cos (A - B) = 1$

$\Rightarrow \cos (A - B) = \cos (0)$

$\Rightarrow A - B = 0$

$\Rightarrow A = B$

Therefore,

$$\sin C = \frac{1 - \cos A \cos B}{\sin A \sin B}$$

$$= \frac{1 - \cos A \cos A}{\sin A \sin A} = \frac{1 - \cos^2 A}{\sin^2 A} = \frac{\sin^2 A}{\sin^2 A} = 1$$

$\Rightarrow C = 90°$

Hence, $A = 45° = B$, $C = 90°$

Now, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\Rightarrow \frac{a}{\sin 45°} = \frac{b}{\sin 45°} = \frac{c}{\sin 90°}$$

$$\Rightarrow \frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{1}{\sqrt{2}}} = \frac{c}{1}$$

$$\Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{\sqrt{2}}$$

$\Rightarrow a:b:c = 1:1:\sqrt{2}$

Hence, the result.

15. We have,

$a(b \cos C - c \cos B)$

$= (ab \cos C - ac \cos B)$

$$= ab\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - ac\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

$$= \left(\frac{a^2 + b^2 - c^2}{2}\right) - \left(\frac{a^2 + c^2 - b^2}{2}\right)$$

$$= \frac{1}{2}(a^2 + b^2 - c^2 - a^2 - c^2 + b^2)$$

$$= \frac{1}{2}(b^2 - c^2 - c^2 + b^2)$$

$$= \frac{1}{2}(2b^2 - 2c^2)$$

$= (b^2 - c^2)$

Hence, the result.

16. We have $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$

$$=\frac{1}{2abc}(b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2)$$

$$=\frac{(a^2+b^2+c^2)}{2abc}$$

17. We have

$$(a-b)^2\cos^2\left(\frac{C}{2}\right)+(a+b)^2\sin^2\left(\frac{C}{2}\right)$$

$$=(a^2+b^2)\left(\cos^2\left(\frac{C}{2}\right)+\sin^2\left(\frac{C}{2}\right)\right)$$

$$-2ab\left(\cos^2\left(\frac{C}{2}\right)-\sin^2\left(\frac{C}{2}\right)\right)$$

$$=(a+b^2)-2ab\cos C$$

$$=(a+b^2)-2ab\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$=(a+b^2)-(a^2+b^2-c^2)$$

$$=c^2$$

18. We have

$(a+b+c)(a-b+c)=3ac$

$\Rightarrow\ (a+c)^2-b^2=3ac$

$\Rightarrow\ a^2+c^2-b^2=3ac-2ac=ac$

$\Rightarrow\ \frac{a^2+c^2-b^2}{2ac}=\frac{ac}{2ac}=\frac{1}{2}$

$\Rightarrow\ \cos B=\frac{1}{2}$

$\Rightarrow\ B=\frac{\pi}{3}$

Hence, the angle $B$ is 60°.

19. We have $2\cos B=\frac{a}{c}$

$\Rightarrow\ 2\left(\frac{a^2+c^2-b^2}{2ac}\right)=\frac{a}{c}$

$\Rightarrow\ \left(\frac{a^2+c^2-b^2}{a}\right)=a$

$\Rightarrow\ (a^2+c^2-b^2)=a^2$

$\Rightarrow\ (c^2-b^2)=0$

$\Rightarrow\ c^2=b^2$

$\Rightarrow\ c=b$

Thus, the triangle is isosceles.

20. Given, $(a+b+c)(b+c-a)=\lambda bc$

$\{(b+c)^2-a^2\}=\lambda bc$

$(b^2+c^2-a^2)=(\lambda-2)bc$

$\frac{(b^2+c^2-a^2)}{2bc}=\frac{(\lambda-2)bc}{2bc}=\frac{\lambda-2}{2}$

$\cos A=\frac{\lambda-2}{2}$

$-1\le\frac{\lambda-2}{2}\le 1$

$-2\le\lambda-2\le 2$

$0\le\lambda\le 4$

21. Given, $A$, $B$, $C$ are in AP

$2B=A+C$

$3B=A+B+C=\pi$

$B=\frac{\pi}{3}$

$\cos B=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

$\frac{a^2+c^2-b^2}{2ac}=\frac{1}{2}$

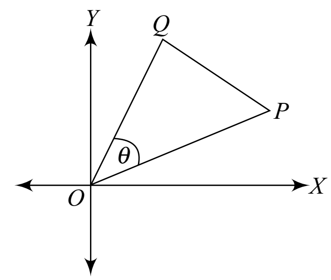
$a^2+c^2-b^2=ac$

$a^2+c^2-b^2=b^2$

$2b^2=a^2+c^2$

$a^2, b^2, c^2\in$ AP

22. Given, $O$ be the origin and $\angle POQ=\theta$




Now, $\cos\theta=\frac{OP^2+OQ^2-PQ^2}{2OP\cdot OQ}$

$$\cos\theta=\frac{(a_1^2+b_1^2)+(a_2^2+b_2^2)-\{(a_1-a_2)^2+(b_1-b_2)^2\}}{2\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}$$

$$=\frac{2(a_1a_2+b_1b_2)}{2\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}$$

$$=\frac{(a_1a_2+b_1b_2)}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}$$

23. Given, $\cot A$, $\cot B$, $\cot C$ are in AP

$\frac{\cos A}{\sin A},\frac{\cos B}{\sin B},\frac{\cos C}{\sin C}\in$ AP

$\frac{b^2+c^2-a^2}{2abck},\frac{a^2+c^2-b^2}{2abck},\frac{a^2+b^2-c^2}{2abck}\in$ AP

$(b^2+c^2-a^2), (a^2+c^2-b^2), (a^2+b^2-c^2)\in$ AP

Subtracting $(a^2+b^2+c^2)$ to each term

$(-2a^2), (-2b^2), (-2c^2)\in$ AP

$a^2, b^2, c^2\in$ AP

24. Let $c=\sqrt{a^2+ab+b^2}$

Clearly, side $c$ is the greatest

Thus, the angle $C$ is the greatest.

Now, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$$= \frac{a^2+b^2-(a^2+ab+b^2)}{2ab}$$

$$= -\frac{ab}{2ab} = -\frac{1}{2}$$

Thus, $\angle C = \dfrac{2\pi}{3}$

25. We have $a \cos A = b \cos B$

$$a\left(\frac{b^2+c^2-a^2}{2bc}\right) = b\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$\Rightarrow$ $a\left(\dfrac{b^2+c^2-a^2}{b}\right) = b\left(\dfrac{a^2+c^2-b^2}{a}\right)$

$\Rightarrow$ $a^2(b^2+c^2-a^2) = b^2(a^2-c^2-b^2)$

$\Rightarrow$ $a^2(c^2-a^2) = b^2(c^2-b^2)$

$\Rightarrow$ $c^2(a^2-b^2) = (a^4-b^4)$

$\Rightarrow$ $c^2(a^2-b^2) = (a^2-b^2)(a^2+b^2)$

$\Rightarrow$ $(a^2-b^2)((a^2+b^2-c^2) = 0$

$\Rightarrow$ $(a^2-b^2) = 0, (a^2+b^2) = c^2$

$\Rightarrow$ $a = b, (a^2+b^2) = c^2$

Thus, the triangle is right angled isosceles.

26. Since the angles are in AP, so $A + C = 2B$

$\Rightarrow$ $A + B + C = 3B$

$\Rightarrow$ $3B = 180°$

$\Rightarrow$ $B = 60°$

$\Rightarrow$ $\cos B = \cos(60°) = \dfrac{1}{2}$

$\Rightarrow$ $\dfrac{a^2+c^2-b^2}{2ac} = \dfrac{1}{2}$

$\Rightarrow$ $a^2+c^2-b^2 = ac$

$\Rightarrow$ $a^2+c^2-ac = b^2$

Now, RHS

$$= \frac{a+c}{\sqrt{a^2-ac+c^2}}$$

$$= \frac{a+c}{b}$$

$$= \frac{k(\sin A + \sin C)}{k \sin B}$$

$$= \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)}$$

$$= \frac{\cos\left(\frac{B}{2}\right)\cos\left(\frac{A-C}{2}\right)}{\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)}$$

$$= \frac{\cos\left(\frac{A-C}{2}\right)}{\sin\left(\frac{B}{2}\right)}$$

$$= \frac{\cos\left(\frac{A-C}{2}\right)}{\sin\left(\frac{60°}{2}\right)}$$

$$= 2\cos\left(\frac{A-C}{2}\right)$$

27. We have

$$\left(\frac{b^2-c^2}{a^2}\right)\sin 2A = \left(\frac{b^2-c^2}{a^2}\right)(2\sin A \cdot \cos A)$$

$$= \left(\frac{b^2-c^2}{a^2}\right)\left(2ka\left(\frac{b^2+c^2-a^2}{2bc}\right)\right)$$

$$= \left(\frac{(b^2-c^2)(b^2+c^2-a^2)}{kabc}\right)$$

$$= \frac{1}{kabc} \times \{(b^4-c^4) - a^2(b^2-c^2)\}$$

Similarly,

$$\left(\frac{c^2-a^2}{b^2}\right) \times \sin 2B = \frac{1}{kabc} \times \{(c^4-a^4) - b^2(c^2-a^2)\}$$

$\left(\dfrac{a^2-b^2}{c^2}\right)\sin 2C$ and

$$= \frac{1}{kabc} \times \{(a^4-b^4) - c^2(a^2-b^2)\}$$

Thus, $\left(\dfrac{b^2-c^2}{a^2}\right)\sin 2A + \left(\dfrac{c^2-a^2}{b^2}\right)\sin 2B$

$$+ \left(\frac{a^2-b^2}{c^2}\right)\sin 2C$$

$$= \frac{1}{kabc} \times \{(b^4-c^4) - a^2(b^2-c^2)\}$$

$$+ \frac{1}{kabc} \times \{(c^4-a^4) - b^2(c^2-a^2)\}$$

$$+ \frac{1}{kabc} \times \{(a^4-b^4) - c^2(a^2-b^2)\}$$

$$= \frac{1}{kabc} \times [(b^4-c^4+c^4-a^4+a^4-b^4)$$

$= a^2(c^2-b^2) + b^2(a^2-c^2) + c^2(b^2-a^2)]$

$= 0$

28. Given,

$\angle A = 60°$

$\Rightarrow$ $\cos A = \cos(60°)$

$\Rightarrow$ $\cos A = \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{b^2+c^2-a^2}{2bc}=\dfrac{1}{2}$

$\Rightarrow \quad (b^2 + c^2 - a^2) = b$ …(i)

Now,

$\left(1+\dfrac{a}{c}+\dfrac{b}{c}\right)\left(1+\dfrac{c}{b}-\dfrac{a}{b}\right)=\left(\dfrac{c+a+b}{c}\right)\left(\dfrac{b+c-a}{b}\right)$

$=\left(\dfrac{b+c+a}{c}\right)\left(\dfrac{b+c-a}{b}\right)$

$=\left(\dfrac{(b+c)^2-a^2}{bc}\right)$

$=\left(\dfrac{b^2+c^2-a^2+2bc}{bc}\right)$

$=\left(\dfrac{bc+2bc}{bc}\right)$, from (i)

$=\left(\dfrac{3bc}{bc}\right)$

$=3$

29. Given, $\dfrac{1}{a+c}+\dfrac{1}{b+c}=\dfrac{3}{a+b+c}$

$\Rightarrow \quad \dfrac{a+b+c}{a+c}+\dfrac{a+b+c}{b+c}=3$

$\Rightarrow \quad 1+\dfrac{b}{a+c}+1+\dfrac{a}{b+c}=3$

$\Rightarrow \quad \dfrac{b}{a+c}+\dfrac{a}{b+c}=1$

$\Rightarrow \quad b(b + c) + a(a + c) = (a + c)(b + c)$

$\Rightarrow \quad b^2 + bc + a^2 + ac = ab + ac + bc + c^2$

$\Rightarrow \quad a^2 + b^2 - c^2 = ab$

Now, $\cos(C)=\left(\dfrac{a^2+b^2-c^2}{2ab}\right)$

$=\dfrac{ab}{2ab}=\dfrac{1}{2}$

$\Rightarrow \quad C=\dfrac{\pi}{3}$

30. We have

$\dfrac{2\cos A}{a}+\dfrac{2\cos B}{b}+\dfrac{2\cos C}{c}=\dfrac{1}{bc}+\dfrac{b}{ca}$

$\Rightarrow \quad \dfrac{2\,bc\cos A}{abc}+\dfrac{ac\cos B}{abc}+\dfrac{2bc\cos C}{abc}$

$=\dfrac{a^2}{abc}+\dfrac{b^2}{abc}$

$\Rightarrow \quad 2bc \cos A + ac \cos B + 2bc \cos C$

$= a^2 + b^2$

$\Rightarrow \quad (b^2+c^2-a^2)+\dfrac{1}{2}(a^2+c^2-b^2)$

$+ (a^2 + b^2 - c^2) = a^2 + b^2$

$\Rightarrow \quad 2b^2 - 2a^2 + c^2 + a^2 - b^2 = 0$

$\Rightarrow \quad a^2 = b^2 + c^2$

$\Delta ABC$ is a right angled triangle at A

Thus, $\angle A = 90°$

31. We have

$2\left(a\sin^2\left(\dfrac{C}{2}\right)+c\sin^2\left(\dfrac{A}{2}\right)\right)$

$=\left(2a\sin^2\left(\dfrac{C}{2}\right)+2\,c\sin^2\left(\dfrac{A}{2}\right)\right)$

$= a(1 - \cos C) + c(1 - \cos A)$

$= a + c - (a \cos C + c \cos A)$

$= (a + c - b)$

32. We have

$2\left(b\cos^2\left(\dfrac{C}{2}\right)+c\cos^2\left(\dfrac{B}{2}\right)\right)$

$=\left(2b\cos^2\left(\dfrac{C}{2}\right)+2\,c\cos^2\left(\dfrac{B}{2}\right)\right)$

$= b(1 + \cos C) + c(1 + \cos B)$

$= b + c + (b \cos C + c \cos B)$

$= (b + c + a)$

$= (a + b + c)$

33. We have

$(b + c) \cos A + (c + a) \cos B + (a + b) \cos C$

$= (b \cos A + a \cos B) + (c \cos A + a \cos C)$
$+ (b \cos C + c \cos B)$

$= (c + b + a)$

$= (a + b + c)$

34. We have

$\dfrac{c-a\cos B}{b-a\cos C}=\dfrac{a\cos B+b\cos A-a\cos B}{c\cos A+a\cos C-a\cos C}$

$=\dfrac{b\cos A}{c\cos A}$

$=\dfrac{b}{c}$

$=\dfrac{k\sin B}{k\sin C}$

$=\dfrac{\sin B}{\sin C}$

35. We have

$2\left(a\sin^2\dfrac{C}{2}+c\sin^2\dfrac{A}{2}\right)$

$=a\left(2\sin^2\left(\dfrac{C}{2}\right)\right)+c\left(2\sin^2\left(\dfrac{A}{2}\right)\right)$

$= a(1 - \cos C) + c(1 - \cos A)$

$= a + c - (a \cos C + c \cos A)$

$= a + c - b$

36. We have

$$\frac{\cos A}{b\cos C + c\cos B} + \frac{\cos B}{c\cos A + a\cos C} + \frac{\cos C}{a\cos B + b\cos A}$$

$$= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \left(\frac{b^2 + c^2 - a^2}{2abc}\right) + \left(\frac{c^2 + a^2 - b^2}{2abc}\right) + \left(\frac{a^2 + b^2 - c^2}{2abc}\right)$$

$$= \frac{1}{2abc}\{(b^2 + c^2 - a^2) + (c^2 + a^2 - b^2) + (a^2 + b^2 - c^2)\}$$

$$= \frac{(a^2 + b^2 + c^2)}{2abc}$$

37. We have

$2(bc \cos A + ca \cos B + ab \cos C)$

$= 2bc \cos A + 2ca \cos B + 2ab \cos C$

$$= 2bc\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + 2\,ca\left(\frac{a^2 + c^2 - b^2}{2ac}\right) + 2ab\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

$= (b^2 + c^2 - a^2) + (a^2 + c^2 - b^2) + (a^2 + b^2 - c^2)$

$= (a^2 + b^2 + c^2)$

38. As we know that,

$$\tan\left(\frac{B - C}{2}\right) = \left(\frac{b - c}{b + c}\right)\cot\left(\frac{A}{2}\right)$$

$$= \left(\frac{(\sqrt{3} + 1) - (\sqrt{3} - 1)}{(\sqrt{3} + 1) + (\sqrt{3} - 1)}\right)\cot\left(\frac{60^\circ}{2}\right)$$

$$= \frac{1}{\sqrt{3}}\cot(30^\circ)$$

$$= \frac{1}{\sqrt{3}} \times \sqrt{3}$$

$$= 1$$

39. As we know that,

$$\tan\left(\frac{B - C}{2}\right) = \left(\frac{b - c}{b + c}\right)\cot\left(\frac{A}{2}\right)$$

$$\Rightarrow \tan\left(\frac{90^\circ}{2}\right) = \left(\frac{\sqrt{3} - 1}{\sqrt{3} + 1}\right)\cot\left(\frac{A}{2}\right)$$

$$\Rightarrow \tan(45^\circ) = (2 - \sqrt{3})\cot\left(\frac{A}{2}\right)$$

$$\Rightarrow \cot\left(\frac{A}{2}\right) = \frac{1}{(2 - \sqrt{3})} = (2 + \sqrt{3})$$

$$\Rightarrow \cot\left(\frac{A}{2}\right) = \cot(15^\circ)$$

$$\Rightarrow A = 30^\circ$$

Hence, the angle $A$ is 30°.

40. Given,

$$\cos(A - B) = \frac{4}{5}$$

$$\Rightarrow 2\cos^2\left(\frac{A - B}{2}\right) - 1 = \frac{4}{5}$$

$$\Rightarrow 2\cos^2\left(\frac{A - B}{2}\right) = 1 + \frac{4}{5} = \frac{9}{5}$$

$$\Rightarrow \cos^2\left(\frac{A - B}{2}\right) = \frac{9}{10}$$

$$\Rightarrow \cos\left(\frac{A - B}{2}\right) = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \tan\left(\frac{A - B}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \left(\frac{a - b}{a + b}\right)\cot\left(\frac{C}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \left(\frac{6 - 3}{6 + 3}\right)\cot\left(\frac{C}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \frac{1}{3}\cot\left(\frac{C}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \cot\left(\frac{C}{2}\right) = 1$$

$$\Rightarrow \frac{C}{2} = \frac{\pi}{4}$$

$$\Rightarrow C = \frac{\pi}{2}$$

Hence, the value of $C$ is $\frac{\pi}{2}$.

43. We have

$2s = a + b + c = 13 + 14 + 15$

$$s = \frac{13 + 14 + 15}{2} = \frac{42}{2} = 21$$

(i) $$\sin\frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}} = \sqrt{\frac{(21 - 14)(21 - 15)}{14.15}} = \sqrt{\frac{7.6}{14.15}} = \frac{1}{\sqrt{5}}$$

(ii) $$\cos\frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}} = \sqrt{\frac{21(21 - 13)}{14.15}} = \sqrt{\frac{21.8}{14.15}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

(iii) $\cos A = 2\cos^2\left(\frac{A}{2}\right) - 1$

$$= 2\left(\frac{4}{5}\right) - 1$$

$$= \frac{8-5}{5} = \frac{3}{5}$$

44. We have $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{b+c}{2c}}$

$\Rightarrow \quad \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{b+c}{2c}}$

$\Rightarrow \quad \frac{s(s-a)}{bc} = \frac{b+c}{2c}$

$\Rightarrow \quad 2s(2s-2a) = 2b(b+c)$

$\Rightarrow \quad (a+b+c)(b+c-a) = 2b(b+c)$

$\Rightarrow \quad ((b+c)^2 - a^2) = 2b(b+c)$

$\Rightarrow \quad b^2 + c^2 - a^2 = 2b^2$

$\Rightarrow \quad a^2 + b^2 = c^2$

Thus, the triangle $ABC$ is right angled at $C$.

45. We have

$$b\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{B}{2}\right)$$

$$= b\left(\frac{s(s-c)}{ab}\right) + c\left(\frac{s(s-b)}{ac}\right)$$

$$= \frac{s}{a}(s-c+s-b)$$

$$= \frac{s}{a}(2s-c-b)$$

$$= \frac{s}{a}(a+b+c-c-b)$$

$$= s$$

46. We have

$$bc\cos^2\left(\frac{A}{2}\right) + ca\cos^2\left(\frac{B}{2}\right) + ab\cos^2\left(\frac{C}{2}\right)$$

$$= bc\left(\frac{s(s-a)}{bc}\right) + ca\left(\frac{s(s-b)}{ca}\right) + ab\left(\frac{s(s-c)}{ab}\right)$$

$$= s(s-a) + s(s-b) + s(s-c)$$

$$= s(3s - (a+b+c))$$

$$= s(3s - 2s)$$

$$= s \times s$$

$$= s^2$$

47. We have $2ac\sin\left(\frac{A-B+C}{2}\right)$

$$= 2ac\sin\left(\frac{A+C-B}{2}\right)$$

$$= 2ac\sin\left(\frac{\pi - B - B}{2}\right)$$

$$= 2ac\sin\left(\frac{\pi}{2} - B\right)$$

$$= 2ac\cos B$$

$$= 2ac\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$= (a^2 + c^2 - b^2)$$

48. We have $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$

$$= \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \times \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \sqrt{\frac{s(s-b)}{(s-a)(s-c)} \times \frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \sqrt{\frac{s^2}{(s-a)^2}}$$

$$= \frac{s}{s-a}$$

$$= \frac{2s}{2s-2a}$$

$$= \frac{a+b+c}{a+b+c-2a}$$

$$= \frac{4a}{4a-2a}$$

$$= 2$$

49. We have $1 - \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)$

$$= 1 - \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$= 1 - \sqrt{\frac{(s-c)^2}{s^2}}$$

$$= 1 - \frac{(s-c)}{s}$$

$$= \frac{c}{s}$$

$$= \frac{2c}{2s}$$

$$= \frac{2c}{(a+b+c)}$$

Hence, the result.

50. We have $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \sqrt{\frac{s^2(s-a)^2}{s(s-a)(s-b)(s-c)}} + \sqrt{\frac{s^2(s-b)^2}{s(s-b)(s-a)(s-c)}}$$

$$+ \sqrt{\frac{s^2(s-c)^2}{s(s-c)(s-a)(s-b)}}$$

$$=\sqrt{\frac{s^2(s-a)^2}{\Delta^2}}+\sqrt{\frac{s^2(s-b)^2}{\Delta^2}}+\sqrt{\frac{s^2(s-c)^2}{\Delta^2}}$$

$$=\frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$$

$$=\frac{s}{\Delta}(s-a+s-b+s-c)$$

$$=\frac{s}{\Delta}(3s-(a+b+c))$$

$$=\frac{s}{\Delta}(3s-2s)$$

$$=\frac{s^2}{\Delta}$$

$$=\frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$=\frac{s}{(s-a)}\times\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$=\frac{2s}{(2s-2a)}\times\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$=\frac{(a+b+c)}{(b+c-a)}\times\cot\left(\frac{A}{2}\right)$$

51. Given, $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in AP

$\Rightarrow \quad 2\cot\left(\frac{B}{2}\right)=\cot\left(\frac{A}{2}\right)+\cot\left(\frac{C}{2}\right)$

$\Rightarrow \quad 2\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$\Rightarrow \quad 2\sqrt{\frac{(s-b)}{(s-a)(s-c)}} = \sqrt{\frac{(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{(s-c)}{(s-a)(s-b)}}$

$\Rightarrow \quad 2\sqrt{\frac{(s-b)^2}{(s-b)(s-a)(s-c)}} = \sqrt{\frac{(s-a)^2}{(s-a)(s-b)(s-c)}} + \sqrt{\frac{(s-c)^2}{(s-a)(s-b)(s-c)}}$

$\Rightarrow \quad 2(s-b)=(s-a)+(s-c)$

$\Rightarrow \quad 2(s-b)=(2s-a-c)$

$\Rightarrow \quad 2b=a+c$

$\Rightarrow \quad a, b, c \in$ AP

52. We have $c(a+b)\cos\frac{B}{2}=b(a+c)\cos\frac{C}{2}$

$\Rightarrow \quad c(a+b)\times\sqrt{\frac{s(s-b)}{ac}}=b(a+c)\times\sqrt{\frac{s(s-c)}{ab}}$

$\Rightarrow \quad c(a+b)\times\sqrt{\frac{(s-b)}{c}}=b(a+c)\times\sqrt{\frac{(s-c)}{b}}$

$\Rightarrow \quad c^2(a+b)^2\times\frac{(s-b)}{c}=b^2(a+c)^2\times\frac{(s-c)}{b}$

$\Rightarrow \quad c(a+b)^2\times(s-b)=b(a+c)^2\times(s-c)$

$\Rightarrow \quad c(a+b)^2\times(s-b)=b(a+c)^2\times(s-c)$

$\Rightarrow \quad c(a+b)^2\times(a+c+b)=b(a+c)^2\times(a+b-c)$

$\Rightarrow \quad \frac{(a+c-b)}{b(a+c)^2}=\frac{(a+b-c)}{c(a+b)^2}$

$\Rightarrow \quad \frac{1}{b(a+c)}-\frac{1}{(a+c)^2}=\frac{1}{c(a+b)}-\frac{1}{(a+b)^2}$

$\Rightarrow \quad \frac{1}{b(a+c)}-\frac{1}{c(a+b)}=\frac{1}{(a+c)^2}-\frac{1}{(a+b)^2}$

$\Rightarrow \quad \frac{ac+c^2-ab-b^2}{bc(a+b)(a+c)}=\frac{(a+b)^2-(a+c)^2}{(a+c)^2(a+b)^2}$

$\Rightarrow \quad \frac{a(c-b)+(c^2-b^2)}{bc}=\frac{2a(b-c)+(b^2-c^2)}{(a+c)(a+b)}$

$\Rightarrow \quad \frac{(c-b)(a+b+c)}{bc}=\frac{(b-c)(2a+b+c)}{(a+c)(a+b)}$

$\Rightarrow \quad (c-b)\left(\frac{(a+b+c)}{bc}+\frac{(2a+b+c)}{(a+c)(a+b)}\right)=0$

$\Rightarrow \quad (c-b)=0$

$$\left(\because\left(\frac{(a+b+c)}{bc}+\frac{(2a+b+c)}{(a+c)(a+b)}\right)\neq 0\right)$$

$\Rightarrow \quad b=c$

$\Rightarrow \quad \Delta$ is isosceles

55. $\Delta ABC=\frac{1}{2}\times\sqrt{2}\times\sqrt{3}=\sqrt{\frac{3}{2}}$ s.u.

56. We have

$$\frac{a^2-b^2}{2}\times\frac{\sin A\sin B}{\sin(A-B)}$$

$$=\frac{k^2(\sin^2 A-\sin^2 B)}{2}\times\frac{\sin A\cdot\sin B}{\sin(A-B)}$$

$$=\frac{k^2\times\sin(A+B)\times\sin(A-B)}{2}\times\frac{\sin A\cdot\sin B}{\sin(A-B)}$$

$$=\frac{k^2\times\sin(A+B)\times\sin A\cdot\sin B}{2}$$

$$=\frac{k^2\times\sin(\pi-C)\times\sin A\cdot\sin B}{2}$$

$$= \frac{k^2 \times \sin C \times \frac{a}{k} \times \frac{b}{k}}{2}$$
$$= \frac{1}{2} \times ab \sin C$$
$$= \Delta$$

57. As we know that, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\Rightarrow \quad \frac{(\sqrt{3}+1)}{\sin (105°)} = \frac{b}{\sin 45°} = \frac{c}{\sin 30°}$$
$$\Rightarrow \quad \frac{(\sqrt{3}+1)}{\cos (15°)} = \frac{b}{\sin 45°} = \frac{c}{\sin 30°}$$
$$\Rightarrow \quad \frac{(\sqrt{3}+1)}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{b}{\frac{1}{\sqrt{2}}} = \frac{c}{\frac{1}{2}}$$
$$\Rightarrow \quad 2\sqrt{2} = b\sqrt{2} = 2c$$
$$\Rightarrow \quad b = 2 \text{ and } c = \sqrt{2}$$

Hence, the area of the triangle is $= \frac{1}{2} bc \sin A$

$$= \frac{1}{2} \times 2 \times \sqrt{2} \times \sin (105°)$$
$$= \frac{1}{2} \times 2 \times \sqrt{2} \times \frac{\sqrt{3}+1}{2\sqrt{2}}$$
$$= \frac{(\sqrt{3}+1)}{2} \text{s.u.}$$

58. We have

$\cot A + \cot B + \cot C$

$$= \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}$$
$$= \frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{a}{k}} + \frac{\frac{c^2+a^2-b^2}{2ac}}{\frac{b}{k}} + \frac{\frac{a^2+b^2-c^2}{2ab}}{\frac{c}{k}}$$
$$= \frac{b^2+c^2-a^2}{2abck} + \frac{c^2+a^2-b^2}{2abck} + \frac{a^2+b^2-c^2}{2abck}$$
$$= \frac{a^2+b^2+c^2}{2abck}$$
$$= \frac{(a^2+b^2+c^2)}{4 \times \left(\frac{1}{2}ab\right) \times \sin C}$$
$$= \frac{(a^2+b^2+c^2)}{4 \times \Delta} \quad \text{...(i)}$$

Also, $\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)$

$$= \frac{(a+b+c)}{(b+c-a)} \times \cot\left(\frac{A}{2}\right)$$
$$= \frac{s^2}{\Delta} \quad \text{...(ii)}$$

Dividing (ii) by (i), we get

$$\frac{\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)}{\cot (A) + \cot (B) + \cot (C)} = \frac{s^2}{4(a^2+b^2+c^2)}$$
$$\Rightarrow \quad \frac{\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)}{\cot (A) + \cot (B) + \cot (C)} = \frac{(a+b+c)^2}{(a^2+b^2+c^2)}$$

59. Let $a$, $b$, and $c$ are the sides of a triangle and $s$ be the semi perimeter.

Let the four quantities are $s$, $(s-a)$, $(s-b)$ and $(s-c)$

Applying, AM ≥ GM, we get

$$\Rightarrow \quad \frac{s+(s-a)+(s-b)+(s-c)}{4} \geq \sqrt[4]{s(s-a)(s-b)(s-c)}$$
$$\Rightarrow \quad \frac{4s-(a+b+c)}{4} \geq \sqrt[4]{\Delta^2}$$
$$\Rightarrow \quad \frac{4s-2s}{4} \geq (\Delta)^{\frac{1}{2}}$$
$$\Rightarrow \quad \frac{s}{2} \geq (\Delta)^{\frac{1}{2}}$$
$$\Rightarrow \quad \Delta < \frac{s^2}{4}$$

60. Let $AD = \alpha$, $BE = \beta$ and $CF = \gamma$

Then, $\Delta = \frac{1}{2} \times a \times AD = \frac{1}{2} \times b \times BE = \frac{1}{2} \times c \times CF$

$$\Rightarrow \quad AD = \frac{2\Delta}{a}, BE = \frac{2\Delta}{b}, CF = \frac{2\Delta}{c}$$
$$\Rightarrow \quad \alpha = \frac{2\Delta}{a}, \beta = \frac{2\Delta}{b}, \gamma = \frac{2\Delta}{c} -$$

Now, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

$$= \frac{a^2}{4\Delta^2} + \frac{b^2}{4\Delta^2} + \frac{c^2}{4\Delta^2}$$
$$= \frac{(a^2+b^2+c^2)}{4\Delta^2}$$
$$= \frac{1}{\Delta} \times \frac{(a^2+b^2+c^2)}{4\Delta}$$
$$= \frac{1}{\Delta} \times (\cot A + \cot B + \cot C)$$
$$= \frac{(\cot A + \cot B + \cot C)}{\Delta}$$

Hence, the result.

61. Let $AD = p_1$, $BE = p_2$ and $CF = p_3$

Then, $\Delta = \frac{1}{2} \times a \times p_1 = \frac{1}{2} \times b \times p_2 = \frac{1}{2} \times c \times p_3$

$$\Rightarrow \quad p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

Now,

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{a}{2\Delta}+\frac{b}{2\Delta}-\frac{c}{2\Delta}$$
$$=\frac{(a+b-c)}{2\Delta}$$
$$=\frac{(a+b+c-2c)}{2\Delta}$$
$$=\frac{(2s-2c)}{2\Delta}$$
$$=\frac{(s-c)}{\Delta}$$
$$=\frac{2ab\times s(s-c)}{\Delta\times s}\times\frac{1}{2ab}$$
$$=\frac{2ab}{\Delta\times s}\times\frac{s(s-c)}{2ab}$$
$$=\frac{2ab}{(a+b+c)\Delta}\times\cos^2\left(\frac{C}{2}\right)$$

62. We have

$(a-b)^2+(b-c)^2+(c-d)^2\ge 0$

$\Rightarrow$ $2(a^2+b^2+c^2)\ge 2(ab+bc+ca)$

$\Rightarrow$ $3(a^2+b^2+c^2)\ge(a^2+b^2+c^2)+(2ab+bc+ca)$

$\Rightarrow$ $3(a^2+b^2+c^2)>(a+b+c)^2>d^2$

$$\Rightarrow \quad \frac{3(a^2+b^2+c^2)}{d^2}>1$$

$$\Rightarrow \quad \frac{(a^2+b^2+c^2)}{d^2}>\frac{1}{3}$$

Thus, the minimum value of $\dfrac{(a^2+b^2+c^2)}{d^2}$ – is $\dfrac{1}{3}$

63. We have

$$\cos A+\cos B+\cos C=\frac{3}{2}$$

$$\Rightarrow \quad 1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)=\frac{3}{2}$$

$$\Rightarrow \quad \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)=\frac{1}{8}$$

It is possible only when

$$\sin\left(\frac{A}{2}\right)=\frac{1}{2},\ \sin\left(\frac{B}{2}\right)=\frac{1}{2},\ \sin\left(\frac{C}{2}\right)=\frac{1}{2}$$

$$\Rightarrow \quad \frac{A}{2}=\frac{\pi}{6},\frac{B}{2}=\frac{\pi}{6},\frac{C}{2}=\frac{\pi}{6}$$

$$\Rightarrow \quad A=\frac{\pi}{3},B=\frac{\pi}{3},C=\frac{\pi}{3}$$

$\Rightarrow$ $\Delta$ is an equilateral.

64. Here,

$$\Delta=\frac{1}{2}\times p\times p_1=\frac{1}{2}\times q\times p_2=\frac{1}{2}\times r\times p_3$$

$$\Rightarrow \quad p=\frac{2\Delta}{p_1},q=\frac{2\Delta}{p_2},r=\frac{2\Delta}{p_3}$$

From sine rule of a triangle,

$$\frac{\sin P}{p}=\frac{\sin Q}{q}=\frac{\sin R}{r}$$

Given $\sin P$, $\sin Q$, $\sin R$ are in AP

$\Rightarrow$ $p, q, r\in$ AP

$$\Rightarrow \quad \frac{2\Delta}{p_1},\frac{2\Delta}{p_2},\frac{2\Delta}{p_3}\in \text{AP}$$

65. In $\Delta ABC$, $\Delta=\dfrac{1}{2}\times ac\sin(\angle B)$

$$\Rightarrow \quad (6+2\sqrt{3})=\frac{1}{2}\times 2(\sqrt{3}+1)\times c\times\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad c=\frac{\sqrt{2}(6+2\sqrt{3})}{(\sqrt{3}+1)}$$

$$\Rightarrow \quad c=\frac{2\sqrt{3}\sqrt{2}(\sqrt{3}+1)}{(\sqrt{3}+1)}$$

$$\Rightarrow \quad c=2\sqrt{6}$$

$$\text{Now, } \cos B=\frac{a^2+c^2-b^2}{2ac}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}=\frac{4(\sqrt{3}+1)^2+24-b^2}{2.2(\sqrt{3}+1)\cdot 2\sqrt{6}}$$

$$\Rightarrow \quad 4(\sqrt{3}+1)^2+24-b^2=8\sqrt{3}(\sqrt{3}+1)$$

$$\Rightarrow \quad 4(4+2\sqrt{3})+24-b^2=8(3+\sqrt{3})$$

$\Rightarrow$ $b^2=16$

$\Rightarrow$ $b=4$

66. Let $\angle B=30°$, $\angle C=45°$

So, $\angle A=180°-(30°+45°)=105°$

From sine formula, we can write

$$\frac{a}{\sin(105°)}=\frac{b}{\sin(30°)}=\frac{c}{\sin(45°)}$$

$$\frac{(\sqrt{3}+1)}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}=\frac{b}{\frac{1}{2}}=\frac{c}{\frac{1}{\sqrt{2}}}$$

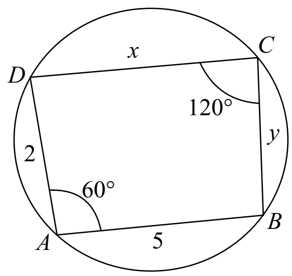
$$2\sqrt{2}=2b=c\sqrt{2}$$

$$b=\sqrt{2},\ c=2$$

Thus, area of triangle $ABC$ is

$$=\frac{1}{2}bc\sin A$$
$$=\frac{1}{2}\times 2\times\sqrt{2}\times\sin(105°)$$
$$=\frac{1}{2}\times 2\times\sqrt{2}\times\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$
$$=\left(\frac{\sqrt{3}+1}{2}\right)\text{s.u.}$$

67. Suppose, $AC=2$, $AB=5$, $BC=x$, $CD=y$ and $\angle BAD=60°$




Area of $\Delta ABC = \frac{1}{2} \cdot 5 \cdot 2 \cdot \sin(60°)$

$= \frac{5\sqrt{3}}{2}$

Also, from $\Delta ABC$,

$\cos(60°) = \frac{25 + 4 - BD^2}{2 \cdot 5 \cdot 2}$

$= \frac{29 - BD^2}{20}$

$\Rightarrow \quad \frac{29 - BD^2}{20} = \frac{1}{2}$

$\Rightarrow \quad BD^2 = 19$

$\Rightarrow \quad BD = \sqrt{19}$

Since $A, B, C, D$ are concyclic, so

$\angle BCD = 180° - 60° = 120°$

Then, from $\Delta BCD$,

$\cos(120°) = \frac{x^2 + y^2 - (\sqrt{19})^2}{2xy}$

$\Rightarrow \quad \frac{x^2 + y^2 - (\sqrt{19})^2}{2xy} = -\frac{1}{2}$

$\Rightarrow \quad \frac{x^2 + y^2 - (\sqrt{19})^2}{xy} = -1$

$\Rightarrow \quad x^2 + y^2 + xy = 19$ …(i)

Again, area of $\Delta BCD$

$= \frac{1}{2} \cdot x \cdot y \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}\,xy}{4}$.

Thus, area of quad. $ABCD = 4\sqrt{3}$

$\Rightarrow \quad \frac{5\sqrt{3}}{2} + \frac{\sqrt{3}\,xy}{4} = 4\sqrt{3}$

$\Rightarrow \quad \frac{5}{2} + \frac{xy}{4} = 4$

$\Rightarrow \quad \frac{xy}{4} = 4 - \frac{5}{2} = \frac{3}{2}$

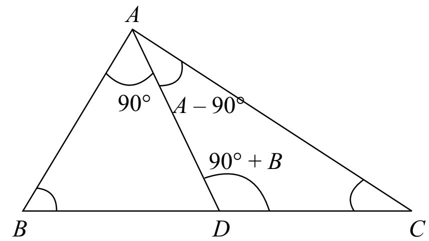
$\Rightarrow \quad xy = 6$

From (i), we get

$x^2 + y^2 = 13$

$\Rightarrow \quad x = 3, y = 2$

68. Since $AD$ is the median, so $BD : DC = 1 : 1$




Clearly, $\angle ADC = 90° + B$.

Now, applying $m : n$ rule, we get,

$(1 + 1)\cot(90° + B) = 1 \cdot \cot(90°) - 1 \cdot \cot(A - 90°)$

$\Rightarrow \quad -2\tan B = 0 - (-\tan A)$

$\Rightarrow \quad -2\tan B = \tan A$

$\Rightarrow \quad \tan A + 2\tan B = 0$

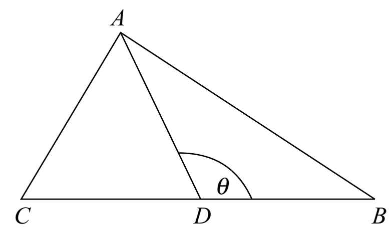
Hence, the result.

$\Rightarrow \quad \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3} \in$ AP

$\Rightarrow \quad p_1, p_2, p_3 \in$ HP

Thus, the altitudes are in HP

69.




By $m : n$ rule, we get

$(1 + 1)\cot\theta = 1 \cdot \cot C - 1 \cdot \cot B$

$\Rightarrow \quad 2\cot\theta = \cot C - \cot B$

$\Rightarrow \quad 2\cot\theta = \frac{a^2 + b^2 - c^2}{2ab\sin C} - \frac{a^2 + c^2 - b^2}{2ab\sin B}$

$\Rightarrow \quad 2\cot\theta = \frac{a^2 + b^2 - c^2}{4\Delta} - \frac{a^2 + c^2 - b^2}{4\Delta}$

$\Rightarrow \quad 2\cot\theta = \frac{2(b^2 - c^2)}{4\Delta}$

$\Rightarrow \quad \cot\theta = \frac{(b^2 - c^2)}{4\Delta}$

Hence, the result.

70. Clearly, the triangle is right angled.

($\because\ 18^2 + 24^2 = 30^2$)

Thus, the area of the triangle

$= \frac{1}{2} \times 24 \times 18 = 12 \times 18$

Therefore, the circum-radius

$= R$

$= \frac{abc}{4\Delta}$

$= \frac{18 \times 24 \times 30}{4 \times 12 \times 18} = 15$

71. As we know that,

$$\frac{a}{\sin A} = 2R$$

$$\Rightarrow \quad 2R = \frac{2\sqrt{3}}{\sin (60°)}$$

$$\Rightarrow \quad R = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} = 2$$

Hence, the circum-radius is 2.

72. Let $a = 3$, $b = 4$ and $c = 5$

Clearly, it is a right angled triangle

Thus, $\Delta = \frac{1}{2} \times 4 \times 3 = 6$ sq.u

Hence, the circum-radius $R$

$$= \frac{abc}{4\Delta}$$

$$= \frac{3 \times 4 \times 5}{6}$$

$$= 10$$

73. We have

$8R^2 = a^2 + b^2 + c^2$ $8R^2 = (2R \sin A)^2 + (2R \sin B)^2 + (2R \sin C)^2$

$\Rightarrow$ $\sin^2 A + \sin^2 B + \sin^2 C = 2$

$\Rightarrow$ $1 - \cos^2 A + 1 - \cos^2 B + \sin^2 C = 2$

$\Rightarrow$ $\cos 2A - \sin^2 C + \cos^2 B = 0$

$\Rightarrow$ $\cos (A + C) \cos (A - C) + \cos^2 B - 0$

$\Rightarrow$ $\cos (\pi - B) \cos (A - C) + \cos^2 B = 0$

$\Rightarrow$ $\cos B \cos (A - C) - \cos^2 B = 0$

$\Rightarrow$ $\cos B (\cos (A - C) - \cos B) = 0$

$\Rightarrow$ $\cos B(\cos (A - C) + \cos (A + C)) = 0$

$\Rightarrow$ $\cos B \cdot 2 \cos A \cos C = 0$

$\Rightarrow$ $\cos A = 0, \cos B = 0, \cos C = 0$

$\Rightarrow$ $A = \frac{\pi}{2}$ or $B = \frac{\pi}{2}$ or $C = \frac{\pi}{2}$

Thus, the triangle is right angled.

74. We have $a \cos A + b \cos B + c \cos C$

$= 2R(\sin A \cdot \cos A + \sin B \cdot \cos B + \sin C \cdot \cos C)$

$= \frac{2R}{2} [2 \sin A \cdot \cos A + 2 \sin B \cdot \cos B + 2 \sin C \cos C]$

$= R(\sin 2A + \sin 2B + \sin 2C)$

$= R(4 \sin A \cdot \sin B \cdot \sin C)$

$= 4R \sin A \cdot \sin B \cdot \sin C$

75. We have

$$\Delta = \frac{1}{2} \times a \times b \times \sin C$$

$$\Rightarrow \quad \Delta = \frac{1}{2} \times 2R \sin A \times 2R \sin B \times \sin C$$

$$\Rightarrow \quad \Delta = 2R^2 \cdot \sin A \cdot \sin B \cdot \sin C$$

76. We have

$$\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c}$$

$$= \frac{\sin A}{2R \sin A} + \frac{\sin B}{2R \sin B} + \frac{\sin C}{2R \sin C}$$

$$= \frac{1}{2R} + \frac{1}{2R} + \frac{1}{2R}$$

$$= \frac{3}{2R}$$

77. Let $AD = p_1$, $BE = p_2$ and $CF = p_3$.

Then,

$$\Delta = \frac{1}{2} \times a \times p_1 = \frac{1}{2} \times b \times p_2 = \frac{1}{2} \times c \times p_3$$

$$\Rightarrow \quad p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

Now,

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a+b+c}{2\Delta}$$

$$= \frac{2R(\sin A + \sin B + \sin C)}{2\Delta}$$

$$\leq \frac{3R}{\Delta} \quad (\because \sin A \leq 1, \sin B \leq 1, \sin C \leq 1)$$

78. Here, $\Delta = \frac{1}{2} ap_1 = \frac{1}{2} bp_2 = \frac{1}{2} cp_3$

$$\Rightarrow \quad p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

Now,

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a+b+c}{2\Delta}$$

$$= \frac{2s}{2\Delta}$$

$$= \frac{s}{\Delta}$$

$$= \frac{1}{r}$$

79. We have

$$\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3}$$

$$= \frac{1}{2\Delta}(a \cos A + b \cos B + c \cos C)$$

$$= \frac{1}{2\Delta} [2R \sin A \cos A + 2R \sin B \cos B + 2R \sin C \cos C]$$

$$= \frac{R}{2\Delta}(\sin 2A + \sin 2B + \sin 2C)$$
$$= \frac{R}{2\Delta}(4\sin A \sin B \sin C)$$
$$= \frac{2R}{\Delta}(\sin A \sin B \sin C)$$
$$= \frac{2R}{\Delta} \times \left(\frac{a}{2R}\right) \times \left(\frac{b}{2R}\right) \times (\sin C)$$
$$= \frac{1}{\Delta R}\left(\frac{1}{2} ab \sin C\right)$$
$$= \frac{1}{\Delta R} \times \Delta$$
$$= \frac{1}{R}$$

80. We have

$$\frac{\cos C}{\sqrt{4R^2 - c^2}} = \frac{\cos C}{\sqrt{4R^2 - 4R^2\sin^2 C}}$$
$$= \frac{\cos C}{\sqrt{4R^2(1-\sin^2 C)}}$$
$$= \frac{\cos C}{2R\cos C} = \frac{1}{2R}$$

81. We have

$$\Delta = \frac{1}{2} \times p_1 \times a$$
$$p_1 = \frac{2\Delta}{a}$$

Similarly,

$$p_2 = \frac{2\Delta}{b}, \; p_3 = \frac{2\Delta}{c}$$

Now,

$$p_1 p_2 p_3 = \frac{8\Delta^3}{abc}$$
$$= \frac{8\left(\frac{abc}{4R}\right)^3}{abc}$$
$$= \frac{a^2b^2c^2}{8R^3}$$

82. We have

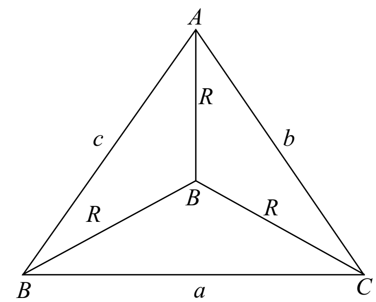
$$\Delta = \frac{1}{2} \times p_1 \times a$$
$$p_1 = \frac{2\Delta}{a}$$

Similarly, $p_2 = \frac{2\Delta}{b}, \; p_3 = \frac{2\Delta}{c}$

Now, $\frac{bp_1}{c} + \frac{cp_2}{a} + \frac{ap_3}{b}$

$$= \frac{2\Delta b}{ac} + \frac{2\Delta c}{ab} + \frac{2\Delta a}{bc}$$
$$= 2\Delta\left(\frac{b}{ac} + \frac{c}{ab} + \frac{a}{bc}\right)$$
$$= \frac{2\Delta(b^2 + c^2 + a^2)}{abc}$$
$$= \frac{(a^2 + b^2 + c^2 +)}{2R}$$

83. Given, $O$ is the circumcentre of $\Delta ABC$




Let, ar($\Delta BOC$) = $\Delta_1$, ar($\Delta AOC$) = $\Delta_2$ and ar($\Delta AOB$) = $\Delta_3$ respectively.

Now, $R_1 = \frac{OB \cdot OC \cdot a}{4(\Delta BOC)} = \frac{aR^2}{\Delta_1}$

$$\frac{a}{R_1} = \frac{4\Delta_1}{R^2}$$

Similarly, $\frac{b}{R_2} = \frac{4\Delta_2}{R^2}, \frac{c}{R_3} = \frac{4\Delta_3}{R^2}$

Now, $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}$

$$= \frac{4}{R^2}(\Delta_1 + \Delta_2 + \Delta_3)$$
$$= \frac{4\Delta}{R^2}$$
$$= \frac{4}{R^2} \times \frac{abc}{4R} = \frac{abc}{R^3}$$

$$\frac{a\sec A + b\sec B + c\sec C}{\tan A \tan B \tan C} = 2$$

84. We have

$$\frac{a\sec A + b\sec B + c\sec C}{\tan A \tan B \tan C}$$
$$= \frac{2R\sin A\sec A + 2R\sin B\sec B + 2R\sin C\sec C}{\tan A \tan B \tan C}$$
$$= \frac{2R\tan A + 2R\tan B + 2R\tan C}{\tan A \tan B \tan C}$$
$$= \frac{2R(\tan A + \tan B + \tan C)}{\tan A \tan B \tan C}$$

$$= \frac{2R\,(\tan A \cdot \tan B \cdot \tan C)}{\tan A \tan B \tan C}$$

$$= 2R$$

85. We have

$a \cos A + b \cos B + c \cos C$

$= 2R(\sin A \cos A + \sin B \cdot \cos B + \sin C \cdot \cos C)$

$= R(2 \sin A \cos A + 2 \sin B \cdot \cos B + 2 \sin C \cdot \cos C$

$= R(\sin 2A + \sin 2B + \sin 2C)$

$= R(4 \sin A \cdot \sin B \cdot \sin C)$

$= 4(R \sin A \cdot \sin B \cdot \sin C)$

86. Here, $s = \frac{4 + 6 + 8}{2} = 9$

Area of a triangle

$= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{9(9-4)(9-6)(9-8)}$

$= \sqrt{9 \times 5 \times 3 \times 1} = 3\sqrt{15}$

Hence, the in-radius

$$= r = \frac{\Delta}{s} = \frac{3\sqrt{15}}{9} = \sqrt{\frac{5}{3}}$$

87. Clearly, it is a right angled triangle

So, its area = $\frac{1}{2} \times 18 \times 24 = 9 \times 24$

and $s = \frac{18 + 24 + 30}{2} = \frac{72}{2} = 36$

Thus, in-radius = $r = \frac{\Delta}{s} = \frac{9 \times 24}{36} = 6$

88. Do yourself.

89. Two sides of a triangle are 2 and $\sqrt{3}$ and the included angle is 30°, then prove that its in-radius is $\frac{1}{2}(\sqrt{3} - 1)$.

89. We have area of the triangle

$= \Delta = \frac{1}{2} \times 2 \times \sqrt{3} \times \sin(30°)$

$= \frac{\sqrt{3}}{2}$

Also, $a^2 = b^2 + c^2 - 2b \cos A$

$a^2 = 4 + 3 - 2 \times 2 \times \sqrt{3} \times \cos(30°)$

$a^2 = 7 - 2 \times 2 \times \sqrt{3} \times \frac{\sqrt{3}}{2}$

$a^2 = 7 - 6 = 1$

$a = 1$

Now, $s = \frac{a + b + c}{2} = \frac{1 + 2 + \sqrt{3}}{2} = \frac{3 + \sqrt{3}}{2}$

Hence, in-radius

$= r = \frac{\Delta}{s}$

$= \frac{\frac{\sqrt{3}}{2}}{\frac{3 + \sqrt{3}}{2}} = \frac{\sqrt{3}}{(\sqrt{3} + 3)}$

$= \frac{1}{(\sqrt{3} + 1)} = \frac{(\sqrt{3} - 1)}{2}$

90. We have

$\frac{R}{r} = \frac{\frac{abc}{4\Delta}}{\frac{\Delta}{s}} = \frac{a^3}{4\Delta} \times \frac{s}{\Delta} = \frac{a^3}{4\Delta} \times \frac{3a}{2\Delta}$

$= \frac{3a^4}{8} \times \frac{1}{\Delta^2}$

$= \frac{3a^4}{8} \times \frac{1}{\frac{3a^4}{16}} = \frac{3a^4}{8} \times \frac{16}{3a^4} = 2$

91. We have

$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{a + b + c}{abc}$

$= \frac{2s}{4\Delta R}$

$= \frac{\frac{\Delta}{r}}{2\Delta R}$

$= \frac{1}{2rR}$

92. We have

$\cos A + \cos B + \cos C$

$= 2 \cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right) + \cos C$

$= 2 \cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A - B}{2}\right) + \cos C$

$= 2 \sin\left(\frac{C}{2}\right)\cos\left(\frac{A - B}{2}\right) + \cos C$

$= 2 \sin\left(\frac{C}{2}\right)\cos\left(\frac{A - B}{2}\right) + 1 - 2 \sin^2\left(\frac{C}{2}\right)$

$= 1 + 2 \sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A - B}{2}\right) - \sin\left(\frac{C}{2}\right)\right)$

$= 1 + 2 \sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A - B}{2}\right) - \cos\left(\frac{A + B}{2}\right)\right)$

$= 1 + 2 \sin\left(\frac{C}{2}\right)\left(2 \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right)$

$= 1 + 4 \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

$= \left(1 + \frac{r}{R}\right)$

93. We have,

$$\sin A + \sin B + \sin C = \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}$$
$$= \frac{a+b+c}{2R}$$
$$= \frac{2s}{2R}$$
$$= \frac{s}{R}$$
$$= \frac{\frac{\Delta}{r}}{R}$$
$$= \frac{\Delta}{rR}$$

94. We have

$a \cot A + b \cot B + c \cot C$

$$= \left[2R \sin A \times \frac{\cos A}{\sin A} + 2R \sin B \times \frac{\cos B}{\sin B} + 2R \sin C \times \frac{\cos C}{\sin C}\right]$$
$$= 2R(\cos A + \cos B + \cos C)$$
$$= 2R\left(1 + \frac{r}{R}\right)$$
$$= 2(R + r)$$

95. We have $r_1 = R$

$$4R \sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) = R$$
$$2 \sin\left(\frac{A}{2}\right)\left\{2 \cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)\right\} = 1$$
$$2 \sin\left(\frac{A}{2}\right)\left\{\cos\left(\frac{B+C}{2}\right) + \cos\left(\frac{B-C}{2}\right)\right\} = 1$$
$$2 \cos\left(\frac{B+C}{2}\right)\left\{\cos\left(\frac{B+C}{2}\right) + \cos\left(\frac{B-C}{2}\right)\right\} = 1$$
$$2 \cos^2\left(\frac{B+C}{2}\right) + \left\{2 \cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\right\} = 1$$

$(1 + \cos(B + C)) = \cos(B) + \cos(C) = 1$

$\cos(B) + \cos(C) = -\cos(B + C) = \cos A$

Hence, the result.

96. We have

$$(b + c) \tan\left(\frac{A}{2}\right)$$
$$= 2R(\sin B + \sin C) \tan\left(\frac{A}{2}\right)$$
$$= 2R \times 2 \sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) \times \tan\left(\frac{A}{2}\right)$$
$$= 2R \times 2 \cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) \times \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$
$$= 2R \times 2 \cos\left(\frac{B-C}{2}\right) \times \sin\left(\frac{A}{2}\right)$$
$$= 2R \times 2 \cos\left(\frac{B-C}{2}\right) \times \cos\left(\frac{B+C}{2}\right)$$
$$= 2R \times (\cos(B) + \cos(C))$$

Thus, LHS

$$= 4R(\cos A + \cos B + \cos C)$$
$$= 4R\left(1 + \frac{r}{R}\right)$$
$$= 4(R + r)$$

97. We have

$$R = \frac{c}{2 \sin C} = \frac{c}{2 \sin(90°)} = \frac{c}{2}$$

$c = 2R$

Also, $r = (s - c) \tan\left(\frac{C}{2}\right)$

$r = (s - c) \tan(45°) = (s - c)$

$2r = (2s - 2c) = (a + b + c - 2c)$

$2r = (a + b - c)$

$2r = (a + b - 2R)$

$2(R + r) = (a + b)$

98. We have

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}$$
$$= \frac{1}{2}\left(2 \cos^2\left(\frac{A}{2}\right) + 2 \cos^2\left(\frac{B}{2}\right) + 2 \cos^2\left(\frac{C}{2}\right)\right)$$
$$= \frac{1}{2}(1 + \cos(A) + 1 + \cos B + 1 + \cos C)$$
$$= \frac{1}{2}(3 + \cos(A) + \cos B + \cos C)$$
$$= \frac{1}{2}\left(3 + \left(1 + \frac{r}{R}\right)\right)$$
$$= \frac{1}{2}\left(4 + \frac{r}{R}\right)$$
$$= \left(2 + \frac{r}{2R}\right)$$

99. Let $O$ is the circum-centre and $OD = x$, $OE = y$, $OF = z$. respectively.

Also, $OA = R = OB = OC$

We have $x = OD = R \cos A$

$$\Rightarrow \quad = \frac{a}{2 \sin A} \cdot \cos A = \frac{a}{2 \tan A}$$

$$\tan A = \frac{a}{2x}$$

Similarly, $\tan B = \frac{b}{2y}$ and $\tan C = \frac{c}{2z}$

As we know that, in a $\Delta ABC$,

$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$

$$\Rightarrow \quad \frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}=\frac{a}{2x}\cdot\frac{b}{2y}\cdot\frac{c}{2z}$$

$$\Rightarrow \quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{a\cdot b\cdot c}{4\cdot x\cdot y\cdot z}$$

100. We have

$$R_1=\frac{OB\cdot OC\cdot BC}{4\Delta OBC}=\frac{R\cdot R\cdot a}{4\Delta_1}=\frac{R^2\cdot a}{4\Delta_1}$$

$$\Rightarrow \quad \frac{a}{R_1}=\frac{4\Delta_1}{R^2}$$

Similarly, $\frac{b}{R_2}=\frac{4\Delta_2}{R^2}$ and $\frac{c}{R_3}=\frac{4\Delta_3}{R^2}$

Thus,

$$\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}=\frac{4\Delta_1}{R^2}+\frac{4\Delta_2}{R^2}+\frac{4\Delta_3}{R^2}$$

$$=\frac{4(\Delta_1+\Delta_2+\Delta_3)}{R^2}$$

$$=\frac{4\Delta}{R^2}$$

$$=\frac{4\Delta}{R^2}$$

101. Let $AD=p_1$, $BE=p_2$ and $CF=p_3$

Then, $\Delta=\frac{1}{2}ap_1=\frac{1}{2}bp_2=\frac{1}{2}cp_3$

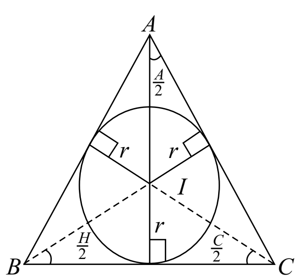
$$\Rightarrow \quad p_1=\frac{2\Delta}{a},\ p_2=\frac{2\Delta}{b},\ p_3=\frac{2\Delta}{c}$$

We have $p_1\cdot p_2\cdot p_3=\frac{2\Delta}{a}\cdot\frac{2\Delta}{b}\cdot\frac{2\Delta}{c}$

$$=\frac{8\Delta^3}{abc}$$

$$=\frac{8\left(\frac{abc}{4R}\right)^3}{abc}$$

102.




Since IA is the internal angle bisector of $\angle A$, so we can write

$$\frac{AB}{AC}=\frac{BD}{DC}$$

$$\Rightarrow \quad \frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$$

$$\Rightarrow \quad \frac{DC}{BD}+1=\frac{b}{c}+1$$

$$\Rightarrow \quad \frac{BC}{BD}=\frac{b+c}{c}$$

$$\Rightarrow \quad BD=\frac{ac}{b+c}$$

In $\Delta ABD$, $\frac{BD}{\sin\left(\frac{A}{2}\right)}=\frac{AD}{\sin B}$

$$\Rightarrow \quad AD=\frac{BD\sin B}{\sin\left(\frac{A}{2}\right)}$$

$$=\frac{ac\sin B}{(b+c)\sin\left(\frac{A}{2}\right)}=\frac{2\Delta}{(b+c)\sin(A/2)}$$

similarly, $BE=\frac{2\Delta}{(c+a)\sin\left(\frac{B}{2}\right)}$

and $CF=\frac{2\Delta}{(a+b)\sin\left(\frac{C}{2}\right)}$

103. Now $2s=a+b+c=18+24+30=72$

$\Rightarrow \quad 2s=72$

$\Rightarrow \quad s=36$

We have $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$

$$=\sqrt{36(36-18)(36-24)(36-30)}$$

$$=\sqrt{36\times18\times12\times6}$$

$$=\sqrt{36\times9\times12\times12}$$

$$=6\times3\times12$$

$$=216$$

Thus, $r_1=\frac{\Delta}{s-a}=\frac{216}{36-18}=\frac{216}{18}=12$

$$r_2=\frac{\Delta}{s-b}=\frac{216}{36-24}=\frac{216}{12}=18$$

and $r_3=\frac{\Delta}{s-c}=\frac{216}{36-30}=\frac{216}{6}=36$

104. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}$

$$=\frac{3s-(a+b+c)}{\Delta}$$

$$=\frac{3s-2s}{\Delta}$$

$$=\frac{s}{\Delta}$$

$$=\frac{1}{r}$$

105. $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}$

$$=\frac{(b-c)(s-a)}{\Delta}+\frac{(c-a)(s-b)}{\Delta}+\frac{(a-b)(s-c)}{\Delta}$$

$$=\frac{1}{\Delta}[s(b-c+c-a+a-b)]$$

$$-\frac{1}{\Delta}(ab-ac+bc-ab+ca-bc)$$

$$=0-0$$

$$=0$$

106. Given, $\frac{s-c}{s-a}=\frac{b-c}{a-b}$

$$\Rightarrow \frac{2s-2c}{2s-2a}=\frac{b-c}{a-b}$$

$$\Rightarrow \frac{a+b-c}{b+c-a}=\frac{b-c}{a-b}$$

$$\Rightarrow \frac{a+b-c}{b-c}=\frac{b+c-a}{a-b}$$

$$\Rightarrow \frac{a}{b-c}+1=\frac{c}{a-b}-1$$

$$\Rightarrow \frac{c}{a-b}+\frac{a}{b-c}=2$$

$$\Rightarrow \frac{c(b-c)+a(a-b)}{(a-b)(b-c)}=2$$

$$\Rightarrow \frac{bc-c^2+a^2-ab}{(a-b)(b-c)}=2$$

$$\Rightarrow bc-c^2+a^2-ab=2(ab-ac-b^2+bc)$$

$$\Rightarrow 2b^2-b-c^2+a^2-3ab+2ac=0$$

107. We have

$$\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$$

$$\left(1-\frac{s-b}{s-a}\right)\left(1-\frac{s-c}{s-a}\right)=2$$

$$\left(\frac{s-a-s+b}{s-a}\right)\left(\frac{s-a-s+c}{s-a}\right)=2$$

$$\left(\frac{a-b}{s-a}\right)\left(\frac{a-c}{s-a}\right)=2$$

$$\frac{(a-b)(a-c)}{(s-a)^2}=2$$

$$(a-b)(a-c)=2(s-a)^2$$

$$2(a-b)(a-c)=4(s-a)^2=(2s-2a)^2$$

$$2(a-b)(a-c)=(b+c-a)^2$$

$$2(a^2-ab-ac+bc)$$
$$=a^2+b^2+c^2+2bc-2ab-2ac$$

$$a^2+b^2=c^2$$

Thus, the $\Delta ABC$ is a right angled.

108. We have

$$r_1+r_2+r_3-r$$

$$=\left\{\frac{\Delta}{(s-a)}+\frac{\Delta}{(s-b)}\right\}+\left\{\frac{\Delta}{(s-c)}-\frac{\Delta}{s}\right\}$$

$$=\left\{\frac{\Delta(s-b+s-a)}{(s-a)(s-b)}\right\}+\left\{\frac{\Delta(s-s-+c)}{s(s-c)}\right\}$$

$$=\left\{\frac{\Delta(2s-b-a)}{(s-a)(s-b)}\right\}+\left\{\frac{\Delta(c)}{s(s-c)}\right\}$$

$$=\left\{\frac{\Delta(a+b+c-b-a)}{(s-a)(s-b)}\right\}+\left\{\frac{\Delta(c)}{s(s-c)}\right\}$$

$$=\left\{\frac{\Delta(c)}{(s-a)(s-b)}\right\}+\left\{\frac{\Delta(c)}{s(s-c)}\right\}$$

$$=\Delta(c)\left\{\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right\}$$

$$=\Delta(c)\left\{\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right\}$$

$$=\Delta(c)\left\{\frac{s^2-cs+s^2-(a+b)s+ab}{\Delta^2}\right\}$$

$$=c\times\left\{\frac{2s^2-(a+b+c)s+ab}{\Delta}\right\}$$

$$=c\times\left\{\frac{2s^2-2s^2+ab}{\Delta}\right\}$$

$$=\frac{abc}{\Delta}=4R$$

109. $r_1r_2+r_2r_3+r_3r_1$

$$=\frac{\Delta}{(s-a)}\cdot\frac{\Delta}{(s-b)}+\frac{\Delta}{(s-b)}$$
$$\cdot\frac{\Delta}{(s-c)}+\frac{\Delta}{(s-c)}\cdot\frac{\Delta}{(s-a)}$$

$$=\frac{\Delta^2(s-c+s-a+s-b)}{(s-a)(s-b)(s-c)}$$

$$=\frac{\Delta^2(3s-(a+b+c))}{(s-a)(s-b)(s-c)}$$

$$=\frac{\Delta^2(3s-2s)}{(s-a)(s-b)(s-c)}$$

$$=\frac{\Delta^2\times s}{(s-a)(s-b)(s-c)}$$

$$=\frac{\Delta^2\times s^2}{s(s-a)(s-b)(s-c)}$$

$$=\frac{\Delta^2\times s^2}{\Delta^2}$$

$$=s^2$$

110. $r_1 + r_2 - r_3 + r$

$$= (r_1 + r_2) - (r_3 - r)$$

$$= \left(\frac{\Delta}{(s-a)} + \frac{\Delta}{(s-b)}\right) - \left(\frac{\Delta}{(s-c)} - \frac{\Delta}{s}\right)$$

$$= \Delta\left\{\left(\frac{1}{(s-a)} + \frac{1}{(s-b)}\right) - \left(\frac{1}{(s-c)} - \frac{1}{s}\right)\right\}$$

$$= \Delta\left\{\left(\frac{(s-a+s-b)}{(s-a)(s-b)}\right) - \left(\frac{s-(s-c)}{s(s-c)}\right)\right\}$$

$$= \Delta\left\{\left(\frac{(2s-(a+b))}{(s-a)(s-b)}\right) - \left(\frac{c}{s(s-c)}\right)\right\}$$

$$= \Delta\left\{\left(\frac{((a+b+c)-(a+b))}{(s-a)(s-b)}\right) - \left(\frac{c}{s(s-c)}\right)\right\}$$

$$= \Delta c\left\{\left(\frac{1}{(s-a)(s-b)} - \frac{1}{s(s-c)}\right)\right\}$$

$$= \Delta c\left\{\left(\frac{s(s-c)-(s-a)(s-b)}{(s-a)(s-b)}\right)\right\}$$

$$= \Delta c\left\{\left(\frac{s^2 - cs - (s^2 - (a+b)s + ab)}{s(s-a)(s-b)(s-c)}\right)\right\}$$

$$= \frac{c}{\Delta} \times ((a+b-c)s - ab)$$

$$= \frac{c}{2\Delta} \times ((a+b-c)2s - 2ab)$$

$$= \frac{c}{2\Delta} \times ((a+b-c)(a+b+c) - 2ab)$$

$$= \frac{c}{2\Delta} \times ((a+b)^2 - c^2 - 2ab)$$

$$= \frac{c}{2\Delta} \times (a^2 + b^2 - c^2)$$

$$= \frac{2abc}{2\Delta} \times \left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$= \frac{abc}{\Delta} \times \cos(C)$$

$$= 4R\cos(C)$$

111. Given, $r_1, r_2, r_3$ are in HP

$$\Rightarrow \quad r_2 = \frac{2r_1 r_3}{r_1 + r_3}$$

$$\Rightarrow \quad \frac{\Delta}{s-b} = \frac{2 \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-c}}{\frac{\Delta}{s-a} + \frac{\Delta}{s-c}}$$

$$\Rightarrow \quad \frac{1}{s-b} = \frac{2 \cdot \frac{1}{s-a} \cdot \frac{1}{s-c}}{\frac{1}{s-a} + \frac{1}{s-c}}$$

$$\Rightarrow \quad \frac{1}{s-b}\left(\frac{1}{s-a} + \frac{1}{s-c}\right) = 2 \cdot \frac{1}{s-a} \cdot \frac{1}{s-c}$$

$$\Rightarrow \quad \frac{1}{s-b}\left(\frac{s-c+s-a}{(s-a)(s-c)}\right) = \frac{2}{(s-a)(s-c)}$$

$$\Rightarrow \quad (2s - a - c) = 2(s - b)$$

$$\Rightarrow \quad a + c = 2b$$

$$\Rightarrow \quad a, b, c \in \text{AP}$$

112. Since $a, b, c$ are in AP as well as in GP, so $a = b = c$

Now, $r_1 = \dfrac{\Delta}{s-a} = r_2 = r_3$

Thus, $\left(\dfrac{r_1}{r_2} - \dfrac{r_2}{r_3} + 10\right)$

$$= 1 - 1 + 10 = 10$$

113. We have

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2}$$

$$= \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2} + \frac{s^2}{\Delta^2}$$

$$= \frac{1}{\Delta^2}[4s^2 - 2(a+b+c)s + (a^2+b^2+c^2)]$$

$$= \frac{1}{\Delta^2}[4s^2 - 2 \cdot 2s \cdot s + (a^2+b^2+c^2)]$$

$$= \frac{1}{\Delta^2}[4s^2 - 4s^2 + (a^2+b^2+c^2)]$$

$$= \frac{(a^2+b^2+c^2)}{\Delta^2}$$

114. We have

$$(r_1 - r)(r_2 - r)(r_3 - r)$$

$$= \left(\frac{\Delta}{(s-a)} - \frac{\Delta}{s}\right)\left(\frac{\Delta}{(s-b)} - \frac{\Delta}{s}\right)\left(\frac{\Delta}{(s-c)} - \frac{\Delta}{s}\right)$$

$$= \Delta^3\left(\frac{1}{(s-a)} - \frac{1}{s}\right)\left(\frac{1}{(s-b)} - \frac{1}{s}\right)\left(\frac{1}{(s-c)} - \frac{1}{s}\right)$$

$$= \Delta^3\left(\frac{s-s+a}{s(s-a)}\right)\left(\frac{s-s+b}{s(s-b)}\right)\left(\frac{s-s+c}{s(s-c)}\right)$$

$$= \frac{\Delta^3}{s^2}\left(\frac{abc}{s(s-a)(s-b)(s-c)}\right)$$

$$= \frac{\Delta^3}{s^2}\left(\frac{abc}{\Delta^2}\right)$$

$$= \frac{\Delta \times abc}{s^2}$$

$$= 4 \times \left(\frac{\Delta}{s}\right)^2 \times \left(\frac{abc}{4\Delta}\right)$$

$$= 4r^2 R$$

115. We have

$$r_1 = \frac{\Delta}{s-a} = 1, r_2 = \frac{\Delta}{s-b} = 2$$

and $r_3 = \frac{\Delta}{s-c}$

$\Rightarrow \quad s - a = \Delta, \ s - b = \frac{\Delta}{2}$

and $\quad s - c = \frac{\Delta}{r_3}$

$$\Rightarrow \quad c = \Delta\left(1 + \frac{1}{2}\right), a = \Delta\left(\frac{1}{2} + \frac{1}{r_3}\right),$$

$$b = \Delta\left(1 + \frac{1}{r_3}\right)$$

Since triangle is right angled, so

$$a^2 + b^2 = c^2$$

$$\Rightarrow \quad \Delta^2\left(\frac{3}{2}\right)^2 = \frac{\Delta^2(r_3+2)^2}{4(r_3)^2} + \Delta^2\left(\frac{r_3+1}{r_3}\right)^2$$

$$\Rightarrow \quad \left(\frac{3}{2}\right)^2 = \frac{(r_3+2)^2}{4(r_3)^2} + \left(\frac{r_3+1}{r_3}\right)^2$$

$$\Rightarrow \quad 9r_3^2 = (r_3+2)^2 + 4(r_3+1)^2$$

$$\Rightarrow \quad 4r_3^2 - 12r_3 - 8 = 0$$

$$\Rightarrow \quad r_3^2 - 3r_3 - 2 = 0$$

$$\Rightarrow \quad r_3 = \frac{3 \pm \sqrt{17}}{2} = \frac{3 + \sqrt{17}}{2}$$

as $r_3$ is positive.

116. Let $a$, $b$ be the sides of a triangle.
Then $a + b = 5$ and $ab = 3$

Now, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$$\Rightarrow \quad \cos\left(\frac{\pi}{3}\right) = \frac{19 - c^2}{6}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{19 - c^2}{6}$$

$\Rightarrow \quad 19 - c^2 = 3$

$\Rightarrow \quad c = 4$

Thus, $r \cdot R = \frac{\Delta}{s} \times \frac{abc}{4\Delta}$

$$= \frac{abc}{4s} = \frac{abc}{2(a+b+c)} = \frac{3.4}{2(5+4)} = \frac{12}{18} = \frac{2}{3}$$

117. By sine rule, $\frac{a}{\sin(120°)} = \frac{b}{\sin(30°)}$

$$\Rightarrow \quad \frac{a}{\sqrt{3}/2} = \frac{b}{1/2}$$

$$\Rightarrow \quad a = b\sqrt{3}$$

Also, from the above figure, $r = \sqrt{3}$

$$\Rightarrow \quad \frac{\sqrt{3}}{a/2} = \tan(15°)$$

$$\Rightarrow \quad \frac{2\sqrt{3}}{a} = (2 - \sqrt{3})$$

$$\Rightarrow \quad a = \frac{2\sqrt{3}}{(2-\sqrt{3})}$$

Now, $b = \frac{a}{\sqrt{3}} = \frac{2\sqrt{3}}{(2-\sqrt{3})} \times \frac{1}{\sqrt{3}} = \frac{2}{(2-\sqrt{3})}$

Thus, the required area

$$= \frac{1}{2} \times ab \times \sin(30°)$$

$$= \frac{1}{2} \times \frac{2\sqrt{3}}{(2-\sqrt{3})} \times \frac{2}{(2-\sqrt{3})} \times \frac{1}{2}$$

$$= \sqrt{3} \times (2 + \sqrt{3})^2$$

$$= \sqrt{3} \times (7 + 4\sqrt{3})$$

$$= (12 + 7\sqrt{3}) \text{ sq. u.}$$

118. Given, $r = r_1 - r_2 - r_3$

$\Rightarrow \quad r_1 - r = r_2 + r_3$

$$\Rightarrow \quad \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$

$$\Rightarrow \quad \frac{1}{s-a} - \frac{1}{s} = \frac{1}{s-b} + \frac{1}{s-c}$$

$$\Rightarrow \quad \frac{(s-a-s)}{(s-a)} = \frac{(s-c+s-b)}{(s-b)(s-c)}$$

$$\Rightarrow \quad \frac{a}{(s-a)} = \frac{a}{(s-b)(s-c)}$$

$$\Rightarrow \quad \frac{(s-b)(s-c)}{(s-a)} = 1$$

$$\Rightarrow \quad \tan^2\left(\frac{A}{2}\right) = 1$$

$$\Rightarrow \quad \tan\left(\frac{A}{2}\right) = 1$$

$$\Rightarrow \quad \tan\left(\frac{A}{2}\right) = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad A = \frac{\pi}{2}$$

Thus, the triangle is right angled.

119. We have $r \cdot r_1 \cdot r_2 \cdot r_3$

$$= \frac{\Delta}{s} \cdot \frac{\Delta}{(s-a)} \cdot \frac{\Delta}{(s-b)} \cdot \frac{\Delta}{(s-c)}$$

$$= \frac{\Delta^4}{s(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta^4}{\Delta^2}$$

$$= \Delta^2$$

120. We have

$$\frac{r_1+r_2}{1+\cos C} = \frac{\frac{\Delta}{s-a}+\frac{\Delta}{s-b}}{2\cos^2\left(\frac{C}{2}\right)}$$

$$= \frac{\frac{\Delta(s-a+s-b)}{(s-a)(s-b)}}{2\cos^2\left(\frac{C}{2}\right)}$$

$$= \frac{\Delta(2s-a-b)}{(s-a)(s-b)\times 2\times\frac{s(s-c)}{ab}}$$

$$= \frac{\Delta\times abc}{s(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta\times abc}{\Delta^2} = \frac{abc}{\Delta}$$

Similarly, we can easily proved that,

$$\frac{(r_2+r_3)}{1+\cos A} = \frac{abc}{\Delta} \text{ and } \frac{(r_3+r_1)}{1+\cos B} = \frac{abc}{\Delta}$$

Thus, $\frac{(r_2+r_3)}{1+\cos A} = \frac{abc}{\Delta}$ and $\frac{(r_3+r_1)}{1+\cos B} = \frac{abc}{\Delta}$

121. We have

$$\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)$$

$$= \frac{1}{\Delta^3}(s-s+a)(s-s+b)(s-s+c)$$

$$= \frac{1}{\Delta^3}\times abc$$

$$= 16\times\frac{abc}{4\Delta}\times\frac{s^2}{\Delta^2(2s)^2}.$$

$$= \frac{16R}{r^2(a+b+c)^2}$$

122. We have

$$\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right)$$

$$= \left(\frac{s-a}{\Delta}+\frac{s-b}{\Delta}\right)\left(\frac{s-b}{\Delta}+\frac{s-c}{\Delta}\right)\left(\frac{s-c}{\Delta}+\frac{s-a}{\Delta}\right)$$

$$= \frac{1}{\Delta^3}\times abc$$

$$= \frac{1}{\left(\frac{abc}{4R}\right)^3}\times abc$$

$$= \frac{64R^3}{(abc)^2}$$

123. We have

$$(r_1+r_2+r_3-r)^2$$
$$= r_1^2+r_2^2+r_3^2+r^2-2r(r_1+r_2+r_3) + 2(r_1r_2+r_2r_3+r_3r_1)$$

Now, $(r_1+r_2+r_3-r) = 4R$

$(r_1r_2+r_2r_3+r_3r_1) = s^2$

and $2r(r_1+r_2+r_3)$

$$= 2\times\frac{\Delta}{s}\left(\frac{\Delta}{(s-a)}+\frac{\Delta}{(s-a)}+\frac{\Delta}{(s-a)}\right)$$

$$= 2\times\frac{\Delta^2}{s}\left(\frac{1}{(s-a)}+\frac{1}{(s-b)}+\frac{1}{(s-c)}\right)$$

$$= 2\times\frac{\Delta^2}{\text{s}}\left(\frac{(s-b)(s-c)+(s-a)(s-c)+(s-a)(s-b)}{(s-a)(s-b)(s-c)}\right)$$

$$= 2\times\Delta^2\left(\frac{3s^2-2(a+b+c)s+(ab+bc+ca)}{s(s-a)(s-b)(s-c)}\right)$$

$$= 2\times\Delta^2\left(\frac{3s^2-2\cdot 2s\cdot s+(ab+bc+ca)}{\Delta^2}\right)$$

$$= 2\times((ab+bc+ca)-s^2)$$

Thus, $(r_1^2+r_2^2+r_3^2+r^2)$

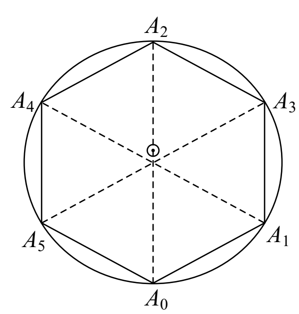
$$= (r_1+r_2+r_3-r)^2+2r(r_1+r_2+r_2) - 2(r_1r_2+r_2r_3+r_3r_1)$$
$$= 16R^2+2(ab+bc+ca-s^2)-2s^2$$
$$= 16R^2+2(ab+bc+ca)-(2s)^2$$
$$= 16R^2+2(ab+bc+ca)-(a+b+c)^2$$
$$= 16R^2-(a^2+b^2+c^2)$$

Hence, the result.

124. In a triangle $\Delta ABC$, prove that

$$\frac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{(r_1r_2+r_2r_3+r_3r_1)} = 4R$$

125.




Here, $OA_0 = OA_1 = OA_2 = \ldots = OA_5 = 1$.

and

$$\angle A_0OA_1 = \frac{2\pi}{6} = \angle A_1OA_2 = \cdots = \angle A_4OA_5$$

Now, $\cos\left(\frac{\pi}{3}\right) = \frac{OA_0^2+OA_1^2-A_1A_2}{2OA_0\cdot OA_1}$

$$\Rightarrow \quad \frac{1}{2} = \frac{1+1-A_1A_2}{2\cdot 1\cdot 1}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{2 - A_0A_1^2}{2}$$

$$\Rightarrow \quad A_0A_1^2 = 1$$

$$\Rightarrow \quad A_0A_1 = 1$$

Also, $\cos\left(\frac{2\pi}{3}\right) = \frac{OA_0^2 + OA_1^2 - A_0A_2^2}{2 \cdot OA_0 \cdot OA_1}$

$$\Rightarrow \quad -\frac{1}{2} = \frac{1 + 1 - A_0A_2^2}{2 \cdot 1 \cdot 1}$$

$$\Rightarrow \quad -\frac{1}{2} = \frac{2 - A_0A_2^2}{2}$$

$$\Rightarrow \quad A_0A_2 = \sqrt{3}$$

Again, $\cos\left(\frac{4\pi}{3}\right) = \frac{OA_0^2 + OA_4^2 - A_0A_4^2}{2 \cdot OA_0 \cdot OA_4}$

$$= \frac{1 + 1 - A_0A_4^2}{2 \cdot 1 \cdot 1}$$

$$= \frac{2 - A_0A_4^2}{2}$$

$$\Rightarrow \quad -\frac{1}{2} = \frac{2 - A_0A_4^2}{2}$$

$$\Rightarrow \quad A_0A_4 = \sqrt{3}$$

Hence, the value of

$$\Rightarrow \quad A_0A_1 \cdot A_0A_2 \cdot A_0A_4 = 1 \cdot \sqrt{3} \cdot \sqrt{3} = 3$$

126. Let $R$ be the radius of the circle.
Then, $A_1 = \pi R^2$

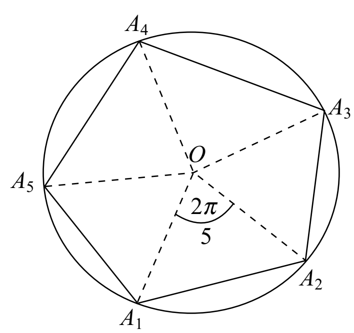
and $\Delta = \frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$

$$A_2 = \frac{5 \cdot R^2}{2}\sin\left(\frac{360°}{5}\right)$$

$$= \frac{5}{2}R^2 \sin(72°) = \frac{5R^2}{2} \times \cos(18°)$$

Now, $\frac{A_1}{A_2} = \frac{\pi R^2}{\frac{5}{2}R^2\cos(18°)} = \frac{2\pi}{5} \times \sec(18°)$

127.




Here, $OA_1 = OA_2 = OA_3 = OA_4 = OA_5 = 1$

and

$$\angle A_1OA_2 = \frac{2\pi}{5} = \angle A_2OA_3 = \cdots = \angle A_4OA_5$$

Now, $\cos\left(\frac{2\pi}{5}\right) = \frac{OA_1^2 + OA_2^2 - A_1A_2^2}{2 \cdot OA_1 \cdot OA_2}$

$$\Rightarrow \quad \sin(18°) = \frac{1 + 1 - A_1A_2^2}{2 \cdot 1 \cdot 1}$$

$$\Rightarrow \quad \frac{\sqrt{5} - 1}{4} = \frac{2 - A_1A_2^2}{2}$$

$$\Rightarrow \quad A_1A_2^2 = 2 - \frac{\sqrt{5} - 1}{2} = \frac{5 - \sqrt{5}}{2}$$

$$\Rightarrow \quad A_1A_2 = \sqrt{\frac{5 - \sqrt{5}}{2}}$$

Similarly, $A_1A_3 = \sqrt{\frac{5 + \sqrt{5}}{2}}$.

Thus, $A_1A_2 \times A_1A_3$

$$= \sqrt{\left(\frac{5 - \sqrt{5}}{2}\right) \times \left(\frac{5 + \sqrt{5}}{2}\right)} = \sqrt{\frac{25 - 5}{4}} = \sqrt{\frac{20}{4}} = \sqrt{5}$$

128. $R = \frac{a}{2}\ \operatorname{cosec}\left(\frac{\pi}{n}\right)$,
As we know that, the circum-radius of $n$ sided regular polygon

$$= \frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{n}\right), \text{ where } a = \text{side}$$

and $n$ = number of sides

$$= 6 \times \operatorname{cosec}\left(\frac{\pi}{12}\right)$$

$$= 6 \times \operatorname{cosec}(15°)$$

$$= \frac{6 \times 2\sqrt{2}}{\sqrt{3} - 1}$$

$$= 6\sqrt{2}(\sqrt{3} + 1)$$

129. Let the perimeter of the pentagon and the decagon be $10x$.
Then each side of the pentagon is $2x$ and the decagon is $x$.
Let $A_1$ = the area of the pentagon

$$= 5x^2 \cot\left(\frac{\pi}{5}\right)$$

and $A_2$ = the area of the decagon

$$= \frac{5}{2}x^2 \cot\left(\frac{\pi}{10}\right)$$

Now, $\frac{A_1}{A_2} = \frac{5x^2\cot\left(\frac{\pi}{5}\right)}{\frac{5}{2}x^2\cot\left(\frac{\pi}{10}\right)} = \frac{2\cot\left(\frac{\pi}{5}\right)}{\cot\left(\frac{\pi}{10}\right)}$

$$=\frac{2\cot(36°)}{\cot(18°)}=\frac{2\cos(36°)\sin(18°)}{\sin(36°)\cos(18°)}$$

$$=\frac{2\cos(36°)\sin(18°)}{2\sin(18°)\cos^2(18°)}$$

$$=\frac{2\cos(36°)}{(1+\cos(36°))}$$

$$=\frac{2\left(\frac{\sqrt{5}+1}{4}\right)}{\left(1+\frac{\sqrt{5}+1}{4}\right)}$$

$$=\frac{2(\sqrt{5}+1)}{(\sqrt{5}+5)}$$

$$=\frac{2(\sqrt{5}+1)}{\sqrt{5}(\sqrt{5}+1)}$$

$$=\frac{2}{\sqrt{5}}$$

130. We have

$$R=\frac{2a}{2}\operatorname{cosec}\left(\frac{\pi}{\mathrm{n}}\right)=a\operatorname{cosec}\left(\frac{\pi}{n}\right)$$

and $r=\frac{2a}{2}\cot\left(\frac{\pi}{n}\right)=a\cot\left(\frac{\pi}{n}\right)$

Now,

$$r+R=a\cot\left(\frac{\pi}{n}\right)+a\operatorname{cosec}\left(\frac{\pi}{n}\right)$$

$$=a\left(\cot\left(\frac{\pi}{n}\right)+\operatorname{cosec}\left(\frac{\pi}{n}\right)\right)$$

$$=a\left(\frac{1+\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)$$

$$=a\left(\frac{2\cos^2\left(\frac{\pi}{2n}\right)}{2\sin\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right)}\right)$$

$$=a\cot\left(\frac{\pi}{2n}\right)$$

131. Let $A_1$ be the area of the regular pentagon and $A_2$ be the area of the regular decagon.
Therefore, $A_1 = A_2$

$$\Rightarrow \quad \frac{5a^2}{4}\cot\left(\frac{\pi}{5}\right)=\frac{6b^2}{4}\cot\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad 5a^2\cot\left(\frac{\pi}{5}\right)=6b^2\cot\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad 5a^2\cot(36°)=6b^2\cot(30°)$$

$$\Rightarrow \quad 5a^2\cot(36°)=6\sqrt{3}\,b^2$$

$$\Rightarrow \quad \frac{a^2}{6\sqrt{3}}=\frac{b^2}{5\cot(36°)}=\lambda$$

Hence the ratio of their perimeters

$$=\frac{5a}{6b}$$

$$=\frac{5}{6}\times\frac{\sqrt{\lambda 6\sqrt{3}}}{\sqrt{5\lambda\cot(36°)}}=\sqrt{\frac{5\sqrt{3}}{6}\tan(36°)}$$

132. Let the perimeter of the two polygons are $nx$ and $2nx$ respectively.
Then each side of the polygons are $2x$ and $x$.
Let $A_1$ = the area of the polygon of $n$ sides

$$=nx^2\cot\left(\frac{\pi}{n}\right)$$

and $A_2$ = the area of the decagon

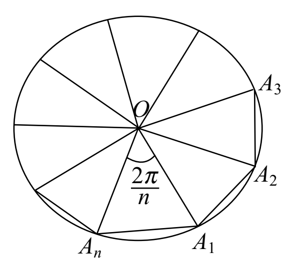
$$=\frac{5}{2}n^2\cot\left(\frac{\pi}{2n}\right)$$

Thus, $\frac{A_1}{A_2}=\frac{5n^2\cot\left(\frac{\pi}{n}\right)}{\frac{5}{2}n^2\cot\left(\frac{\pi}{2n}\right)}=\frac{2\cot\left(\frac{\pi}{n}\right)}{\cot\left(\frac{\pi}{2n}\right)}$

$$=\frac{2\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{2n}\right)}{2\sin\left(\frac{\pi}{2n}\right)\cos^2\left(\frac{\pi}{2n}\right)}$$

$$=\frac{2\cos\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)}$$

133. Let $O$ be the centre and $A_1A_2 \ldots A_n$ be the regular polygon of $n$-sides.




Let $OA_1 = OA_2 = \ldots = OA_n = r$
and $\angle A_1OA_2 = \angle A_2OA_3$

$$=\cdots=\angle A_nOA_1=\frac{2\pi}{n}$$

From the triangle $OA_1A_2$,

$$\cos\left(\frac{2\pi}{n}\right)=\frac{OA_1^2+OA_2^2-A_1A_2^2}{2\cdot OA_1\cdot OA_2}$$

$$=\frac{r^2+r^2-A_1A_2^2}{2\cdot r\cdot r}$$

$\Rightarrow \quad A_1A_2^2 = 2r^2 - 2r^2 \cos\left(\frac{2\pi}{n}\right)$

$\Rightarrow \quad A_1A_2^2 = 2r^2\left(1 - \cos\left(\frac{2\pi}{n}\right)\right)$

$\Rightarrow \quad A_1A_2^2 = 2r^2 \cdot 2\sin^2\left(\frac{2\pi}{n}\right)$

$= 4r^2 \cdot \sin^2\left(\frac{2\pi}{n}\right)$

$\Rightarrow \quad A_1A_2 = 2r \cdot \sin\left(\frac{2\pi}{n}\right)$

Similarly, $A_1A_3 = 2r \cdot \sin\left(\frac{4\pi}{n}\right)$

and $\quad A_1A_4 = 2r \cdot \sin\left(\frac{6\pi}{n}\right)$

Given, $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$

$$\Rightarrow \quad \frac{1}{2r \cdot \sin\left(\frac{2\pi}{n}\right)} = \frac{1}{2r \cdot \sin\left(\frac{4\pi}{n}\right)} + \frac{1}{2r \cdot \sin\left(\frac{6\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{1}{\sin\left(\frac{\pi}{n}\right)} - \frac{1}{\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{\sin\left(\frac{3\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{2\cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \quad 2\cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \sin\left(\frac{4\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \sin\left(\frac{4\pi}{n}\right) = \sin\left(\pi - \frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \left(\frac{4\pi}{n}\right) = \left(\pi - \frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \left(\frac{7\pi}{n}\right) = \pi$$

$\Rightarrow \quad n = 7$

134. Let $r$ be the radius of the in-circle and $r_1$, $r_2$ and $r_3$ are the ex-radii of the given triangle.

Then $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$

$$= \frac{1}{\sqrt{\pi r_1^2}} + \frac{1}{\sqrt{\pi r_1^2}} + \frac{1}{\sqrt{\pi r_1^2}}$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$$

$$= \frac{1}{\sqrt{\pi}} \times \frac{1}{r}$$

$$= \frac{1}{\sqrt{\pi r^2}}$$

$$= \frac{1}{\sqrt{A}}$$

Hence, the result.

135. Let the perimeter of the polygon of $n$ sides = $nx$

Let $\quad A_1$ = the area of the polygon of $n$ sides

$= nx^2 \cot\left(\frac{\pi}{n}\right)$

and $\quad A_2$ = the area of the circle = $\pi x^2$

Now, $\frac{A_2}{A_1} = \frac{\pi x^2}{nx^2 \cot\left(\frac{\pi}{n}\right)} = \frac{\tan\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}$

Thus, $A_2 : A_1 = \tan\left(\frac{\pi}{n}\right) : \left(\frac{\pi}{n}\right)$

136. We have

$$r + R = \frac{a}{2}\cot\left(\frac{a}{n}\right) + \frac{a}{2}\operatorname{cosec}\left(\frac{a}{n}\right)$$

$$= \frac{a}{2}\left[\cot\left(\frac{a}{n}\right) + \operatorname{cosec}\left(\frac{a}{n}\right)\right]$$

$$= \frac{a}{2}\left[\frac{\cos\left(\frac{a}{n}\right)}{\sin\left(\frac{a}{n}\right)} + \frac{1}{\sin\left(\frac{a}{n}\right)}\right]$$

$$= \frac{a}{2}\left[\frac{1 + \cos\left(\frac{a}{n}\right)}{\sin\left(\frac{a}{n}\right)}\right]$$

$$= \frac{a}{2}\left[\frac{2\cos^2\left(\frac{a}{2n}\right)}{2\sin\left(\frac{a}{2n}\right)\cos\left(\frac{a}{2n}\right)}\right]$$

$$= \frac{a}{2}\cot\left(\frac{a}{2n}\right)$$

## Level III

1. Let $a = x^2 + x + 1$, $b = 2x + 1$ and $c = x^2 - 1$
First we have to shown that, which one is greatest amongst the sides $a$, $b$ and $c$.
Clearly, $a > 0$, $b > 0$ and $c > 0$
$\Rightarrow \quad x > 1$
Now,
$$a - b = (x^2 + x + 1) - (2x + 1)$$
$$= x^2 - x$$
$$= x(x - 1) > 0 \text{ as } x > 1$$
and
$$a - c = (x^2 + x + 1) - (x^2 - 1)$$
$$= x + 2 > 0, \text{ as } x > 1$$
Therefore, $a$ is the greatest side
$\Rightarrow A$ is the greatest angle
$$\text{Now, } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{(2x+1)^2 + (x^2-1)^2 - (x^2+x+1)^2}{2(2x+1)(x^2-1)}$$
$$= \frac{(2x+1)^2 + (2x^2+x)(-x-2)}{2(2x+1)(x^2-1)}$$
$$= \frac{(2x+1)^2 + x(2x+1)(-x-2)}{2(2x+1)(x^2-1)}$$
$$= \frac{(2x+1)(2x+1-x^2-2x)}{2(2x+1)(x^2-1)}$$
$$= \frac{(1-x^2)}{2(x^2-1)} = -\frac{1}{2}$$
$$\Rightarrow \quad \cos A = -\frac{1}{2}$$
$$\Rightarrow \quad A = 120°$$

2. Given, $\cos\theta = \dfrac{a}{b+c}$
$$1 + \cos\theta = 1 + \frac{a}{b+c} = \frac{a+b+c}{b+c}$$
$$\Rightarrow \quad 2\cos^2\left(\frac{\theta}{2}\right) = \frac{a+b+c}{b+c}$$
$$\Rightarrow \quad \sec^2\left(\frac{\theta}{2}\right) = \frac{2(b+c)}{(a+b+c)}$$
$$\Rightarrow \quad 1 + \tan^2\left(\frac{\theta}{2}\right) = \frac{2(b+c)}{(a+b+c)}$$
Similarly, we can easily proved that
$$1 + \tan^2\left(\frac{\varphi}{2}\right) = \frac{2(c+a)}{(a+b+c)}$$
$$1 + \tan^2\left(\frac{\psi}{2}\right) = \frac{2(a+b)}{(a+b+c)}$$
Adding, we get,
$$3 + \tan^2\left(\frac{\theta}{2}\right) + \tan^2\left(\frac{\varphi}{2}\right) + \tan^2\left(\frac{\psi}{2}\right)$$
$$= \frac{4(a+b+c)}{(a+b+c)}$$
$$\Rightarrow \quad 3 + \tan^2\left(\frac{\theta}{2}\right) + \tan^2\left(\frac{\varphi}{2}\right) + \tan^2\left(\frac{\psi}{2}\right) = 4$$
$$\Rightarrow \quad \tan^2\left(\frac{\theta}{2}\right) + \tan^2\left(\frac{\varphi}{2}\right) + \tan^2\left(\frac{\psi}{2}\right) = 1$$
Hence, the result.

3. Given, $\sin A \sin B \sin C = p$
$\cos A \cos B \cos C = q$
Thus, $\tan A \tan B \tan C = \dfrac{p}{q}$
Also, $A + B + C = \pi$
$\tan A + \tan B + \tan C = \tan A \tan B \tan C$
$$\tan A + \tan B + \tan C = \frac{p}{q}$$
Also, $\tan A \tan B + \tan B \tan C + \tan C \tan A = \dfrac{1+q}{q}$
Hence, the required equation is
$$x^3 - \left(\frac{p}{q}\right)x^2 + \left(\frac{1+q}{q}\right)x - \left(\frac{p}{q}\right) = 0$$
$$qx^3 - px^2 + (1+q)x - p = 0$$

4. We have
$$\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$$
$$\Rightarrow \quad 2\sin^3\theta = \sin(A-\theta)\{2\sin(B-\theta)\sin(C-\theta)\}$$
$$= \sin(A-\theta)\{\cos(B-C) - \cos(B+C-2\theta)\}$$
$$\Rightarrow \quad 4\sin^3\theta = 2\sin(A-\theta)\{\cos(B-C) - \cos(B+C-2\theta)\}$$
$$= 2\sin(A-\theta)\cos(B-C) - 2\sin(A-\theta)\cos(B+C-2\theta)$$
$$= (\sin(A+B-\theta-C) - \sin(A+C-\theta-B)) - (\sin(A+B+C-3\theta) - \sin(B+C-A-\theta))$$
$$= \sin(\pi-(2C+\theta)) + \sin(\theta-(2B+\theta)) - \sin 3\theta + \sin(\pi-(2A+\theta))$$
$$\Rightarrow \quad \sin 3\theta + 4\sin^3\theta$$
$$= \sin(2A+\theta) + \sin(2B+\theta) + \sin(2C+\theta)$$
$$\Rightarrow \quad 3\sin\theta = (\sin 2A + \sin 2B + \sin 2C)\cos\theta + (\cos 2A + \cos 2B + \cos 2C)\sin\theta$$
$$\Rightarrow \quad (3 - \cos 2A - \cos 2B - \cos 2C)\sin\theta$$
$$= (\sin 2A + \sin 2B + \sin 2C)\cos\theta$$
$$\Rightarrow \quad \{(1-\cos 2A) + (1-\cos 2B) + (1-\cos 2C)\}\sin\theta$$
$$= 4\sin A \sin B \sin C \cos\theta$$
$$\Rightarrow \quad (2\sin^2 A + 2\sin^2 B + 2\sin^2 C)\sin\theta$$
$$= 4\sin A \sin B \sin C \cos\theta$$
$$\Rightarrow \quad 2[(\sin^2 A + \sin^2 B - \sin^2 C) + (\sin^2 B + \sin^2 C - \sin^2 A) + (\sin^2 C + \sin^2 A - \sin^2 B)]\sin\theta$$

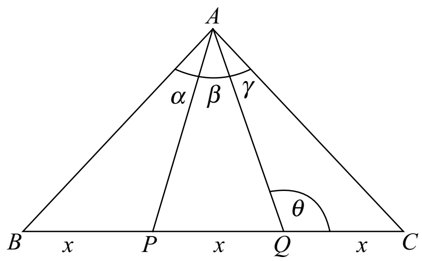
$= 4 \sin A \sin B \sin C \cos \theta$

$\Rightarrow$ $4[\sin A \sin B \sin C + \sin B \sin C \cos A + \sin C \sin A \cos B]\sin \theta$

$= 4 \sin A \sin B \sin C \cos \theta$

$\Rightarrow$ $\cot \theta = \cot A + \cot B + \cot C$

5.




Let, $BP = PQ = QC = x$ and also

let $\angle BAP = \alpha, \angle PAQ = \beta, \angle QAC = \gamma$

It is given that $\tan \alpha = t_1$, $\tan \beta = t_2$ and $\tan \gamma = t_3$

Applying $m : n$ rule in $\Delta ABC$, we get,

$$(2x + x)\cot \theta = 2x \cot(\alpha + \beta) - x \cot \gamma \quad \text{...(i)}$$

From $\Delta APC$, we get,

$$(x + x)\cot \theta = x \cot \beta - x \cot \gamma \quad \text{...(ii)}$$

Dividing (i) by (ii), we get,

$$\frac{2 \cot (\alpha + \beta) - \cot \gamma}{\cot \beta - \cot \gamma} = \frac{3}{2}$$

$\Rightarrow$ $4 \cot(\alpha + \beta) - 2 \cot \gamma = 3(\cot \beta - \cot \gamma)$

$\Rightarrow$ $4 \cot(\alpha + \beta) = 3 \cot \beta - \cot \gamma$

$$\Rightarrow \frac{1}{4 \cot (\alpha + \beta)} = \frac{1}{3 \cot \beta - \cot \gamma}$$

$$\Rightarrow \frac{\tan (\alpha + \beta)}{4} = \frac{1}{\frac{3}{\tan \beta} - \frac{1}{\tan \gamma}}$$

$$\Rightarrow \frac{\tan \alpha + \tan \beta}{4(1 - \tan \alpha \tan \beta)} = \frac{\tan \beta \tan \gamma}{3 \tan \gamma - \tan \beta}$$

$$\Rightarrow \frac{t_1 + t_2}{4(1 - t_1 t_2)} = \frac{t_2 t_3}{3t_3 - t_2}$$

$$\Rightarrow 4(t_2 t_3 - t_1 t_2^2 t_3) = 3t_1 t_3 - t_1 t_2 + 3t_2 t_3 - t_2^2$$

$$\Rightarrow (t_2 t_3 + t_1 t_2 + t_1 t_3 + t_2^2) = 4t_1 t_3 + 4t_1 t_2^2 t_3$$

$$\Rightarrow (t_3(t_2 + t_1) + t_2(t_1 + t_2)) = 4t_1 t_3(1 + t_2^2)$$

$$\Rightarrow (t_1 + t_2)(t_3 + t_2) = 4t_1 t_3 t_2^2 \left(1 + \frac{1}{t_2^2}\right)$$

$$\Rightarrow \left(\frac{t_1 + t_2}{t_1 t_2}\right)\left(\frac{t_3 + t_2}{t_2 t_3}\right) = 4\left(1 + \frac{1}{t_2^2}\right)$$

$$\Rightarrow \left(\frac{1}{t_1} + \frac{1}{t_2}\right)\left(\frac{1}{t_2} + \frac{1}{t_3}\right) = 4\left(1 + \frac{1}{t_2^2}\right)$$

6. We have

$\cos A + \cos B + \cos C$

$$= 2 \cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right) + \cos C$$

$$= 2 \cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A - B}{2}\right) + \cos C$$

$$= 2 \sin\left(\frac{C}{2}\right)\cos\left(\frac{A - B}{2}\right) + \cos C$$

$$= 2 \sin\left(\frac{C}{2}\right)\cos\left(\frac{A - B}{2}\right) + 1 - 2 \sin^2\left(\frac{C}{2}\right)$$

$$= 1 + 2 \sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A - B}{2}\right) - \sin\left(\frac{C}{2}\right)\right)$$

$$= 1 + 2 \sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A - B}{2}\right) - \cos\left(\frac{A + B}{2}\right)\right)$$

$$= 1 + 2 \sin\left(\frac{C}{2}\right)\left(2 \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right)$$

$$= 1 + 4 \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$= \left(1 + \frac{r}{R}\right)$$

7. We have

$$\sin A + \sin B + \sin C = \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}$$

$$= \frac{a + b + c}{2R}$$

$$= \frac{2s}{2R}$$

$$= \frac{s}{R}$$

$$= \frac{\frac{\Delta}{r}}{R}$$

$$= \frac{\Delta}{rR}$$

8. We have

$a \cot A + b \cot B + c \cot C$

$$= \left[2R \sin A \times \frac{\cos A}{\sin A} + 2R \sin B \times \frac{\cos B}{\sin B} + 2R \sin C \times \frac{\cos C}{\sin C}\right]$$

$$= 2R (\cos A + \cos B + \cos C)$$

$$= 2R\left(1 + \frac{r}{R}\right)$$

$$= 2(R + r)$$

9. We have,

$$\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}$$

$$= \frac{1}{2}\left(2\cos^2\left(\frac{A}{2}\right) + 2\cos^2\left(\frac{B}{2}\right) + 2\cos^2\left(\frac{C}{2}\right)\right)$$

$$= \frac{1}{2}(1 + \cos(A) + 1 + \cos B + 1 + \cos C)$$

$$= \frac{1}{2}(3 + \cos(A) + \cos B + \cos C)$$

$$= \frac{1}{2}\left(3 + \left(1 + \frac{r}{R}\right)\right)$$

$$= \frac{1}{2}\left(4 + \frac{r}{R}\right)$$

$$= \left(2 + \frac{r}{2R}\right)$$

10. We have

$$\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3}$$

$$= \frac{1}{2\Delta}(a \cos A + b \cos B + c \cos C)$$

$$= \frac{1}{2\Delta}[2R \sin A \cos A + 2R \sin B \cos B + 2R \sin C \cos C]$$

$$= \frac{R}{2\Delta}(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{R}{2\Delta}(4 \sin A \sin B \sin C)$$

$$= \frac{2R}{\Delta}(\sin A \sin B \sin C)$$

$$= \frac{2R}{\Delta} \times \left(\frac{a}{2R}\right) \times \left(\frac{b}{2R}\right) \times (\sin C)$$

$$= \frac{1}{\Delta R}\left(\frac{1}{2}ab \sin C\right)$$

$$= \frac{1}{\Delta R} \times \Delta$$

$$= \frac{1}{R}$$

11. Let, $O$ be the circum-centre and $OD = x$, $OE = y$, $OF = z$. respectively.

Also, $OA = R = OB = OC$

We have, $x = OD = R \cos A$

$$= \frac{a}{2 \sin A} \cdot \cos A = \frac{a}{2 \tan A}$$

$$\Rightarrow \quad \tan A = \frac{a}{2x}$$

Similarly, $\tan B = \frac{b}{2y}$ and $\tan C = \frac{c}{2z}$

As we know that, in a $\Delta ABC$,

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$$

$$\Rightarrow \quad \frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = \frac{a}{2x} \cdot \frac{b}{2y} \cdot \frac{c}{2z}$$

$$\Rightarrow \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{a \cdot b \cdot c}{4 \cdot x \cdot y \cdot z}$$

12. We have

$$R_1 = \frac{OB \cdot OC \cdot BC}{4\Delta OBC} = \frac{R \cdot R \cdot a}{4\Delta_1} = \frac{R^2 \cdot a}{4\Delta_1}$$

$$\Rightarrow \quad \frac{a}{R_1} = \frac{4\Delta_1}{R^2}$$

Similarly, $\frac{b}{R_2} = \frac{4\Delta_2}{R^2}$ and $\frac{c}{R_3} = \frac{4\Delta_3}{R^2}$

Thus,

$$\frac{a}{R_1} + \frac{b}{R_2} + \frac{\mathrm{c}}{R_3} = \frac{4\Delta_1}{R^2} + \frac{4\Delta_2}{R^2} + \frac{4\Delta_3}{R^2}$$

$$= \frac{4(\Delta_1 + \Delta_2 + \Delta_3)}{R^2}$$

$$= \frac{4\Delta}{R^2}$$

$$= \frac{4\Delta}{R^2}$$

$$= \frac{abc}{4R^3}$$

Hence, the result.

13. We have

$$\frac{r_1}{bc} = \frac{4R \sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}{(2R \sin B)(2R \sin C)}$$

$$= \frac{\sin\left(\frac{A}{2}\right)}{4R \sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$= \frac{\sin^2\left(\frac{A}{2}\right)}{4R \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$= \frac{\sin^2\left(\frac{A}{2}\right)}{r}$$

Similarly, $\frac{r_2}{ca} = \frac{\sin^2\left(\frac{B}{2}\right)}{r}$

and $\frac{r_3}{ab} = \frac{\sin^2\left(\frac{C}{2}\right)}{r}$

Now, $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab}$

$$= \frac{1}{r}\left(\sin^2\left(\frac{A}{2}\right) + \sin^2\left(\frac{B}{2}\right) + \sin^2\left(\frac{C}{2}\right)\right)$$
$$= \frac{1}{2r}\left(2\sin^2\left(\frac{A}{2}\right) + 2\sin^2\left(\frac{B}{2}\right) + 2\sin^2\left(\frac{C}{2}\right)\right)$$
$$= \frac{1}{2r}(1 - \cos(A) + 1 - \cos(B) + 1 - \cos(C))$$
$$= \frac{1}{2r}(3 - (\cos(A) + \cos(B) + \cos(C)))$$
$$= \frac{1}{2r}\left(3 - \left(1 + \frac{r}{R}\right)\right)$$
$$= \frac{1}{2r}\left(2 - \frac{r}{R}\right)$$
$$= \left(\frac{1}{r} - \frac{1}{2R}\right)$$

14. We have

$$\frac{bc}{r_1} + \frac{ca}{r_2} + \frac{ab}{r_3}$$
$$= abc\left(\frac{1}{ar_1} + \frac{1}{br_2} + \frac{1}{cr_3}\right)$$
$$= abc\left(\frac{s-a}{a\Delta} + \frac{s-b}{b\Delta} + \frac{s-c}{c\Delta}\right)$$
$$= \frac{abc}{\Delta}\left(\frac{s-a}{a} + \frac{s-b}{b} + \frac{s-c}{c}\right)$$
$$= \frac{abc}{2\Delta}\left(\frac{2s-2a}{a} + \frac{2s-2b}{b} + \frac{2s-2c}{c}\right)$$
$$= \frac{abc}{2\Delta}\left(\frac{b+c}{a} - 1 + \frac{c+a}{b} - 1 + \frac{a+b}{c} - 1\right)$$
$$= \frac{abc}{2\Delta}\left(\left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{a}{c} + \frac{c}{a}\right) - 3\right)$$
$$= 2R\left(\left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{a}{c} + \frac{c}{a}\right) - 3\right)$$

15. We have

$$(r + r_1)\tan\left(\frac{B-C}{2}\right)$$
$$= \left(\frac{\Delta}{s} + \frac{\Delta}{s-a}\right) \times \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$
$$= \Delta\left(\frac{s-a+s}{s(s-a)}\right) \times \left(\frac{b-c}{b+c}\right) \times \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$
$$= \Delta\left(\frac{b+c}{\sqrt{s(s-a)}}\right) \times \left(\frac{b-c}{b+c}\right) \times \sqrt{\frac{1}{(s-b)(s-c)}}$$
$$= \Delta \times (b+c) \times \left(\frac{b-c}{b+c}\right) \times \frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}$$
$$= \Delta \times (b-c) \times \frac{1}{\Delta}$$
$$= b - c$$

Similarly, $(r + r_2)\tan\left(\frac{C-A}{2}\right) - = c - a$

and $(r + r_3)\tan\left(\frac{A-B}{2}\right) = a - b$

Now, $(r + r_1)\tan\left(\frac{B-C}{2}\right) + (r + r_2)\tan\left(\frac{C-A}{2}\right) + (r + r_3)\tan\left(\frac{A-B}{2}\right)$

$= b - c + c - a + a - b$
$= 0$

16. Let $ABC$ be a right angled triangle in a circle of radius $R$. Therefore, $BC = 2R$ = diameter.

Now, $\Delta = \frac{1}{2} \times AB \times AC \times \sin(90°)$

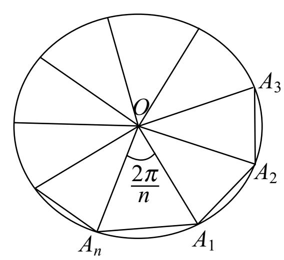
$= \frac{1}{2} \times AB \times AC$

It will be maximum, when $AB = AC$
Thus, $AB^2 + AC^2 = BC^2 = 4R^2$
$\Rightarrow \quad 2AB^2 = 4R^2$
$\Rightarrow \quad AB^2 = 2R^2$
Now, $2s = AB + BC + CA = 2AB + BC$
$= 2R(1 + \sqrt{2})$

and $r = \frac{\Delta}{s} = \frac{R^2}{R(\sqrt{2}+1)} = \frac{R}{(\sqrt{2}+1)}$

Therefore, $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r} = \frac{(\sqrt{2}+1)}{R}$

17. Let $O$ be the centre and $A_1A_2 \ldots A_n$ be the regular polygon of $n$-sides.




Let $OA_1 = OA_2 = \ldots = OA_n = r$
and $\angle A_1OA_2 = \angle A_2OA_3$

$= \cdots = \angle A_nOA_1 = \frac{2\pi}{n}$

From the $\Delta OA_1A_2$,

$$\cos\left(\frac{2\pi}{n}\right) = \frac{OA_1^2 + OA_2^2 - A_1A_2^2}{2 \cdot OA_1 \cdot OA_2}$$
$$= \frac{r^2 + r^2 - A_1A_2^2}{2 \cdot r \cdot r}$$

$$\Rightarrow \quad A_1A_2^2 = 2r^2 - 2r^2\cos\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow \quad A_1A_2^2 = 2r^2\left(1-\cos\left(\frac{2\pi}{n}\right)\right)$$

$$\Rightarrow \quad A_1A_2^2 = 2r^2 \cdot 2\sin^2\left(\frac{2\pi}{n}\right)$$

$$= 4r^2 \cdot \sin^2\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow \quad A_1A_2 = 2r \cdot \sin\left(\frac{2\pi}{n}\right)$$

Similarly, $A_1A_3 = 2r \cdot \sin\left(\frac{4\pi}{n}\right)$

and $A_1A_4 = 2r \cdot \sin\left(\frac{6\pi}{n}\right)$

Given, $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$

$$\Rightarrow \quad \frac{1}{2r\cdot\sin\left(\frac{2\pi}{n}\right)} = \frac{1}{2r\cdot\sin\left(\frac{4\pi}{n}\right)} + \frac{1}{2r\cdot\sin\left(\frac{6\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{1}{\sin\left(\frac{\pi}{n}\right)} - \frac{1}{\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{\sin\left(\frac{3\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{2\cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \quad 2\cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \sin\left(\frac{4\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \sin\left(\frac{4\pi}{n}\right) = \sin\left(\pi - \frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \left(\frac{4\pi}{n}\right) = \left(\pi - \frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \left(\frac{7\pi}{n}\right) = \pi$$

$\Rightarrow \quad n = 7$

18. Let, $r$ be the radius of the in-circle and $r_1$, $r_2$ and $r_3$ are the ex-radii of the given triangle

Then, $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$

$$= \frac{1}{\sqrt{\pi r_1^2}} + \frac{1}{\sqrt{\pi r_1^2}} + \frac{1}{\sqrt{\pi r_1^2}}$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$$

$$= \frac{1}{\sqrt{\pi}} \times \frac{1}{r}$$

$$= \frac{1}{\sqrt{\pi r^2}}$$

$$= \frac{1}{\sqrt{A}}$$

Hence, the result.

19. Let, $a$, $b$, $c$ be the sides of a triangle such that $a$ and $c$ are the least and the greatest side of $\Delta ABC$. It is given that $a$, $b$, $c$ are in AP.

$\Rightarrow \quad 2b = a + c$

Now, $\cos\theta = \frac{a^2+b^2-c^2}{2ab}$

$$= \frac{a^2-c^2}{2ab} + \frac{b^2}{2ab}$$

$$= \frac{(a+c)(a-c)}{a(a+c)} + \frac{b}{2a}$$

$$= \frac{(a-c)}{a} + \frac{b}{2a}$$

$$= \frac{2a-2c+b}{2a}$$

$$= \frac{4a-4c+a+c}{4a}$$

$$= \frac{5a-3c}{4a}$$

and $\quad \cos\varphi = \frac{b^2+c^2-a^2}{2bc}$

$$= \frac{b^2}{2bc} + \frac{c^2-a^2}{2bc}$$

$$= \frac{b}{2c} + \frac{(c-a)(c+a)}{(c+a)c}$$

$$= \frac{b}{2c} + \frac{(c-a)}{c}$$

$$= \frac{b+2c-2a}{2c}$$

$$= \frac{2b+4c-4a}{4c}$$

$$= \frac{a+c+4c-4a}{4c}$$

$$= \frac{5c-3a}{4c}$$

Now,

$$4(1-\cos\theta)(1-\cos\varphi)$$
$$=4\left(1-\frac{5a-3c}{4a}\right)\left(1-\frac{5c-3a}{4c}\right)$$
$$=4\left(\frac{4a-5a+3c}{4a}\right)\left(\frac{4c-5c+3a}{4c}\right)$$
$$=4\left(\frac{3c-a}{4a}\right)\left(\frac{3a-c}{4c}\right)$$
$$=\frac{1}{4}\left(\frac{3c-a}{a}\right)\left(\frac{3a-c}{c}\right)$$
$$=\frac{1}{4}\left(\frac{(3c-a)(3a-c)}{ac}\right)$$
$$=\frac{9ac-3c^2+ac-3a^2}{4ac}$$
$$=\frac{10ac-3c^2-3a^2}{4ac}$$
$$=\frac{5ac-3c^2+5ac-3a^2}{4ac}$$
$$=\frac{c(5a-3c)+a(5c-3a)}{4ac}$$
$$=\left(\frac{(5c-3a)}{4a}+\frac{(5a-3c)}{4c}\right)$$
$$=\cos\theta+\cos\varphi$$

Hence, the result.

20. Let $AD=\alpha$, $BE=\beta$ and $CF=\gamma$

Then, $\Delta=\frac{1}{2}\times a\times AD=\frac{1}{2}\times b\times BE=\frac{1}{2}\times c\times CF$

$$\Rightarrow \quad AD=\frac{2\Delta}{a}, BE=\frac{2\Delta}{b}, CF=\frac{2\Delta}{c}$$
$$\Rightarrow \quad \alpha=\frac{2\Delta}{a}, \beta=\frac{2\Delta}{b}, \gamma=\frac{2\Delta}{c}$$

Now,

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{a^2}{4\Delta^2}+\frac{b^2}{4\Delta^2}+\frac{c^2}{4\Delta^2}$$
$$=\frac{(a^2+b^2+c^2)}{4\Delta^2}$$
$$=\frac{1}{\Delta}\times\frac{(a^2+b^2+c^2)}{4\Delta}$$
$$=\frac{1}{\Delta}\times(\cot A+\cot B+\cot C)$$
$$=\frac{(\cot A+\cot B+\cot C)}{\Delta}$$

Hence, the result.

21. Let $AD=p_1$, $BE=p_2$ and $CF=p_3$

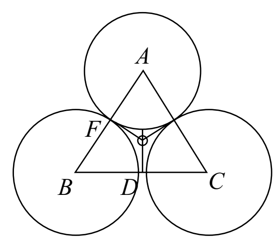
Then, $\Delta=\frac{1}{2}\times a\times p_1=\frac{1}{2}\times b\times p_2=\frac{1}{2}\times c\times p_3$

$$\Rightarrow \quad p_1=\frac{2\Delta}{a}, p_2=\frac{2\Delta}{b}, p_3=\frac{2\Delta}{c}$$

Now,

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{a}{2\Delta}+\frac{b}{2\Delta}-\frac{c}{2\Delta}$$
$$=\frac{(a+b-c)}{2\Delta}$$
$$=\frac{(a+b+c-2c)}{2\Delta}$$
$$=\frac{(2s-2c)}{2\Delta}$$
$$=\frac{(s-c)}{\Delta}$$
$$=\frac{2ab\times s(s-c)}{\Delta\times s}\times\frac{1}{2ab}$$
$$=\frac{2ab}{\Delta\times s}\times\frac{s(s-c)}{2ab}$$
$$=\frac{2ab}{(a+b+c)\Delta}\times\cos^2\left(\frac{C}{2}\right)$$

22.




Let $A$, $B$, $C$ be the centres of three circles whose radii are $a$, $b$ and $c$ respectively.

Clearly, $OD=OF=OE$

So, $O$ will be its in-centre.

Let $OD=OF=OE=r$

Let s be the semi parameter of $\Delta ABC$

Thus, $s=\frac{a+b+b+c+c+a}{2}=a+b+c$

Area of a $\Delta ABC$

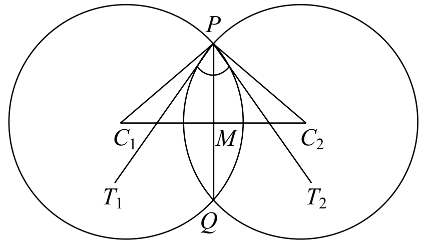
$$=\sqrt{s(s-a-b)(s-b-c)(s-c-a)}$$
$$=\sqrt{(a+b+c)abc}$$

Now, $OD=r=\frac{\Delta}{s}$

$$=\frac{\sqrt{(a+b+c)abc}}{(a+b+c)}$$
$$=\sqrt{\frac{abc}{(a+b+c)}}$$

Hence, the result.

23.




Let $C_1PC_2 = \theta$

Thus, $\cos(180° - \theta) = \dfrac{a^2 + b^2 - (C_1C_2)^2}{2ab}$

$\Rightarrow \quad (C_1C_2)^2 = a^2 + b^2 + 2ab\cos\theta$

$\Rightarrow \quad (C_1C_2) = \sqrt{a^2 + b^2 + 2ab\cos\theta}$

Area of $\Delta C_1PC_2 = \dfrac{1}{2}ab\sin\theta$

So, area of $\Delta C_1PC_2 = \dfrac{1}{2}\cdot C_1C_2 \cdot PM$

$\Rightarrow \quad \dfrac{1}{2}C_1C_2\cdot PM = \dfrac{1}{2}ab\sin\theta$

$\Rightarrow \quad PM = \dfrac{ab\sin\theta}{C_1C_2}$

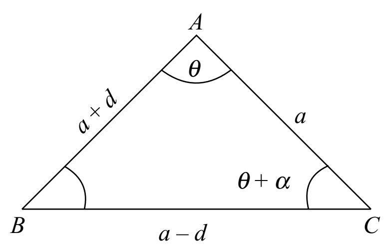
$\Rightarrow \quad PM = \dfrac{ab\sin\theta}{\sqrt{a^2 + b^2 + 2\,ab\cos\theta}}$

Hence, the length of the common chord

$= PQ = 2PM$

$= \dfrac{2\,ab\sin\theta}{\sqrt{a^2 + b^2 + 2\,ab\cos\theta}}$

24. Let the sides be $a - d$, $a$, $a + d$




Consider d > 0.
Thus, the greatest side is $a + d$ and the smallest side is $a - d$.
Let $\angle A = \theta$, $\angle C = \theta + \alpha$ and
$\angle B = 180° - (2\theta + \alpha)$
Applying sine rule, we get

$$\frac{a-d}{\sin\theta} = \frac{a}{\sin(180° - (2\theta + \alpha))} = \frac{a+d}{\sin(\theta + \alpha)}$$

$$\Rightarrow \quad \frac{a-d}{\sin\theta} = \frac{a}{\sin(2\theta + \alpha)} = \frac{a+d}{\sin(\theta + \alpha)}$$

$$= \frac{2a}{\sin\theta + \sin(\theta + \alpha)}$$

Now, $\dfrac{a-d}{\sin\theta} = \dfrac{a+d}{\sin(\theta+\alpha)}$

$$\Rightarrow \quad \frac{a-d}{a+d} = \frac{\sin\theta}{\sin(\theta+\alpha)}$$

$$\Rightarrow \quad \frac{2a}{2d} = \frac{\sin\theta + \sin(\theta+\alpha)}{\sin\theta - \sin(\theta+\alpha)}$$

$$\Rightarrow \quad \frac{2a}{2d} = \frac{2\sin\left(\theta + \frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2\cos\left(\theta + \frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)}$$

$$\Rightarrow \quad \frac{a}{d} = \frac{\tan\left(\theta + \frac{\alpha}{2}\right)}{\tan\left(\frac{\alpha}{2}\right)}$$

$$\Rightarrow \quad \frac{d}{a} = \frac{\tan\left(\frac{\alpha}{2}\right)}{\tan\left(\theta + \frac{\alpha}{2}\right)} \qquad \ldots\text{(i)}$$

Also, $\dfrac{a}{\sin(2\theta+\alpha)} = \dfrac{2a}{\sin\theta + \sin(\theta+\alpha)}$

$$\Rightarrow \quad \frac{\sin\theta + \sin(\theta+\alpha)}{\sin(2\theta+\alpha)} = 2$$

$$\Rightarrow \quad \frac{2\sin\left(\theta + \frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2\sin\left(\theta + \frac{\alpha}{2}\right)\cos\left(\theta + \frac{\alpha}{2}\right)} = 2$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\alpha}{2}\right)}{\cos\left(\theta + \frac{\alpha}{2}\right)} = 2$$

$$\Rightarrow \quad \cos\left(\theta + \frac{\alpha}{2}\right) = \frac{\cos\left(\frac{\alpha}{2}\right)}{2}$$

$$\Rightarrow \quad \tan\left(\theta + \frac{\alpha}{2}\right) = \frac{\sqrt{4 - \cos^2\left(\frac{\alpha}{2}\right)}}{\cos\left(\frac{\alpha}{2}\right)} \qquad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$\frac{d}{a} = \frac{\tan\left(\frac{\alpha}{2}\right)}{\dfrac{\sqrt{4 - \cos^2\left(\frac{\alpha}{2}\right)}}{\cos\left(\frac{\alpha}{2}\right)}} = \frac{\sin\left(\frac{\alpha}{2}\right)}{\sqrt{4 - \cos^2\left(\frac{\alpha}{2}\right)}}$$

$$= \sqrt{\frac{\sin^2\left(\frac{\alpha}{2}\right)}{4 - \cos^2\left(\frac{\alpha}{2}\right)}}$$

$$=\sqrt{\frac{\frac{1-\cos\alpha}{2}}{4-\frac{1+\cos\alpha}{2}}}$$

$$=\sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}=x$$

Hence, the required ratio is

$= a-d : a : a+d$

$= 1-\frac{d}{a}:1:1+\frac{d}{a}$

$= 1-x : 1 : 1+x$

Hence, the result.

25. Given equation is $x^3 - px^2 + qx - r = 0$

Thus, $a+b+c = p$, $ab+bc+ca = q$, $abc = r$

Now,

$$\Delta^2 = s(s-a)(s-b)(s-c)$$

$$=\frac{p}{2}\left(\frac{p}{2}-a\right)\left(\frac{p}{2}-b\right)\left(\frac{p}{2}-c\right)$$

$$=\frac{p}{2}\left[\left(\frac{p}{2}\right)^2-(a+b+c)\left(\frac{p}{2}\right)^2 + (ab+bc+ca)\left(\frac{p}{2}\right)-abc\right]$$

$$=\frac{p}{2}\left[\left(\frac{p}{2}\right)^3-\left(\frac{p}{2}\right)^2 p+\left(\frac{p}{2}\right)q-r\right]$$

$$=\frac{p}{2}\left(\frac{p^3-2p^3+4pq-8r}{8}\right)$$

$$=\frac{p}{16}(4pq-p^3-8r)$$

$$\Delta=\frac{1}{4}\sqrt{p(4pq-p^3-8r)}$$

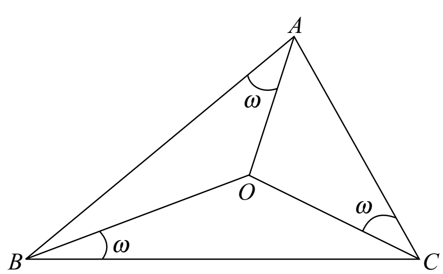
Thus, area of a triangle $= \Delta$

$$=\frac{1}{4}\sqrt{p(4pq-p^3-8r)}$$

Hence, the result

26. Let $\angle OCB = C-\omega$

and $\angle BOC = 180° - \omega - (C-\omega) = 180° - C$




Similarly, $\angle AOB = 180° - B$

Now from $\Delta OAB$, we have

$$\frac{OB}{\sin\omega}=\frac{AB}{\sin(180°-B)}$$

$$\Rightarrow \frac{OB}{\sin\omega}=\frac{c}{\sin B}$$

$$\Rightarrow OB=\frac{c\sin\omega}{\sin B} \quad \ldots\text{(i)}$$

Now, from $\Delta OBC$, we get

$$\frac{OB}{\sin(C-\omega)}=\frac{BC}{\sin(180°-C)}$$

$$\Rightarrow \frac{OB}{\sin(C-\omega)}=\frac{a}{\sin C}$$

$$\Rightarrow OB=\frac{a\sin(C-\omega)}{\sin C} \quad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$\frac{c\sin\omega}{\sin B}=\frac{a\sin(C-\omega)}{\sin C}$$

$$\Rightarrow \frac{k\sin C\sin\omega}{\sin B}=\frac{k\sin A\sin(C-\omega)}{\sin C}$$

$$\Rightarrow \frac{\sin C\sin\omega}{\sin B}=\frac{\sin A\sin(C-\omega)}{\sin(A+B)}$$

$$\Rightarrow \frac{\sin C\sin\omega}{\sin A\sin B}=\frac{\sin(C-\omega)}{\sin A\cos B+\cos A\sin B}$$

$$\Rightarrow \frac{\sin C\sin\omega}{\sin A\sin B}=\frac{\sin C\cos\omega-\cos C\sin\omega}{\sin A\cos B+\cos A\sin B}$$

$\Rightarrow$ $\sin C\sin A\cos B\sin\omega + \sin C\cos A\sin B\sin\omega$
$= \sin A\sin B\sin C\cos\omega - \sin A\sin B\cos C\sin\omega$

Dividing both the sides by $\sin A\sin B\sin C\sin\omega$

we get

$\cot B + \cot A = \cot\omega - \cot C$

$\Rightarrow$ $\cot A + \cot B + \cot C = \cot\omega$

(ii) We have

$\cot A + \cot B + \cot C = \cot\omega$

$\Rightarrow$ $(\cot A + \cot B + \cot C)^2 = \cot^2\omega$

$\Rightarrow$ $(\cot^2 A + \cot^2 B + \cot^2 C) + 2 = \cot^2\omega$

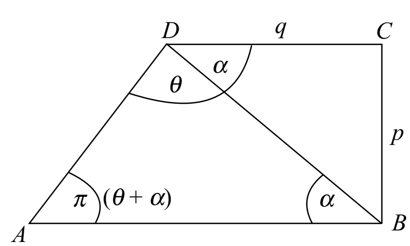
$\Rightarrow$ $\operatorname{cosec}^2 A - 1 + \operatorname{cosec}^2 B - 1$
$+ \operatorname{cosec}^2 C - 1 + 2 = \operatorname{cosec}^2\omega - 1$

$\Rightarrow$ $\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C = \operatorname{cosec}^2\omega$

$\Rightarrow$ $\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$

Hence, the result.

27.




From $\Delta BCD$, we get, $BD=\sqrt{p^2+q^2}$

Let $\angle ABD = \angle BDC = \alpha$
then $\angle DAB = \pi - (\theta + \alpha)$
Now, from $\Delta ABD$, we have

$$\frac{AB}{\sin\theta} = \frac{BD}{\sin(\pi - (\theta+\alpha))}$$

$$\Rightarrow \quad \frac{AB}{\sin\theta} = \frac{BD}{\sin(\theta+\alpha)}$$

$$\Rightarrow \quad AB = \frac{BD\sin\theta}{\sin(\theta+\alpha)}$$

$$\Rightarrow \quad AB = \frac{BD^2\sin\theta}{BD\sin(\theta+\alpha)}$$

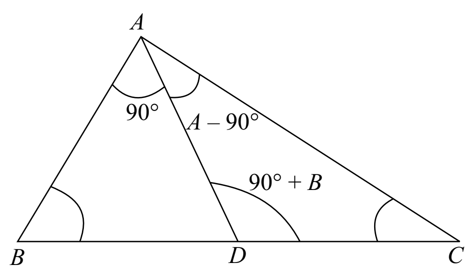
$$= \frac{BD^2\sin\theta}{BD\sin\theta\cos\alpha + BD\cos\theta\sin\alpha}$$

$$\Rightarrow \quad AB = \frac{(p^2+q^2)\sin\theta}{q\sin\theta + p\cos\theta}$$

Hence, the result.

28. We have
$c\cos(A-\theta) + a\cos(C+\theta)$
$= k\sin C\cos(A-\theta) + k\sin A\cos(C+\theta)$
$= k[\sin C\cos(A-\theta) + \sin A\cos(C+\theta)]$
$= k[\sin C\cos A\cos\theta + \sin C\sin A\sin\theta$
$+ \sin A\cos C\cos\theta - \sin A\sin C\sin\theta]$
$= k[\cos\theta(\sin A\cos C + \cos A\sin C)]$
$= k\cos\theta\sin(A+C)$
$= k\cos\theta\sin(A+C)$
$= k\cos\theta\sin B$
$= (k\sin B)\cos\theta$
$= b\cos\theta$

29. Since $AD$ is the median, so $BD : DC = 1 : 1$.




Clearly, $\angle ADC = 90° + B$.
Now, applying $m : n$ rule, we get,
$(1+1)\cot(90° + B) = 1 \cdot \cos(90°) - 1 \cdot \cot(A - 90°)$
$\Rightarrow \quad -2\tan B = 0 - (-\tan A)$
$\Rightarrow \quad -2\tan B = \tan A$
$\Rightarrow \quad \tan A + 2\tan B = 0$
Hence, the result.

30. Let $u = \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

$$\Rightarrow \quad 2u = \left(2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right)\sin\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad 2u = \left(\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right)\sin\left(\frac{C}{2}\right)$$

$$= \left(\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right)\right)\sin\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \sin^2\left(\frac{C}{2}\right) - \cos\left(\frac{A-B}{2}\right)\cdot\sin\left(\frac{C}{2}\right) + 2u = 0$$

Since $\sin\left(\frac{C}{2}\right)$ is real, so $D \geq 0$

$$\Rightarrow \quad \cos^2\left(\frac{A-B}{2}\right) - 8u \geq 0$$

$$\Rightarrow \quad \cos^2\left(\frac{A-B}{2}\right) \geq 8u$$

$$\Rightarrow \quad u \leq \frac{1}{8}\cos^2\left(\frac{A-B}{2}\right)$$

$$\Rightarrow \quad u \leq \frac{1}{8}$$

Hence, $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \leq \frac{1}{8}$

31. Let $x = \tan\left(\frac{A}{2}\right), y = \tan\left(\frac{B}{2}\right), z = \tan\left(\frac{C}{2}\right)$

To prove, $x^2 + y^2 + z^2 \geq 1$
Now, $x^2 + y^2 + z^2 - xy - yz - zx$

$$= \frac{1}{2}[(x-y)^2 + (y-z)^2 + (z-x)^2] \geq 0$$

Thus, $x^2 + y^2 + z^2 \geq xy + yz + zx$ …(i)
Since in $\Delta ABC$,

$$A + B + C = \pi$$

$$\Rightarrow \quad \frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$$

$$\Rightarrow \quad \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$\Rightarrow \quad \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right)}{1 - \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)} = \frac{1}{\tan\left(\frac{C}{2}\right)}$$

$$\Rightarrow \quad \tan\left(\frac{A}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) = 1$$

$$\Rightarrow \quad \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) + \tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right) = 1$$

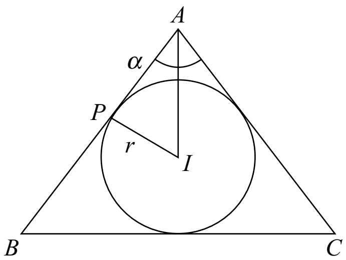
$$\Rightarrow \quad xy + yz + zx = 1$$

Therefore, from (i), we get

$$x^2 + y^2 + z^2 \geq 1$$

$$\Rightarrow \quad \tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \geq 1$$

32. Let the incircle touches the side $AB$ at $P$ where $AP = \alpha$




Let $I$ be its incentre and $AI$ bisects $\angle BAC$

Now, from $\Delta IPA$

$$\tan\left(\frac{A}{2}\right) = \frac{r}{\alpha}$$

$$\Rightarrow \quad \alpha = r\cot\left(\frac{A}{2}\right)$$

Similarly, $\beta = r\cot\left(\frac{B}{2}\right), \gamma = r\cot\left(\frac{C}{2}\right)$

In a $\Delta ABC$, we have

$$\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)$$

$$= \cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$$

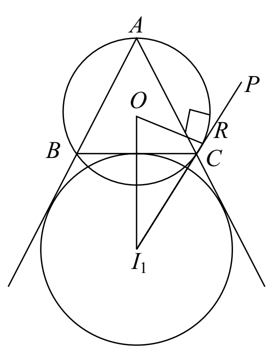
$$\Rightarrow \quad \frac{\alpha}{r} + \frac{\beta}{r} + \frac{\gamma}{r} = \frac{\alpha}{r} \cdot \frac{\beta}{r} \cdot \frac{\gamma}{r}$$

$$\Rightarrow \quad \frac{\alpha + \beta + \gamma}{r} = \frac{\alpha\beta\gamma}{r^3}$$

$$\Rightarrow \quad r^2 = \frac{\alpha\beta\gamma}{\alpha + \beta + \gamma}$$

Hence, the result.

33.




Let $O$ and $I_1$ be respectively the centres of the circum-circle and the ex-circle touching the line $BC$.

Clearly, $OI_1 = \sqrt{R^2 + 2Rr_1}$

$$\Rightarrow \quad \sqrt{R^2 + t_1^2} = \sqrt{R^2 + 2Rr_1}$$

$$\Rightarrow \quad (R^2 + t_1^2) = (R^2 + 2Rr_1)$$

$$\Rightarrow \quad t_1^2 = 2Rr_1$$

$$\Rightarrow \quad \frac{1}{t_1^2} = \frac{1}{2Rr_1}$$

Similarly, $\frac{1}{t_2^2} = \frac{1}{2Rr_2}, \frac{1}{t_3^2} = \frac{1}{2Rr_3}$

Now,

$$\frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} = \frac{1}{2Rr_1} + \frac{1}{2Rr_2} + \frac{1}{2Rr_3}$$

$$= \frac{1}{2R}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$$

$$= \frac{1}{2R} \cdot \frac{1}{r}$$

$$= \frac{1}{2R} \cdot \frac{s}{\Delta}$$

$$= \frac{1}{2 \cdot \frac{abc}{4\Delta}} \cdot \frac{(a + b + c)}{2\Delta}$$

$$= \frac{(a + b + c)}{abc}$$

Hence, the result.

34. We have

$$\frac{\sum \cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right)}{\prod \cot^2\left(\frac{A}{2}\right)}$$

$$= \frac{\left(\cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right) + \cot^2\left(\frac{B}{2}\right) \cot^2\left(\frac{C}{2}\right) + \cot^2\left(\frac{C}{2}\right)\cot^2\left(\frac{A}{2}\right)\right)}{\cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right)\cot^2\left(\frac{C}{2}\right)}$$

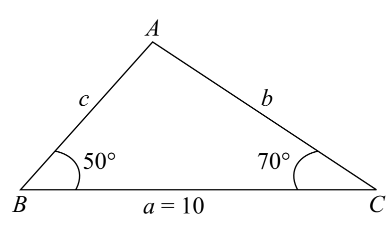
$$= \frac{1}{\cot^2\left(\frac{A}{2}\right)} + \frac{1}{\cot^2\left(\frac{B}{2}\right)} + \frac{1}{\cot^2\left(\frac{C}{2}\right)}$$

$$= \tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right)$$

$$\geq 1$$

Hence, the minimum value is 1.

35. Let $BC = a = 10$



From sine rule, we have

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{a}{\sin(60°)}=\frac{b}{\sin(50°)}=\frac{c}{\sin(70°)}$$

$$=\frac{a+b+c}{\sin(60°)+\sin(50°)+\sin(70°)}$$

$$=\frac{a+b+c}{\frac{\sqrt{3}}{2}+2\sin(60°)\cos(10°)}$$

$$=\frac{a+b+c}{\frac{\sqrt{3}}{2}+\sqrt{3}\cos(10°)}$$

$$\Rightarrow \quad \frac{a}{\sin(60°)}=\frac{a+b+c}{\frac{\sqrt{3}}{2}+\sqrt{3}\cos(10°)}$$

$$\Rightarrow \quad \frac{10}{\frac{\sqrt{3}}{2}}=\frac{a+b+c}{\frac{\sqrt{3}}{2}+\sqrt{3}\cos(10°)}$$

$$\Rightarrow \quad \frac{10}{1}=\frac{a+b+c}{1+2\cos(10°)}$$

$\Rightarrow \quad a+b+c=10+20\cos(10°)$

$\Rightarrow \quad a+b+c=x+y\cos(z°)$

$\Rightarrow \quad x=10, y=20, z=10$

Hence, the value of $(x+y+z)$

$=10+20+10=40$

36. We have

$$\frac{a\cos A+b\cos B+c\cos C}{a+b+c}$$

$$=\frac{1}{2}\left(\frac{2\sin A\cos A+2\sin B\cos B+2\sin C\cos C}{\sin A+\sin B+\sin C}\right)$$

$$=\frac{1}{2}\left(\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}\right)$$

$$=\frac{1}{2}\left(\frac{4\sin A\sin B\sin C}{\sin A+\sin B+\sin C}\right)$$

$$=\left(\frac{2\sin A\sin B\sin C}{\sin A+\sin B+\sin C}\right)$$

$$=\left(\frac{2\sin A\sin B\sin C}{4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}\right)$$

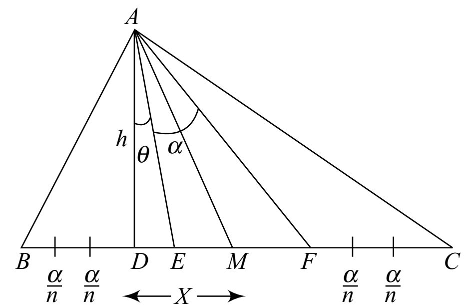
$$=\left(\frac{16\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)}{4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}\right)$$

$$=4\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)$$

$$=\frac{1}{R}\times 4R\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right)$$

$$=\frac{r}{R}$$

37.




Each part of the base $BC$ be $\frac{\alpha}{n}$.

Let $AD=h$ is the altitude.

The $n$th part $EF$ which contains the middle point $M$ subtends an angle $\alpha$ at $A$.

Let $\angle DAE=\theta$ and $\angle EAF=\alpha$

$\therefore \angle DAF=\theta+\alpha$

Also, let $DM=x$

Then $DE=x-\frac{\alpha}{2n}$ and $DF=x+\frac{\alpha}{2n}$

In $\Delta ADF$, $\tan\theta=\frac{DE}{AD}=\frac{x-\frac{a}{2n}}{h}=\frac{2nx-a}{2nh}$

$$2nh\tan\theta=(2nx-a) \qquad \ldots\text{(i)}$$

Also, in $\Delta ADF$

$$\tan(\alpha+\theta)=\frac{x+\frac{a}{2n}}{h}=\frac{2nx+a}{2nh} \qquad \ldots\text{(ii)}$$

Eliminating $\theta$ from (i) and (ii), we get,

$$2nh\tan\alpha+2nx-a$$

$$\Rightarrow \quad =2nx+a-(2nx+a)\left(\frac{(2nx-a)\tan\alpha}{2nx}\right)$$

$$2nh\tan\alpha=2a-\frac{(4n^2x^2-a^2)}{2nh}\tan\alpha$$

$$\Rightarrow \quad \left(2nh+\frac{(4n^2x^2-a^2)}{2nh}\right)\tan\alpha=2a$$

$$\Rightarrow \quad \left(\frac{4n^2h^2+(4n^2x^2-a^2)}{2nh}\right)\tan\alpha=2a$$

$$\Rightarrow \quad \tan\alpha=\frac{4anh}{4n^2h^2+(4n^2x^2-a^2)}$$

$$\Rightarrow \quad \tan\alpha=\frac{4anh}{4n^2(h^2+x^2)-a^2}$$

$$\Rightarrow \quad \tan\alpha=\frac{4anh}{4n^2\left(\frac{a^2}{4}\right)-a^2}$$

$$\left(\because AM = BM = \frac{a}{2}\right)$$

$\Rightarrow \quad \tan\alpha = \frac{4anh}{(n^2-1)a^2}$

$\Rightarrow \quad \tan\alpha = \frac{4nh}{(n^2-1)a}$

38. We have

$$\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right) = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \times \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$
$$= \sqrt{\frac{s(s-b)s(s-c)}{(s-a)(s-c)(s-a)(s-b)}}$$
$$= \sqrt{\frac{s^2}{(s-a)^2}}$$
$$= \frac{s}{s-a}$$
$$= \frac{2s}{2s-2a}$$
$$= \frac{(a+b+c)}{(a+b+c)-2a}$$
$$= \frac{(a+b+c)}{(b+c-a)}$$
$$= \frac{(a+3a)}{(3a-a)} = \frac{4a}{2a} = 2$$

39. We have

$$\frac{a^2-b^2}{a^2+b^2} = \frac{\sin(A-B)}{\sin(A+B)}.$$

$\Rightarrow \quad \frac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B} = \frac{\sin(A-B)}{\sin(A+B)}$

$\Rightarrow \quad \frac{\sin(A+B)\sin(A-B)}{\sin^2 A + \sin^2 B} = \frac{\sin(A-B)}{\sin(A+B)}$

$\Rightarrow \quad \sin(A-B) = 0,$

and $\left(\frac{\sin(A+B)}{\sin^2 A + \sin^2 B} - \frac{1}{\sin(A+B)}\right) = 0$

Now, $\sin(A-B) = 0$

$\Rightarrow \quad A = B$

$\Rightarrow \quad \Delta$ is isosceles

and $\left(\frac{\sin(A+B)}{\sin^2 A + \sin^2 B} - \frac{1}{\sin(A+B)}\right) = 0$

$\Rightarrow \quad \sin^2 A + \sin^2 B = \sin^2(A+B)$

$\Rightarrow \quad \sin^2 A + \sin^2 B = \sin^2 A\cos^2 B)$
$\qquad + \cos^2 A\sin^2 B + 2\sin A\sin B\cos A\cos B$

$\Rightarrow \quad -2\sin^2 A\sin^2 B$
$\qquad + 2\sin A\sin B\cos A\cos B = 0$

$\Rightarrow \quad \sin A\sin B = \cos A\cos B$

$\Rightarrow \quad \tan A\tan B = 1$

$\Rightarrow \quad \tan A = \cot B$

$\Rightarrow \quad \tan A = \tan\left(\frac{\pi}{2} - B\right)$

$\Rightarrow \quad A = \frac{\pi}{2} - B$

$\Rightarrow \quad A + B = \frac{\pi}{2}$

$\Rightarrow \quad C = \pi - (A+B) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

Thus, the triangle $\Delta ABC$ is right angled.

40. We have

$$\left(\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right)\right)\left(a\sin^2\left(\frac{B}{2}\right) + b\sin^2\left(\frac{A}{2}\right)\right)$$
$$= \left(\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\right)$$
$$\times\left(a\cdot\frac{(s-a)(s-c)}{ac} + b\cdot\frac{(s-b)(s-c)}{bc}\right)$$
$$= \sqrt{\frac{s}{(s-c)}}\left[\sqrt{\frac{(s-a)}{(s-b)}} + \sqrt{\frac{(s-b)}{(s-a)}}\right]$$
$$\times\left(\frac{(s-a)(s-c)}{c} + \frac{(s-b)(s-c)}{c}\right)$$
$$= \sqrt{\frac{s}{(s-c)}}\left(\frac{s-a+s-b}{\sqrt{(s-a)(s-b)}}\right)$$
$$\times\frac{s-c}{c}((s-a)+(s-b))$$
$$= \left(\frac{c\sqrt{s}}{\sqrt{(s-c)(s-a)(s-b)}}\right)\times\frac{s-c}{c}(2s-(a+b))$$
$$= \left(\frac{c\sqrt{s}}{\sqrt{(s-c)(s-a)(s-b)}}\right)\times(s-c)$$
$$= c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = c\cot\left(\frac{C}{2}\right)$$

41. From $\Delta ABC$,

$$\Delta = \frac{1}{2}ax = \frac{1}{2}ay = \frac{1}{2}az$$

$\Rightarrow \quad \frac{1}{x} = \frac{a}{2\Delta}, \frac{1}{y} = \frac{\mathrm{b}}{2\Delta}, \frac{1}{z} = \frac{c}{2\Delta}$

Now, LHS

$$= \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$$
$$= \frac{a^2+b^2+c^2}{4\Delta^2}$$

and RHS

$$= \frac{1}{\Delta}(\cot A + \cot B + \cot C)$$

$$= \frac{1}{\Delta k}\left(\frac{b^2+c^2-a^2}{2abc} + \frac{a^2+c^2-b^2}{2bac} + \frac{a^2+b^2-c^2}{2abc}\right)$$

$$= \frac{1}{\Delta k}\left(\frac{a^2+b^2+c^2}{2abc}\right)$$

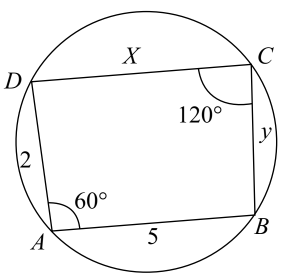
$$= \frac{1}{\Delta}\cdot\left(\frac{a^2+b^2+c^2}{2.2\Delta}\right)$$

$$\left(\because \Delta = \frac{1}{2}ab\sin C = \frac{1}{2}(abck)\right)$$

$$= \left(\frac{a^2+b^2+c^2}{4\Delta^2}\right)$$

Hence, the result.

42. Suppose $AC = 2$, $AB = 5$, $BC = x$, $CD = y$ and $\angle BAD = 60°$




Area of $\Delta ABC = \frac{1}{2}\cdot 5\cdot 2\cdot \sin(60°)$

$$= \frac{5\sqrt{3}}{2}$$

Also, from $\Delta ABC$

$$\cos(60°) = \frac{25+4-BD^2}{2\cdot 5\cdot 2}$$

$$= \frac{29-BD^2}{20}$$

$$\Rightarrow \quad \frac{29-BD^2}{20} = \frac{1}{2}$$

$\Rightarrow \quad BD^2 = 19$

$\Rightarrow \quad BD = \sqrt{19}$

Since $A, B, C, D$ are concyclic, so

$\angle BCD = 180° - 60° = 120°$

Then from $\Delta BCD$,

$$\cos(120°) = \frac{x^2+y^2-(\sqrt{19})^2}{2xy}$$

$$\Rightarrow \quad \frac{x^2+y^2-(\sqrt{19})^2}{2xy} = -\frac{1}{2}$$

$$\Rightarrow \quad \frac{x^2+y^2-(\sqrt{19})^2}{xy} = -1$$

$\Rightarrow \quad x^2 + y^2 + xy = 19$ …(i)

Again, area of $\Delta BCD$

$$= \frac{1}{2}\cdot x\cdot y\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}\,xy}{4}.$$

Thus, area of quad. $ABCD = 4\sqrt{3}$

$$\Rightarrow \quad \frac{5\sqrt{3}}{2} + \frac{\sqrt{3}\,xy}{4} = 4\sqrt{3}$$

$$\Rightarrow \quad \frac{5}{2} + \frac{xy}{4} = 4$$

$$\Rightarrow \quad \frac{xy}{4} = 4 - \frac{5}{2} = \frac{3}{2}$$

$\Rightarrow \quad xy = 6$

From (i), we get

$x^2 + y^2 = 13$

$\Rightarrow \quad x = 3, y = 2$

43. As we know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Now we have

$8R^2 = a^2 + b^2 + c^2$

$\Rightarrow \quad 8R^2 = 4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$

$\Rightarrow \quad (\sin^2 A + \sin^2 B + \sin^2 C) = 2$

$\Rightarrow \quad (\cos^2 A - \sin^2 C) + \cos^2 B = 0$

$\Rightarrow \quad \cos(A + C)\cos(A - C) + \cos^2 B = 0$

$\Rightarrow \quad \cos(\pi - B)\cos(A - C) + \cos^2 B = 0$

$\Rightarrow \quad -\cos B\cos(A - C) + \cos^2 B = 0$

$\Rightarrow \quad \cos B(\cos(A - C) + \cos^2 B = 0$

$\Rightarrow \quad \cos B\cdot 2\cos A\cos C = 0$

$\Rightarrow \quad \cos A = 0, \cos B = 0, \cos C = 0$

$$\Rightarrow \quad A = \frac{\pi}{2} = B = C$$

Thus, the triangle is right angled.

44. We have

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{C}{2}\right)$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \sqrt{\frac{(s-b)}{s}}\left(\sqrt{\frac{(s-c)}{(s-a)}} + \sqrt{\frac{(s-a)}{(s-c)}}\right)$$

$$= \sqrt{\frac{(s-b)}{s}}\left(\frac{s-c+s-a}{\sqrt{(s-a)(s-c)}}\right)$$

$$= \sqrt{\frac{(s-b)}{s}}\left(\frac{2s-c-a}{\sqrt{(s-a)(s-c)}}\right)$$

$$= \sqrt{\frac{(s-b)}{s(s-a)(s-c)}} \times (a+b+c-c-a)$$

$$= b \times \sqrt{\frac{(s-b)}{s(s-a)(s-c)}}$$

$$= \frac{2s}{3} \times \sqrt{\frac{(s-b)}{s(s-a)(s-c)}}$$

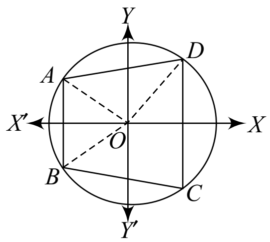
$$= \frac{2}{3} \times \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}$$

$$= \frac{2}{3} \times \cot\left(\frac{B}{2}\right)$$

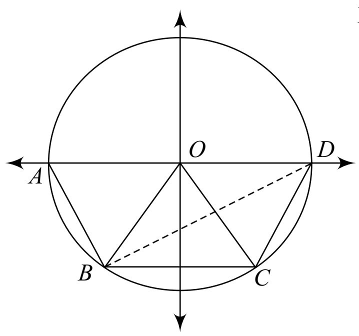
Hence, the result.

***Note:*** No questions asked in 1994.

45.




In $\Delta AOB$,
$OA = OB = 1$
and $AB = 1$ (given)
Thus, $\angle OAB = \angle OBA$
$= \angle AOB = 60°$




In $\Delta BOD$, $OA = 1$, $OB = 1$, $OB = 1$

and $BD = \sqrt{3}$

Let $\angle BOD = \theta$

Then $\cos\theta = \dfrac{OB^2 + OD^2 - BD^2}{2 \cdot OB \cdot OD}$

$\Rightarrow \quad \cos\theta = \dfrac{1+1-3}{2 \cdot 1 \cdot 1} = -\dfrac{1}{2}$

$\Rightarrow \quad \theta = 120°$

Therefore, $\angle AOD = \angle AOB + \angle BOD$
$= 60° + 120° = 180°$

$\Rightarrow$ $AOD$ is a straight line

$\Rightarrow$ $AD$ = diameter = 2

If $\angle BOC = \varphi$, then $\angle COD = \dfrac{2\pi}{3} - \varphi$

Area of the cyclic parallelogram $ABCD$
= ar of($\Delta AOB + \Delta BOC + \Delta BOD$)

$$\Rightarrow \quad \frac{3\sqrt{3}}{4} = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin(60°) + \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin\varphi + \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin\left(\frac{2\pi}{3} - \varphi\right)$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \sin\varphi + \sin\left(\frac{2\pi}{3} - \varphi\right)\right) = \frac{3\sqrt{3}}{4}$$

$$\Rightarrow \quad \frac{1}{2}\left(\sin\varphi + \sin\left(\frac{2\pi}{3} - \varphi\right)\right) = \frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4}$$

$$\Rightarrow \quad \frac{1}{2}\left(\sin\varphi + \sin\left(\frac{2\pi}{3} - \varphi\right)\right) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \left(\sin\varphi + \sin\left(\frac{2\pi}{3} - \varphi\right)\right) = \sqrt{3}$$

$$\Rightarrow \quad 2\sin\left(\frac{\pi}{3}\right)\cos\left(\varphi - \frac{\pi}{3}\right) = \sqrt{3}$$

$$\Rightarrow \quad 2 \times \frac{\sqrt{3}}{2} \times \cos\left(\varphi - \frac{\pi}{3}\right) = \sqrt{3}$$

$$\Rightarrow \quad \cos\left(\varphi - \frac{\pi}{3}\right) = 1$$

$$\Rightarrow \quad \left(\varphi - \frac{\pi}{3}\right) = 0$$

$$\Rightarrow \quad \varphi = \frac{\pi}{3}$$

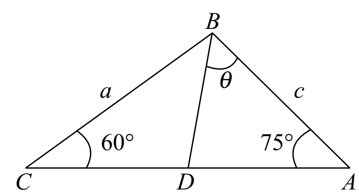
Thus, $\angle BOC = 60°$
and $\angle COD = 120° - 60° = 60°$
In $\Delta BOC$,

$$\frac{BC}{\sin 60°} = \frac{OB}{\sin 60°}$$

$\Rightarrow \quad BC = OB = 1$

Similarly, we can prove that $CD = 1$
Hence, $AB = 1$, $BC = 1$, $CD = 1$ and $AD = 2$.

46.




Here, $\angle B = 180° - (60° + 75°) = 45°$
Let $\angle ABD = \theta$

It is given that, $\dfrac{ar(\Delta BAD)}{ar(\Delta BCD)} = \sqrt{3}$

$$\Rightarrow \quad \frac{\frac{1}{2}cx\sin\theta}{\frac{1}{2}ax\sin(45° - \theta)} = \sqrt{3}$$

$$\Rightarrow \quad \frac{c\sin\theta}{a\sin(45° - \theta)} = \sqrt{3}$$

$$\Rightarrow \quad \frac{\sin C\sin\theta}{\sin A\sin(45° - \theta)} = \sqrt{3}$$

$$\Rightarrow \quad \frac{\sin 60°\sin\theta}{\sin 75°\sin(45° - \theta)} = \sqrt{3}$$

$$\Rightarrow \quad \frac{\frac{\sqrt{3}}{2}\sin\theta}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\sin(45^\circ-\theta)}=\sqrt{3}$$

$$\Rightarrow \quad \sqrt{2}\sin\theta=(\sqrt{3}+1)\sin(45^\circ-\theta)$$

$$\Rightarrow \quad \sqrt{2}\sin\theta=(\sqrt{3}+1)\frac{1}{\sqrt{2}}(\cos\theta-\sin\theta)$$

$$\Rightarrow \quad 2\sin\theta=(\sqrt{3}+1)(\cos\theta-\sin\theta)$$

$$\Rightarrow \quad (3+\sqrt{3})\sin\theta=(\sqrt{3}+1)\cos\theta$$

$$\Rightarrow \quad \frac{\sin\theta}{\cos\theta}=\frac{(\sqrt{3}+1)}{(3+\sqrt{3})}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \tan\theta=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta=\frac{\pi}{6}$$

Hence, $\angle ABD=\frac{\pi}{6}$

47. Given, $\cos(A-B)=\frac{4}{5}$

$$\Rightarrow \quad 2\cos^2\left(\frac{A-B}{2}\right)-1=\frac{4}{5}$$

$$\Rightarrow \quad 2\cos^2\left(\frac{A-B}{2}\right)=1+\frac{4}{5}$$

$$\Rightarrow \quad \cos^2\left(\frac{A-B}{2}\right)=\frac{9}{10}$$

$$\Rightarrow \quad \cos\left(\frac{A-B}{2}\right)=\frac{3}{\sqrt{10}}$$

$$\Rightarrow \quad \tan\left(\frac{A-B}{2}\right)=\frac{1}{3}$$

$$\Rightarrow \quad \left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right)=\frac{1}{3}$$

$$\Rightarrow \quad \left(\frac{6-3}{6+3}\right)\cot\left(\frac{C}{2}\right)=\frac{1}{3}$$

$$\Rightarrow \quad \frac{1}{3}\cot\left(\frac{C}{2}\right)=\frac{1}{3}$$

$$\Rightarrow \quad \cot\left(\frac{C}{2}\right)=1$$

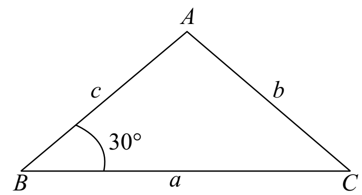
$$\Rightarrow \quad \cot\left(\frac{C}{2}\right)=\cot\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad C=\frac{\pi}{2}$$

Thus,

$$\text{ar}(\Delta ABC)=\frac{1}{2}ab\sin C$$
$$=\frac{1}{2}\times 6\times 3\times\sin(90^\circ)$$
$$=9 \text{ sq units.}$$

48.




We have

$$\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{\sqrt{6}}{\sin(30^\circ)}=\frac{4}{\sin C}$$

$$\Rightarrow \quad \sin C=\frac{4}{\sqrt{6}}\times\frac{1}{2}=\frac{2}{\sqrt{6}}<1$$

$C$ may be acute or obtuse

Also, we observe that $b<c$

$\Rightarrow \quad B<C$

i.e. if $C$ is obtuse, then $B$ should be acute.

$\Rightarrow$ It is possible.

Thus, two triangles are possible in this case.

Now, applying cosine rule,

$$\cos B=\frac{c^2+a^2-b^2}{2ac}$$

$$\Rightarrow \quad \cos(30^\circ)=\frac{16+a^2-6}{2\cdot 4\cdot a}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}=\frac{16+a^2-6}{2\cdot 4\cdot a}$$

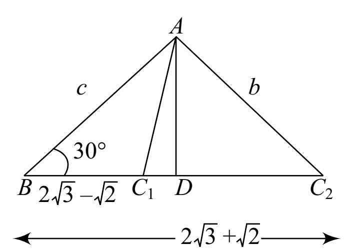
$$\Rightarrow \quad \frac{a^2+10}{8a}=\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad a^2-4\sqrt{3}\,a+10=0$$

$$\Rightarrow \quad (a-2\sqrt{3})^2=12-10=2$$

$$\Rightarrow \quad (a-2\sqrt{3})=\pm\sqrt{2}$$

$$\Rightarrow \quad a=2\sqrt{3}\pm\sqrt{2}$$




Now,

$$\text{ar}(\Delta ABC_1)=\frac{1}{2}\times 4\times(2\sqrt{3}-\sqrt{2})\times\sin(30^\circ)$$
$$=(2\sqrt{3}-\sqrt{2}) \text{ sq units}$$

and

$$\text{ar}(\Delta ABC_2)=\frac{1}{2}\times 4\times(2\sqrt{3}+\sqrt{2})\times\sin(30^\circ)$$
$$=(2\sqrt{3}+\sqrt{2}) \text{ sq. units}$$

49. Given, $2s = 24$

$\Rightarrow \quad s = 12$

Also, $r_1, r_2, r_3 \in$ HP

$\Rightarrow \quad \dfrac{\Delta}{s-a}, \dfrac{\Delta}{s-b}, \dfrac{\Delta}{s-c} \in \text{HP}$

$\Rightarrow \quad \dfrac{\Delta}{s-a}, \dfrac{\ddot{A}}{s-b}, \dfrac{\Delta}{s-c} \in \text{HP}$

$\Rightarrow \quad \dfrac{1}{s-a}, \dfrac{1}{s-b}, \dfrac{1}{s-c} \in \text{HP}$

$\Rightarrow \quad (s-a), (s-b), (s-c) \in$ AP

$\Rightarrow \quad a, b, c \in$ AP

$\Rightarrow \quad 2b = a + c$

$\Rightarrow \quad 3b = a + b + c = 24$

$\Rightarrow \quad b = 8$

Again, $r_1, r_2, r_3 \in$ HP

$\Rightarrow \quad \dfrac{1}{r_1}, \dfrac{1}{r_2}, \dfrac{1}{r_3} \in \text{AP}$

$\Rightarrow \quad \dfrac{2}{r_2} = \dfrac{1}{r_1} + \dfrac{1}{r_3}$

$\Rightarrow \quad r_2 = \dfrac{2r_1r_3}{r_1 + r_3}$

$\Rightarrow \quad r_1r_2 + r_2r_3 = 2r_1r_3$

$\Rightarrow \quad r_1r_2 + r_1r_3 + r_2r_3 = 2r_1r_3 + r_1r_3 = 3r_1r_3$

$\Rightarrow \quad 3r_1r_3 = s^2 = 144$

$\Rightarrow \quad r_1r_3 = \dfrac{144}{3} = 48$

$\Rightarrow \quad \dfrac{\Delta}{(s-a)} \times \dfrac{\Delta}{(s-c)} = 48$

$\Rightarrow \quad \dfrac{24}{(12-a)} \times \dfrac{24}{(12-c)} = 48$

$\Rightarrow \quad (12-a)(12-c) = 12$

$\Rightarrow \quad 144 - 12(a+c) = 12$

$\Rightarrow \quad 144 - 12.16 + c(16-c) = 12$

$\Rightarrow \quad 16c - c^2 - 64 = 0$

$\Rightarrow \quad c^2 - 16c + 64 = 0$

$\Rightarrow \quad (c-8)^2 = 0$

$\Rightarrow \quad c = 8$

So, the lengths of the sides are 8 cm, 8 cm, 8 cm.

50. Given equation is $x^3 - 11x^2 + 38x - 40 = 0$

It is also given that a, b and c are its roots

Thus, $a + b + c = 11$,

$ab + bc + ca = 38$

and $\quad abc = 40$

Now,

$$\frac{m}{n} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc}$$

$$= \frac{(a+b+c)^2 - 2(ab+bc+ca)}{2abc}$$

$$= \frac{(11)^2 - 2.38}{2.40}$$

$$= \frac{121-76}{80}$$

$$= \frac{45}{80} = \frac{9}{16}$$

Thus, $m = 9$ and $n = 16$

Hence, the value of $m + n$

$= 9 + 16 = 25$

## Level IV

1. Given, $\dfrac{\cos A + 2\cos C}{\cos A + 2\cos B} = \dfrac{\sin B}{\sin C}$

$\cos A(\sin B - \sin C) + (\sin 2B - \sin 2C) = 0$

$\cos A(\sin B - \sin C) + 2\cos(B+C)\sin(B-C) = 0$

$\cos A(\sin B - \sin C) + 2\cos(\pi - A)\sin(B-C) = 0$

$\cos A(\sin B - \sin C) + 2\cos A\sin(B-C) = 0$

$\cos A[(\sin B - \sin C) - 2\sin(B-C)] = 0$

$\cos A[(\sin B - \sin C) - 2\sin(B-C)] = 0$

$\cos A = 0$, $[(\sin B - \sin C) - 2\sin(B-C)] = 0$

when $\cos A = 0$

$\angle A = 90°$

$\Delta$ is right angled.

when $[(\sin B - \sin C) - 2\sin(B-C)] = 0$

$[(\sin B - \sin C) - 2(\sin B\cos C - \cos B\sin C)] = 0$

$$\left[(b-c) - 2\left(b\frac{a^2+b^2-c^2}{2ab} - c\frac{a^2+c^2-b^2}{2ac}\right)\right] = 0$$

$$\left[(b-c) - \left(\frac{a^2+b^2-c^2}{a} - \frac{a^2+c^2-b^2}{a}\right)\right] = 0$$

$$\left[(b-c) - \frac{1}{a}(a^2+b^2-c^2-a^2-c^2+b^2)\right] = 0$$

$$\left[(b-c) - \frac{2}{a}(b^2-c^2)\right] = 0$$

$[a(b-c) - 2(b^2-c^2)] = 0$

$(b-c)[a - 2(b+c)] = 0$

$(b-c) = 0$

$b = c$

$\Delta$ is isoseles

2. We have

a $\tan A + b\tan B = (a+b)\tan\left(\dfrac{A+B}{2}\right)$

a $\tan A + b\tan B = (a+b)\cot\left(\dfrac{C}{2}\right)$

$$a\left(\tan A - \cot\left(\frac{C}{2}\right)\right) = b\left(\cot\left(\frac{C}{2}\right) - \tan B\right)$$

$$a\left(\frac{\sin A}{\cos A} - \frac{\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)}\right) = b\left(\frac{\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)} - \frac{\sin B}{\cos B}\right)$$

$$a\left(\frac{\sin A \sin\left(\frac{C}{2}\right) - \cos A \cos\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)\cos A}\right)$$

$$= b\left(\frac{\cos B \cos\left(\frac{C}{2}\right) - \sin B \sin\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)\cos B}\right)$$

$$-\frac{a\cos\left(A + \frac{C}{2}\right)}{\cos A \sin\left(\frac{C}{2}\right)} = \frac{b\cos\left(B + \frac{C}{2}\right)}{\cos B \sin\left(\frac{C}{2}\right)}$$

$$-\frac{a\cos\left(A + \frac{C}{2}\right)}{\cos A} = \frac{b\cos\left(B + \frac{C}{2}\right)}{\cos B}$$

$$-\frac{\sin A\cos\left(A + \frac{C}{2}\right)}{\cos A} = \frac{\sin B\cos\left(B + \frac{C}{2}\right)}{\cos B}$$

$$-2\sin A \cos B \cos\left(A + \frac{C}{2}\right)$$

$$= 2\sin B \cos A \cos\left(B + \frac{C}{2}\right)$$

$$-(\sin(A + B) + \sin(A - B))\cos\left(A + \frac{C}{2}\right)$$

$$= (\sin(A + B) - \sin(A - B))\cos\left(B + \frac{C}{2}\right)$$

$$\sin(A + B)\left\{\cos\left(B + \frac{C}{2}\right) + \cos\left(A + \frac{C}{2}\right)\right\}$$

$$= \sin(A - B)\left\{\cos\left(A + \frac{C}{2}\right) - \cos\left(B + \frac{C}{2}\right)\right\}$$

$$\sin(A + B)\left\{2\cos\left(\frac{A + B + C}{2}\right)\cos\left(\frac{B - A}{2}\right)\right\}$$

$$= \sin(A - B)\left\{2\sin\left(\frac{A + B + C}{2}\right)\sin\left(\frac{B - A}{2}\right)\right\}$$

$$\sin(A - B)\left\{2\sin\left(\frac{A + B + C}{2}\right)\sin\left(\frac{B - A}{2}\right)\right\} = 0$$

$$\sin(A - B) = 0, \left\{2\sin\left(\frac{A+B+C}{2}\right)\sin\left(\frac{B-A}{2}\right)\right\} = 0$$

$$\sin(A - B) = 0$$

$$(A - B) = 0$$

$$A = B$$

Thus, the triangle is isoseles.

3. We have

$$a(rr_1 + r_2r_3) = a\left(\frac{\Delta}{s}\cdot\frac{\Delta}{(s-a)} + \frac{\Delta}{(s-b)}\cdot\frac{\Delta}{(s-c)}\right)$$

$$= a\Delta^2\left(\frac{1}{s(s-a)} + \frac{1}{(s-b)(s-c)}\right)$$

$$= a\Delta^2\left(\frac{(s-b)(s-c) + s(s-a)}{s(s-a)(s-b)(s-c)}\right)$$

$$= a\Delta^2\left(\frac{2s^2 - (a+b+c)s + bc}{s(s-a)(s-b)(s-c)}\right)$$

$$= a\Delta^2\left(\frac{2s^2 - 2s\cdot s + bc}{s(s-a)(s-b)(s-c)}\right)$$

$$= a\Delta^2\left(\frac{2s^2 - 2s^2 + bc}{\Delta^2}\right)$$

$$= abc$$

Also,

$$b(rr_2 + r_1r_3) = b\left(\frac{\Delta}{s}\cdot\frac{\Delta}{(s-b)} + \frac{\Delta}{(s-a)}\cdot\frac{\Delta}{(s-c)}\right)$$

$$= b\Delta^2\left(\frac{1}{s}\cdot\frac{1}{(s-b)} + \frac{1}{(s-a)}\cdot\frac{1}{(s-c)}\right)$$

$$= b\Delta^2\left(\frac{1}{s(s-b)} + \frac{1}{(s-a)(s-c)}\right)$$

$$= b\Delta^2\left(\frac{(s-a)(s-c) + s(s-b)}{s(s-a)(s-b)(s-c)}\right)$$

$$= b\Delta^2\left(\frac{s^2 - (a+b+c)s + ac + s^2}{s(s-a)(s-b)(s-c)}\right)$$

$$= b\Delta^2\left(\frac{2s^2 - 2s\cdot s + ac}{s(s-a)(s-b)(s-c)}\right)$$

$$= b\Delta^2\left(\frac{2s^2 - 2s^2 + ac}{\Delta^2}\right)$$

$$= b\Delta^2\left(\frac{ac}{\Delta^2}\right)$$

$$= abc$$

Similarly, $c(rr_3 + r_1r_2) = abc$

Hence, the result.

4. Now, $(r + r_1)\tan\left(\frac{B-C}{2}\right)$

$$= \left(\frac{\Delta}{s} + \frac{\Delta}{(s-a)}\right)\left(\frac{b-c}{b+c}\right)\cot\left(\frac{C}{2}\right)$$

$$= \Delta\left(\frac{1}{s} + \frac{1}{(s-a)}\right)\left(\frac{b-c}{b+c}\right)\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \Delta\left(\frac{s-a+s}{s(s-a)}\right)\left(\frac{b-c}{b+c}\right)\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \Delta(2s-a)\left(\frac{b-c}{b+c}\right)\sqrt{\frac{1}{s(s-a)(s-b)(s-c)}}$$

$$= \Delta(a+b+c-a)\left(\frac{b-c}{b+c}\right)\times\frac{1}{\Delta}$$

$$= (b+c)\left(\frac{b-c}{b+c}\right)$$

$$= (b-c)$$

Thus, L.H.S.

$= (b-c)+(c-a)+(a-b)$

$= 0.$

5. We have

$$\frac{\tan\left(\frac{A}{2}\right)}{(a-b)(a-c)}+\frac{\tan\left(\frac{B}{2}\right)}{(b-a)(b-c)}+\frac{\tan\left(\frac{C}{2}\right)}{(c-a)(c-b)}$$

Now, $$\frac{\tan\left(\frac{A}{2}\right)}{(a-b)(a-c)}=\frac{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}{(a-b)(a-c)}$$

$$=\frac{\Delta}{s(s-a)(a-b)(a-c)}$$

So, LHS

$$=\frac{\Delta}{s}\left[\frac{1}{(s-a)(a-b)(a-c)}-\frac{1}{(s-b)(a-b)(b-c)}-\frac{1}{(s-c)(c-a)(b-c)}\right]$$

$$=\frac{\Delta}{s(a-b)(b-c)(c-a)}\left[\frac{(b-c)}{(s-a)}+\frac{(c-a)}{(s-b)}+\frac{(a-b)}{(s-b)}\right]$$

$$=\frac{\Delta}{s(a-b)(b-c)(c-a)}\left[\frac{\Sigma(b-c)\{s^2-(b+c)s+bc\}}{(s-a)(s-b)(s-c)}\right]$$

$$=-\frac{\Delta}{s(a-b)(b-c)(c-a)}\times\left[\frac{s^2\Sigma(b-c)-s\,\Sigma(b^2-c^2)+\Sigma bc(b-c)}{(s-a)(s-b)(s-c)}\right]$$

$$=-\frac{\Delta}{s(a-b)(b-c)(c-a)}\left[\frac{\Sigma bc(b-c)}{(s-a)(s-b)(s-c)}\right]$$

$$=\left[\frac{\Delta}{s(a-b)(b-c)(c-a)}\frac{(a-b)(b-c)(c-a)}{(s-a)(s-b)(s-c)}\right]$$

$$=\left[\frac{\Delta}{s(s-a)(s-b)(s-c)}\right]$$

$$=\left[\frac{\Delta}{\Delta^2}\right]$$

$$=\frac{1}{\Delta}$$

Hence, the result.

6. We have

$$\frac{\text{area of the incircle}}{\text{area of triangle } ABC}=\frac{\pi r^2}{\Delta}$$

$$=\frac{\pi r^2}{\Delta}$$

$$=\frac{\pi r^2}{\Delta}=\frac{\pi}{\Delta}\times\frac{\Delta^2}{s^2}=\frac{\pi\Delta}{s^2}$$

Now, $\cot\left(\frac{A}{2}\right)\cdot\cot\left(\frac{B}{2}\right)\cdot\cot\left(\frac{C}{2}\right)$

$$=\left[\frac{s(s-a)}{(s-b)(s-c)}\frac{s(s-b)}{(s-a)(s-c)}\frac{s(s-c)}{(s-a)(s-b)}\right]^{1/2}$$

$$=\left[\frac{s^3}{(s-a)(s-b)(s-c)}\right]^{1/2}$$

$$=\left[\frac{s^4}{s(s-a)(s-b)(s-c)}\right]^{1/2}$$

$$=\left[\frac{s^4}{\Delta^2}\right]^{1/2}$$

$$=\frac{s^2}{\Delta}$$

Hence, the area

$$=\frac{\pi}{\cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)}$$

7. We have

$$\cos A\cdot\cot\left(\frac{A}{2}\right)=\left(1-2\sin^2\left(\frac{A}{2}\right)\right)\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$=\cot\left(\frac{A}{2}\right)-\sin A$$

Now, $a$, $b$, $c$ are in AP

$\sin A$, $\sin B$, $\sin C$ are also in A.P and

and $\cot\left(\frac{A}{2}\right), \cot\left(\frac{B}{2}\right), \cot\left(\frac{C}{2}\right) \in$ AP

Thus, their differences are also in A.P.

8. Let $BC = a = c$, $CA = b = 10\,c$ and $AB = c$

Clearly, $a^2 + b^2 = 101c^2$

Applying, sine rule, we get,

$$\frac{\sin A}{c}=\frac{\sin B}{10c}=\frac{\sin C}{c}$$

$$\frac{\sin A}{c}=\frac{\sin B}{10c}=\frac{\sin C}{c}=k(\text{say})$$

Now,

$$\frac{\cot C}{\cot A+\cot B}=\frac{\frac{\cos C}{\sin C}}{\frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}}$$

$$=\frac{\frac{\cos C}{\sin C}}{\frac{\sin(A+B)}{\sin A\sin B}}$$

$$=\frac{\frac{\cos C}{\sin C}}{\frac{\sin C}{\sin A\sin B}}$$

$$=\frac{\cos C}{\sin C}\times\sin B$$

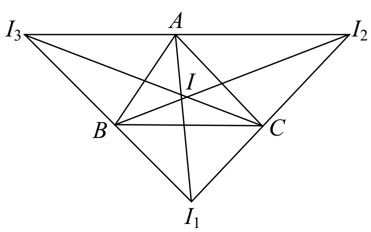
$$=\cos C\times\frac{\sin B}{\sin C}$$

$$=\left(\frac{a^2+b^2-c^2}{2ab}\right)\times\frac{\sin B}{\sin C}$$

$$=k^2\left(\frac{c^2+100c^2-c^2}{2\cdot ck\cdot 10ck}\right)\times\frac{10ck}{ck}$$

$$=5\times10=50$$

9. Do yourself.
10. We have




Area of the triangle $\Delta I_1I_2I_3$

$\frac{1}{2}\times$ (product of two sides) $\times$ (sine of included angles)

$$=\frac{1}{2}\times\left(4R\cos\left(\frac{B}{2}\right)\right)\times\left(4R\cos\left(\frac{C}{2}\right)\right)\times\sin\left(90^\circ-\frac{A}{2}\right)$$

$$=8R^2\times\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$$

$$=8R^2\times\sqrt{\frac{s(s-a)}{bc}}\times\sqrt{\frac{s(s-b)}{ca}}\times\sqrt{\frac{s(s-c)}{ab}}$$

$$=\frac{8R^2}{abc}\times s\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\frac{8R^2s}{abc}\times\Delta$$

$$=\frac{8}{abc}\times\left(\frac{abc}{4\Delta}\right)^2\times\Delta s$$

$$=\frac{8abc}{16}\times\frac{s}{\Delta}$$

$$=\frac{abc}{2r}$$

11. Clearly, $r_1+r_2+r_3=r+4R$

and $r_1r_2+r_2r_3+r_3r_1=s^2$

and $r_1r_2r_3=\dfrac{\Delta^2}{(s-a)(s-b)(s-c)}=\Delta s=s^2r$

Hence, the required equation is

$$x^3-(r+4R)x^2+s^2x-rs^2=0$$

12. Let $r$ be the in-radius and $R$ be the circum radius of an equilateral triangle

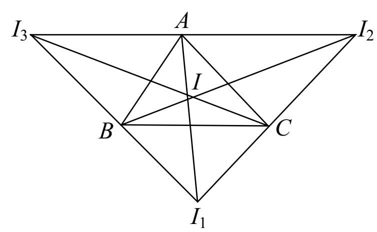
Now, $r=\dfrac{\Delta}{s}=\dfrac{\sqrt{3}a^2}{4}\times\dfrac{2}{3a}=\dfrac{a}{2\sqrt{3}}$

And, $R=\dfrac{abc}{4\Delta}=\dfrac{a^3}{4\times\frac{\sqrt{3}a^2}{4}}=\dfrac{a}{\sqrt{3}}$

Thus, $\dfrac{r}{R}=\dfrac{\frac{a}{2\sqrt{3}}}{\frac{a}{\sqrt{3}}}=\dfrac{1}{2}$

Hence, the result.

13. We have




Hence, the area

$=\frac{1}{2}\times$ (product of two sides) $\times$ (sine of included angles)

$$=\frac{1}{2}\times\left(4R\cos\left(\frac{B}{2}\right)\right)\times\left(4R\cos\left(\frac{C}{2}\right)\right)\times\sin\left(90^\circ-\frac{A}{2}\right)$$

$$=\frac{1}{2}\times\left(4R\cos\left(\frac{B}{2}\right)\right)\times\left(4R\cos\left(\frac{C}{2}\right)\right)\times\cos\left(\frac{A}{2}\right)$$

$$=8R^2\times\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$$

14. Hence, the circum-radius

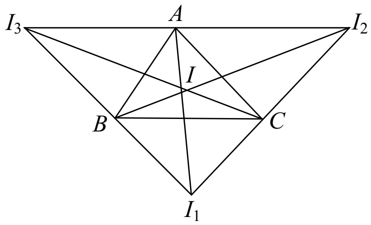
$$=\frac{I_2I_3}{2\sin(I_2I_1I_3)}$$

$$=\frac{4R\cos\left(\frac{A}{2}\right)}{2\sin\left(90^\circ-\frac{A}{2}\right)}$$

$$= \frac{4R\cos\left(\frac{A}{2}\right)}{2\cos\left(\frac{A}{2}\right)}$$

$$= 2R$$

15. From the figure,




$\angle IBI_1$, $\angle ICI_1$ are right angles

Here, $II_1$ is the diameter of the circum-circle of the triangle $\Delta BCI_1$

Thus, $II_1 = \dfrac{BC}{\sin(\angle BI_1C)} = \dfrac{a}{\sin\left(90° - \frac{A}{2}\right)}$

$$= \frac{a}{\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2R\sin A}{\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2R \times 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

$$= 4R\sin\left(\frac{A}{2}\right)$$

Similarly, $II_2 = 4R\sin\left(\frac{B}{2}\right)$,

$II_3 = 4R\sin\left(\frac{C}{2}\right)$.

16. We have

$a^3\cos(B - C)$

$= a^2k\sin A\cos(B - C)$

$= a^2k\sin(B + C)\cos(B - C)$

$= -\dfrac{a^2k}{2}[2\sin(B + C)\cos(B - C)]$

$= -\dfrac{a^2k}{2}[\sin 2B + \sin 2C]$

$= \dfrac{k^3}{2}[\sin^2 A\sin 2B + \sin^2 A\sin 2C]$

$= k^3[\sin^2 A\sin B\cos B + \sin^2 A\sin C\cos C]$

Similarly, $b^3\cos(C - A)$

$= k^3[\sin^2 B\sin C\cos C + \sin^2 B\sin A\cos A]$

and $c^3\cos(A - B)$

$= k^3[\sin^2 C\sin A\cos A + \sin^2 C\sin B\cos B]$

Adding all we get,

$k^3[\sin A\sin B(\sin A\cos B + \cos A\sin B)$
$+ \sin B\sin C(\sin B\cos C + \cos B\sin C)$
$+ \sin C\sin A(\sin C\cos A + \cos C\sin A)]$

$= k^3[\sin A\sin B\sin(A + B) + \sin B\sin C\sin(B + C)$
$+ \sin C\sin A\sin(C + A)]$

$= k^3[\sin A\sin B\sin C + \sin A\sin B\sin C$
$+ \sin A\sin B\sin C]$

$= k^3[3\sin A\sin B\sin C]$

$= 3(k\sin A)(k\sin B)(k\sin C)$

$= 3abc$

17. Do yourself.

18. We have $c\cos(A - \theta) + a\cos(C + \theta)$

$= c(\cos A\cos\theta + \sin A\sin\theta)$
$+ a(\cos C\cos\theta - \sin C\sin\theta)$

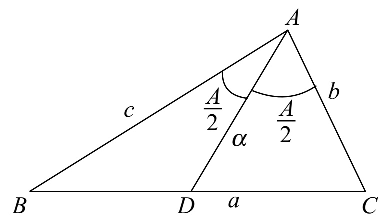
$= \cos\theta(c\cos A + a\cos C)$
$+ \sin\theta(c\sin A - a\sin C)$

$= b\cos\theta + \sin\theta(k\sin C\sin A - k\sin C\sin A)$

$= b\cos\theta + \sin\theta \times 0$

$= b\cos\theta$

19. Let, $BC = a$, $AC = b$, $AB = c$




Clearly,

$ar(\Delta ABC) = ar(\Delta ABD) + ar(\Delta ACD)$

$$\frac{1}{2}bc\sin A = \frac{1}{2}c\alpha\sin\left(\frac{A}{2}\right) + \frac{1}{2}b\alpha\sin\left(\frac{A}{2}\right)$$

$$bc\sin A = c\alpha\sin\left(\frac{A}{2}\right) + b\alpha\sin\left(\frac{A}{2}\right)$$

$$\frac{1}{\alpha}\cos\left(\frac{A}{2}\right) = \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right)$$

Similarly, $\dfrac{1}{\beta}\cos\left(\dfrac{B}{2}\right) = \dfrac{1}{2}\left(\dfrac{1}{c} + \dfrac{1}{a}\right)$

And $\dfrac{1}{\gamma}\cos\left(\dfrac{C}{2}\right) = \dfrac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$

Adding, all we get,

$$\frac{1}{\alpha}\cos\left(\frac{A}{2}\right) + \frac{1}{\beta}\cos\left(\frac{B}{2}\right) + \frac{1}{\gamma}\cos\left(\frac{C}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right) + \frac{1}{2}\left(\frac{1}{c} + \frac{1}{a}\right)$$

$$= \frac{1}{2}\left(\frac{2}{a} + \frac{2}{b} + \frac{2}{c}\right)$$

$$= \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Hence,

$$\frac{1}{AD}\cos\left(\frac{A}{2}\right)+\frac{1}{BE}\cos\left(\frac{B}{2}\right)+\frac{1}{CF}\cos\left(\frac{C}{2}\right)$$
$$=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

20. Do yourself.
21. Clearly, $2s = a + b + c$

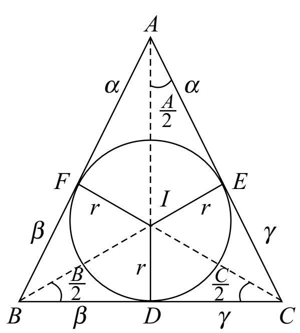
As we know that, AM ≥ GM

$$\frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$
$$\frac{3s-(a+b+c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$
$$\frac{3s-2s}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$
$$\frac{s}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$
$$\left(\frac{s}{3}\right)^3 \geq (s-a)(s-b)(s-c)$$
$$s^4 \geq 27 \times s(s-a)(s-b)(s-c)$$
$$s^4 \geq 27 \times \Delta^2$$
$$s^2 \geq 3\sqrt{3} \times \Delta$$
$$\Delta \leq \frac{s^2}{3\sqrt{3}}$$

22. Do yourself
23. We have

$$2\alpha + 2\beta + 2\gamma = a + b + c = 2\text{ s}$$
$$\alpha + \beta + \gamma = s$$

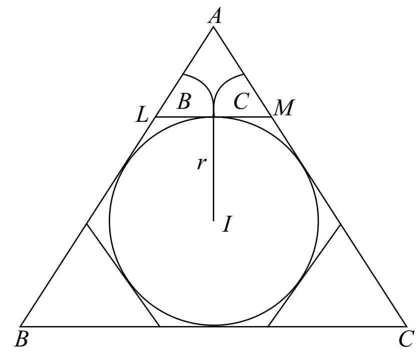
and $\alpha = s - a, \beta = s - b, \gamma = s - c$




Now, $r^2 = \dfrac{\Delta^2}{s^2} = \dfrac{s(s-a)(s-b)(s-c)}{s^2}$

$$= \frac{(s-a)(s-b)(s-c)}{s}$$
$$= \frac{\alpha \cdot \beta \cdot \gamma}{\alpha + \beta + \gamma}$$

24.




In $\Delta ALM$, we have

$$\frac{x}{\sin A} = \frac{AM}{\sin B} = \frac{AL}{\sin C}$$
$$AL = x \cdot \frac{\sin C}{\sin A} = \frac{cx}{a}$$
$$AM = x \cdot \frac{\sin B}{\sin A} = \frac{bx}{a}$$

From the figure, it is clear that

$r = ex$ – radius of $\Delta ALM$

$$r = \left(\frac{x + AL + AM}{2}\right)\tan\left(\frac{A}{2}\right)$$
$$r = \left(\frac{x + \frac{cx}{2} + \frac{bx}{a}}{2}\right)\tan\left(\frac{A}{2}\right)$$
$$r = \left(\frac{a+b+c}{2a}\right)x\tan\left(\frac{A}{2}\right) = \frac{sx}{a}\tan\left(\frac{A}{2}\right)$$
$$(s-a)\tan\left(\frac{A}{2}\right) = \frac{sx}{a}\tan\left(\frac{A}{2}\right)$$
$$(s-a) = \frac{sx}{a}$$

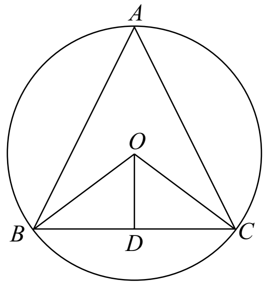
Similarly, $(s-b) = \dfrac{sx}{b}, (s-c) = \dfrac{sx}{c}$

Adding, we get,

$$(s-a)+(s-b)+(s-c) = \frac{sx}{a}+\frac{sx}{b}+\frac{sx}{c}$$
$$3s-(a+b+c) = \frac{sx}{a}+\frac{sx}{b}+\frac{sx}{c}$$
$$3s-2s = \frac{sx}{a}+\frac{sx}{b}+\frac{sx}{c}$$
$$\frac{sx}{a}+\frac{sx}{b}+\frac{sx}{c} = s$$
$$\frac{x}{a}+\frac{x}{b}+\frac{x}{c} = 1$$

Hence, the result.

25. It is given that,




$$x = OD = R\cos A$$
$$= \frac{a}{2\sin A}\cdot\cos A$$
$$= \frac{a}{2\tan A}$$
$$\frac{a}{x} = 2\tan A$$

Similarly, $\frac{b}{y} = 2\tan B$, $\frac{c}{z} = 2\tan C$

As we know that,

$\tan A + \tan B + \tan C = \tan A\tan B\tan C$

$$\frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z} = \frac{a}{2x}\cdot\frac{b}{2y}\cdot\frac{c}{2z}$$
$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z} = \frac{abc}{4xyz}$$

Hence, the result.

26. Given, $\sin A\sin B\sin C = p$

and $\cos A\cos B\cos C = q$

Here, $A + B + C = \pi$

$\tan(A + B + C) = \tan(\pi) = 0$

$\tan A + \tan B + \tan C = \tan A\cdot\tan B\tan C$

$$= \frac{p}{q}$$

Also, $(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A)$

$$= \frac{\cos(A+B+C)}{\cos A\cos B\cos C} = -\frac{1}{q}$$

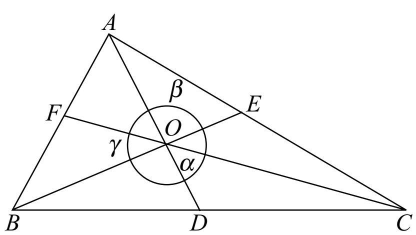
Thus, $\tan A\tan B + \tan B\tan C + \tan C\tan A$

$$= 1 + \frac{1}{q} = \left(\frac{q+1}{q}\right)$$

Hence, the required equation is

$$x^3 - \left(\frac{p}{q}\right)x^2 + \left(\frac{q+1}{q}\right)x - \left(\frac{p}{q}\right) = 0$$
$$qx^3 - px^2 + (q+1)x - p = 0$$

27. Let the medians $AD$, $BE$ and $CF$ meet at $O$ such that $\angle BOC = \alpha$, $\angle AOC = \beta$, $\angle AOB = \gamma$




Let $AD = p_1$, $BE = p_2$, $CF = p_3$

Clearly, $OA = \frac{2}{3}p_1$, $OB = \frac{2}{3}p_2$, $OC = \frac{2}{3}p_3$

From $\Delta AOC$, we get

$$\cos\beta = \frac{OA^2 + OC^2 - AC^2}{2OA\cdot OC}$$
$$\Rightarrow \quad \cos\beta = \frac{\frac{4}{9}p_1^2 + \frac{4}{9}p_3^2 - b^2}{2\cdot\frac{2}{3}p_1\cdot\frac{2}{3}p_3}$$
$$\Rightarrow \quad \cos\beta = \frac{4p_1^2 + 4p_3^2 - 9b^2}{8p_1p_3} \qquad \ldots\text{(i)}$$

Also, $ar(\Delta AOC) = \frac{1}{2}OA\cdot OC\cdot\sin\beta$

$$\Rightarrow \quad \frac{1}{3}\Delta = \frac{1}{2}OA\cdot OC\cdot\sin\beta$$
$$\Rightarrow \quad \frac{1}{3}\Delta = \frac{1}{2}\frac{2}{3}p_1\cdot\frac{2}{3}p_3\cdot\sin\beta$$
$$\Rightarrow \quad \sin\beta = \frac{3\Delta}{2p_1p_3}$$

Dividing (i) by (ii), we get

$$\cot\beta = \frac{4p_1^2 + 4p_3^2 - 9b^2}{12\Delta} \qquad \ldots\text{(iii)}$$

Again $AD$ is the median of $\Delta ABC$

So, $AB^2 + AC^2 = 2BD^2 + 2AD^2$

$$\Rightarrow \quad b^2 + c^2 = 2\cdot\left(\frac{a}{2}\right)^2 + 2p_1^2$$
$$\Rightarrow \quad b^2 + c^2 = \frac{a^2}{2} + 2p_1^2$$
$$\Rightarrow \quad p_1^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

Similarly, $p_2^2 = \frac{2a^2 + 2c^2 - b^2}{4}$

And $p_3^2 = \frac{2a^2 + 2b^2 - c^2}{4}$

Now, from (iii), we get,

$$\cot\beta = \frac{4p_1^2 + 4p_3^2 - 9b^2}{12\Delta}$$
$$= \frac{(2b^2 + 2c^2 - a^2) + (2a^2 + 2c^2 - b^2) - 9b^2}{12\Delta}$$
$$= \frac{a^2 + c^2 - 5b^2}{12\Delta}$$

Similarly, $\cot\alpha = \frac{b^2 + c^2 - 5a^2}{12\Delta}$

And $\cot\gamma = \frac{b^2 + a^2 - 5c^2}{12\Delta}$

Now,

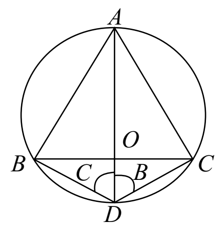
$$\cot\alpha+\cot\beta+\cot\gamma=-\frac{3(a^2+b^2+c^2)}{12\Delta}$$
$$=-\frac{(a^2+b^2+c^2)}{4\Delta}$$

Also, $\cot A+\cot B+\cot C=\dfrac{(a^2+b^2+c^2)}{4\Delta}$

Hence,

$$\cot A+\cot B+\cot C+\cot\alpha+\cot\beta+\cot\gamma=0$$

28. Let $AO$ be perpendicular from $A$ on $BC$. When $AO$ is produced, it meets the circumscribing circle at $D$ such that $OD=\alpha$ since angles in the same segment are equal.




Thus, $\angle ADB=\angle ACB=\angle C$

and $\angle ADC=\angle ABC=\angle B$

From $\Delta BOD$, $\tan(C)=\dfrac{OB}{OD}$ …(i)

From $\Delta COD$, $\tan(B)=\dfrac{OC}{OD}$ …(ii)

Adding (i) and (ii), we get,

$$\tan B+\tan C=\frac{OB+OC}{OD}=\frac{BC}{OD}=\frac{a}{\alpha} \quad \text{…(iii)}$$

Similarly,

$$\tan C+\tan A=\frac{b}{\beta} \quad \text{…(iv)}$$

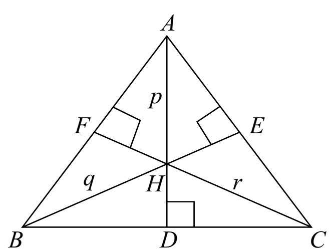
And $\tan A+\tan B=\dfrac{c}{\gamma}$ …(v)

Adding (iii), (iv) and (v), we get,

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=2(\tan A+\tan B+\tan C)$$

Hence, the result.

29. Let $H$ be the orthocentre of the triangle $ABC$ such that $HA=p$, $HB=q$, $HC=r$




From the figure,

$\angle HBD=\angle EBC=90°-C$
$\angle HCD=\angle FCB=90°-B$
$\angle BHC=180°-(\angle HBD+\angle HCD)$
$=180°-(90°-C+90°-B)$
$=(B+C)=18°-A$

Similarly,

$\angle AHC=180°-B$

and $\angle AHB=180°-C$

Now,

$ar(\Delta BHC)+ar(\Delta CHA)+ar(\Delta AHB)$
$=ar(\Delta ABC)$

$$\Rightarrow \frac{1}{2}qr\sin(\angle BHC+\frac{1}{2}pr\sin(\angle AHC)+\frac{1}{2}pq\sin(\angle AHB)=\Delta$$

$$\Rightarrow qr\sin(180°-A)+pr\sin(180°-B)+pq\sin(180°-C)=2\Delta$$

$$\Rightarrow qr\sin A+rp\sin B+pq\sin C=\Delta$$

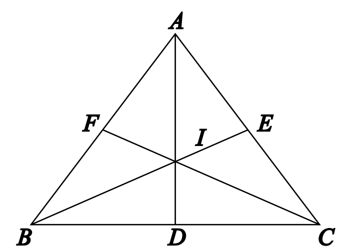
$$\Rightarrow qr\left(\frac{a}{2R}\right)+rp\left(\frac{b}{2R}\right)+pq\left(\frac{c}{2R}\right)=\frac{abc}{4R}$$

$$\Rightarrow aqr+brp+cpq=abc$$

$$\Rightarrow \frac{aqr}{pqr}+\frac{brp}{pqr}+\frac{cpq}{pqr}=\frac{abc}{pqr}$$

$$\Rightarrow \frac{a}{p}+\frac{b}{q}+\frac{c}{r}=\frac{abc}{pqr}$$

30. Here, $AD$ is the internal bisector of the angle $A$




Clearly, $\dfrac{BD}{DC}=\dfrac{c}{b}$

$$\Rightarrow \frac{DC}{BD}=\frac{b}{c}$$

$$\Rightarrow \frac{DC}{BD}+1=\frac{b}{c}+1=\frac{b+c}{c}$$

$$\Rightarrow \frac{DC+BD}{BD}=\frac{b+c}{c}$$

$$\Rightarrow \frac{a}{BD}=\frac{b+c}{c}$$

$$\Rightarrow \frac{BD}{c}=\frac{a}{b+c}$$

Similarly, $\dfrac{BF}{a}=\dfrac{c}{a+b}$

Now,

$$\frac{ar(\Delta BDF)}{ar(\Delta ABC)}=\frac{\frac{1}{2}\cdot BD\cdot BF\cdot\sin B}{\frac{1}{2}ac\sin B}$$

$$=\frac{BD\cdot BF}{ac}=\frac{BD}{a}\cdot\frac{BF}{c}=\frac{ac}{(a+b)(b+c)}$$

Similarly, $\dfrac{ar(\Delta CDE)}{ar(\Delta ABC)}=\dfrac{ac}{(a+b)(b+c)}$

and $\dfrac{ar(\Delta AEF)}{ar(\Delta ABC)} = \dfrac{bc}{(b+c)(a+c)}$

Thus, $\dfrac{ar(\Delta DEF)}{ar(\Delta ABC)}$

$$= \frac{\Delta ABC - (\Delta BDF + \Delta CDE + \Delta AEF)}{\Delta ABC}$$

$$= 1 - \frac{\Delta BDF}{\Delta ABC} - \frac{\Delta CDE}{\Delta ABC} - \frac{\Delta AEF}{\Delta ABC}$$

$$= 1 - \frac{ac}{(a+b)(b+c)} - \frac{ab}{(a+c)(b+c)} - \frac{bc}{(a+b)(a+c)}$$

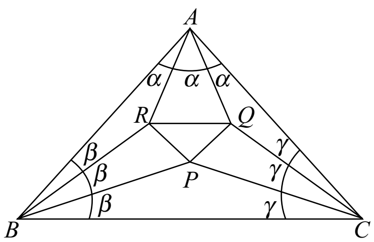
$$= \frac{2abc}{(a+b)(b+c)(c+a)}$$

$$\frac{ar(\Delta DEF)}{ar(\Delta ABC)} = \frac{2abc}{(a+b)(b+c)(c+a)}$$

$$ar(\Delta DEF) = \frac{2\Delta abc}{(a+b)(b+c)(c+a)}$$

Hence, the result.

31.




Since, $A + B + C = 180°$

$3\alpha + 3\beta + 3\gamma = 180°$

$\alpha + \beta + \gamma = 60°$

Clearly, $\angle ARB = 180° - (\alpha + \beta)$

Applying sine rule in triangle $ARB$

$$\frac{AR}{\sin\beta} = \frac{c}{\sin(180° - (\alpha + \beta))}$$

$$\frac{AR}{\sin\beta} = \frac{c}{\sin(\alpha + \beta)}$$

$$AR = \frac{c\sin\beta}{\sin(\alpha+\beta)}$$

$$= \frac{2R\sin C\sin\beta}{\sin(\alpha+\beta)}$$

$$= \frac{2R\sin(3\gamma)\sin\beta}{\sin(\alpha+\beta)}$$

$$= \frac{2R\sin(3\gamma)\sin\beta}{\sin(60° - \gamma)}$$

$$= \frac{2R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)}{\sin(60° - \gamma)}$$

$$= \frac{2R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)}{\sin(60° - \gamma)} \cdot \frac{\cos(30° - \gamma)}{\cos(30° - \gamma)}$$

$$= \frac{4R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{2\sin(60° - \gamma)\cos(30° - \gamma)}$$

$$= \frac{4R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{\sin(90° - 2\gamma) + \sin(30°)}$$

$$= \frac{4R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{\cos(2\gamma) + \frac{1}{2}}$$

$$= \frac{8R\sin\beta\sin\gamma(3 - \sin^2\gamma)\cos(30° - \gamma)}{2\cos(2\gamma) + 1}$$

$$= \frac{8R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{2(1 - 2\sin^2\gamma) + 1}$$

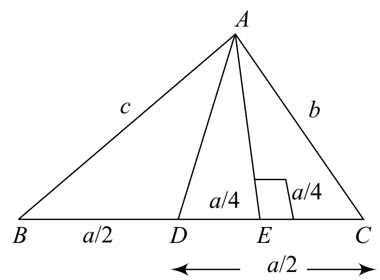
$$= \frac{8R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{(3 - 4\sin^2\gamma)}$$

$$= 8R\sin\beta\sin\gamma \text{ in } \gamma(30° - \gamma)$$

Hence, the result.

32. Since, $AD$ is the median, $BD = DC = \dfrac{a}{2}$

Also, $\angle DAE = \angle CAE = \dfrac{A}{2}$




Applying cosine rule in triangle $ABD$, we get,

$$\cos\left(\frac{A}{3}\right) = \frac{AB^2 + AD^2 - BD^2}{2AB \cdot AD}$$

$$\cos\left(\frac{A}{3}\right) = \frac{c^2 + b^2 - \left(\frac{a}{2}\right)^2}{2bc} = \frac{4b^2 + 4c^2 - a^2}{8bc} \quad \ldots\text{(i)}$$

Applying cosine rule in triangle $ABC$, we get,

$$\cos A = \frac{b^2 + c^2 = a^2}{2bc}$$

$$4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right) = \frac{b^2 + c^2 - a^2}{2bc} \quad \ldots\text{(ii)}$$

Subtracting (ii) from (i), we get

$$4\cos\left(\frac{A}{3}\right) - 4\cos^3\left(\frac{A}{3}\right)$$

$$= \frac{4b^2 + 4c^2 - a^2}{8bc} - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{4b^2 + 4c^2 - a^2 - 4b^2 - 4c^2 + 4a^2}{8bc}$$

$$= \frac{3a^2}{8bc}$$

$$4\cos\left(\frac{A}{3}\right)\left(1 - \cos^2\left(\frac{A}{3}\right)\right) = \frac{3a^2}{8bc}$$

$$4\cos\left(\frac{A}{3}\right)\sin^2\left(\frac{A}{3}\right) = \frac{3a^2}{8bc}$$

$$\cos\left(\frac{A}{3}\right)\sin^2\left(\frac{A}{3}\right) = \frac{3a^2}{32bc}$$

## Integer Type Questions

1. As we know that, in a right angled triangle $a^2 + b^2 + c^2 = 8R^2$

$$\left(\frac{a^2 + b^2 + c^2}{R^2}\right) = 8$$

2. We have, $\left(\frac{r_1 + r_2 + r_3}{r}\right)$

$$= \frac{4R\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\left(\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right)\right)}{4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$= \frac{\left(\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right)\right)}{\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}$$

$$= 1$$

3. As we know that,

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$$

Using, AM ≥ GM

$$\left(\frac{r_1 + r_2 + r_3}{3}\right) \geq \frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}}$$

$$\left(\frac{r_1 + r_2 + r_3}{3}\right) \geq \frac{3}{\frac{1}{r}} = 3r$$

$$\left(\frac{r_1 + r_2 + r_3}{r}\right) \geq 9$$

4. We have

$$\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right) = \left(\sin A + \frac{1}{\sin A} + 1\right)$$

$$= \left(\sin A + \frac{1}{\sin A}\right) + 1$$

$$\geq 2 + 1 = 3$$

Hence, the minimum value is 3.

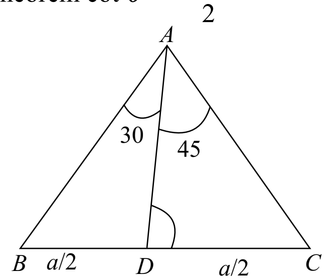
5. We have

$$\frac{1}{8R^2}\{(r^2 + r_1^2 + r_2^2 + r_3^2) + (a^2 + b^2 + c^2)\}$$

$$= \frac{1}{8R^2} \times 16R^2$$

$$= 2$$

6. By $m$-$n$ theorem $\cot\theta = \frac{\sqrt{3} - 1}{2}$




So, $\tan\theta = \frac{2}{(\sqrt{3} - 1)}$

$$\sin\theta = \frac{2}{\sqrt{(8 - 2\sqrt{3})}} \text{ and } \cos\theta = \frac{(\sqrt{3} - 1)}{\sqrt{(8 - 2\sqrt{3})}}$$

From $\Delta ADC$,

$$\frac{a/2}{\sin(45°)} = \frac{AD}{\sin(\pi - (\theta + 45°))} = \frac{AD}{\frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)}$$

$$\frac{a/2}{\sin(45°)} = \frac{AD}{\frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)}$$

$$\frac{a}{2} = \frac{AD}{(\sin\theta + \cos\theta)}$$

$$\frac{a}{2} = \frac{\sqrt{(8 - 2\sqrt{3})}}{(\sqrt{3} + 1)} \times \frac{1}{\sqrt{11 - 6\sqrt{3}}}$$

$$\frac{a}{2} = \frac{\sqrt{(8 - 2\sqrt{3})}}{\sqrt{(4 - 2\sqrt{3})}} \times \frac{1}{\sqrt{11 - 6\sqrt{3}}}$$

$$\frac{a}{2} = \frac{\sqrt{(8 - 2\sqrt{3})}}{\sqrt{(8 - 2\sqrt{3})}} = 1$$

$$a = 2$$

Therefore, the length of the side BC is 2.

7. We have

$$(r_1 + r_2)(r_1 + r_2)(r_1 + r_2) = 4Rs^2$$

Thus, $\frac{(r_1 + r_2)(r_1 + r_2)(r_1 + r_2)}{Rs^2} = 4$

8. We have

$\cos(A - B) = \frac{4}{5}$

$$\frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)} = \frac{4}{5}$$

$$4\left(1+\tan^2\left(\frac{A+B}{2}\right)\right) = 5\left(1-\tan^2\left(\frac{A-B}{2}\right)\right)$$

$$9\tan^2\left(\frac{A-B}{2}\right) = 1$$

$$\tan^2\left(\frac{A-B}{2}\right) = \frac{1}{9}$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$$

$$\left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right) = \frac{1}{3}$$

$$\frac{1}{3}\cot\left(\frac{C}{2}\right) = \frac{1}{3}$$

$$\cot\left(\frac{C}{2}\right) = 1$$

$C = 90°$

Hence, the area of the triangle

$$= \frac{1}{2}\,ab\sin(90°)$$

$$= \frac{1}{2}\times 6\times 3 = 9$$

9. We have

$$\frac{10}{\frac{\sqrt{3}}{2}} = \frac{b}{\sin(50°)} = \frac{c}{\sin(70°)}$$

$$\frac{20}{\sqrt{3}} = \frac{b}{\sin(50°)} = \frac{c}{\sin(70°)}$$

Now, perimeter

$= 10 + b + c$

$$= 10 + \frac{20}{\sqrt{3}}\sin(50°) + \frac{20}{\sqrt{3}}\sin(70°)$$

$$= 10 + \frac{20}{\sqrt{3}}[\sin(50°) + \sin(70°)]$$

$$= 10 + \frac{20}{\sqrt{3}}[\cos(40°) + \cos(20°)]$$

$$= 10 + \frac{20}{\sqrt{3}}\times 2\cos(30°)\cos(10°)$$

$$= 10 + \frac{20}{\sqrt{3}}\times 2\times\frac{\sqrt{3}}{2}\times\cos(10°)$$

$= 10 + 20\cos(10°)$

Thus, $x = 10$, $y = 20$, $z = 10$

Hence, the value of $\left(\frac{x+y+z}{y}\right)$ is 2

10. We have

$$\frac{a\cot A + b\cot B + c\cot C}{(r+R)}$$

$$= \frac{\cos A + \cos B + \cos C}{(r+R)}$$

$$= \frac{2(r+R)}{(r+R)}$$

$= 2$

## Previous Years' JEE-Advanced Examinations

1. We have $\Delta = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3$

$$\Rightarrow \quad \frac{1}{p_1} = \frac{a}{2\Delta}, \frac{1}{p_2} = \frac{b}{2\Delta}, \frac{1}{p_3} = \frac{c}{2\Delta}$$

Now,

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{1}{2\Delta}(a+b-c)$$

$$= \frac{(a+b)^2 - c^2}{2\Delta(a+b+c)}$$

$$= \frac{a^2+b^2-c^2+2ab}{2\Delta(a+b+c)}$$

$$= \frac{2ab\cos C + 2ab}{2(a+b+c)\Delta}$$

$$= \frac{ab(\cos C + 1)}{(a+b+c)\Delta}$$

$$= \frac{2ab}{(a+b+c)\Delta}\times\cos^2\left(\frac{C}{2}\right)$$

2. Let $\angle APD = \theta$, as $\angle PAO = \angle PDO = \frac{\pi}{2}$

$\angle AOD = \pi - \theta$

By the law of cosines,

$OA^2 + OD^2 - AD^2 = 2(OA)(OD)\cos(\pi - \theta)$

$\Rightarrow \quad AD^2 = 2r^2 + 2r^2\cos\theta$

$$AD^2 = 2r^2(1+\cos\theta) = 4r^2\cos^2\left(\frac{\theta}{2}\right)$$

Since $ABCD$ is a cyclic quadrilateral, so

$\angle QRS = \pi - \theta$

Thus, $BC^2 = 4r^2\cos^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = 4r^2\sin^2\left(\frac{\theta}{2}\right)$

Therefore,

$$AD^2 + BC^2 = 4r^2\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right) = 4r^2$$

Similarly, we can easily prove that,

$AB^2 + CD^2 = 4r^2$

Hence, $AB^2 + CD^2 = AD^2 + BC^2$

3. By the law of cosines,

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow \quad \cos(60°) = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{a^2+b^2-c^2}{2ab}$$

$\Rightarrow \quad ab = a^2 + b^2 - c^2$

$\Rightarrow \quad c^2 = a^2 + b^2 - ab$

$\Rightarrow \quad c^2 = (1+\sqrt{3})^2 + 2^2 - 2(1+\sqrt{3})$

$\Rightarrow \quad c^2 = 1+3+2\sqrt{3}+4-2-2\sqrt{3} = 6$

$\Rightarrow \quad c = \sqrt{6}$

From sine law, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$$\Rightarrow \quad \sin A = \frac{a}{c}\cdot \sin C$$

$$\Rightarrow \quad \sin A = \frac{(\sqrt{3}+1)}{\sqrt{6}} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin(105°)$$

$\Rightarrow \quad A = 105°, B = 180 - (A+C) = 15°, c = \sqrt{6}$

4. We have

$$r = \frac{\Delta}{s} = \frac{ac}{2s} = \frac{ac}{a+b+c}$$

$$= \frac{ac(a+c-b)}{(a+c)^2 - b^2}$$

$$= \frac{ac(a+c-b)}{a^2+c^2+2ac-b^2}$$

$$= \frac{1}{2}(a+c-b) \; (\because a^2+c^2 = b^2)$$

$$\Rightarrow \quad r = \frac{1}{2}(a+c-b)$$

$\Rightarrow \quad 2r = (a+c-b) = AB + BC - AC$

Hence, the result.

5. We have

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \quad \ldots(i)$$

$$\Rightarrow \quad \cos C = \frac{a^2+b^2-c^2}{2ab} \quad \ldots(ii)$$

From, $\cos C = \frac{AC}{DC} = \frac{b}{a/2} = \frac{2b}{a}$ …(iii)

From (ii) and (iii), we get

$$\frac{a^2+b^2-c^2}{2ab} = \frac{2b}{a}$$

$\Rightarrow \quad a^2 - c^2 = 3b^2$

From (i), we get

$$\cos A = \frac{b^2-3b^2}{2bc} = -\frac{2b^2}{2bc} = -\frac{b}{c}$$

Thus, $\cos A \cdot \cos C = \left(\frac{2b}{a}\right)\left(-\frac{b}{c}\right) = \frac{2(c^2-a^2)}{3ac}$

6. Let $\angle A = \angle B - \alpha$ and $\angle C = \angle B + \alpha$

We have $\angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad \angle B = 60°$

From sine laws, $\frac{b}{\sin B} = \frac{c}{\sin C}$

$$\Rightarrow \quad \sin C = \frac{c}{b}\cdot \sin B = \left(\sqrt{\frac{2}{3}}.\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{2}}$$

$\Rightarrow \quad C = 45°$

Thus, $\angle A = 180° - (B + C)$

$= 180° - (60° + 45°) = 75°$

7. No questions asked in 1982.

8. We have $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

Given $r_1, r_2, r_3 \in$ HP

$$\Rightarrow \quad \frac{\Delta}{s-a}, \frac{\Delta}{s-b}, \frac{\Delta}{s-c} \in \text{HP}$$

$$\Rightarrow \quad \frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c} \in \text{HP}$$

$\Rightarrow \quad s-a, s-b, s-c \in$ AP

$\Rightarrow \quad -a, -b, -c \in$ AP

$\Rightarrow \quad a, b, c \in$ AP

Hence, the result.

9. We have

$$\cos A + \cos B + \cos C = \frac{3}{2}$$

$$\Rightarrow \quad 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos C = \frac{3}{2}$$

$$\Rightarrow \quad 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\left(\frac{C}{2}\right) = \frac{3}{2}$$

$$\Rightarrow \quad 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin^2\left(\frac{C}{2}\right) = \frac{3}{2} - 1 = \frac{1}{2}$$

$$\Rightarrow \quad 2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right)\right) = \frac{1}{2}$$

$$\Rightarrow \quad 2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right) = \frac{1}{2}$$

$$\Rightarrow \quad 2\sin\left(\frac{C}{2}\right) \times 2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right) = \frac{1}{2}$$

$$\Rightarrow \quad \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = \frac{1}{8}$$

It is possible only when,

$$\sin\left(\frac{A}{2}\right) = \frac{1}{2}, \sin\left(\frac{B}{2}\right) = \frac{1}{2}, \sin\left(\frac{C}{2}\right) = \frac{1}{2}$$

$$\Rightarrow \quad \left(\frac{A}{2}\right) = \frac{\pi}{6}, \left(\frac{B}{2}\right) = \frac{\pi}{6}, \left(\frac{C}{2}\right) = \frac{\pi}{6}$$

$$\Rightarrow \quad A = \frac{\pi}{3}, B = \frac{\pi}{3}, C = \frac{\pi}{3}$$

Thus, $\Delta$ is equilateral.

10. We have

$$\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}=\frac{2(a+b+c)}{36}$$

$$=\frac{(a+b+c)}{18}=\lambda$$

Thus, $a = 7\lambda$, $b = 6\lambda$ and $c = 5\lambda$

$$\Rightarrow \quad \cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{36\lambda^2+25\lambda^2-49\lambda^2}{2(6\lambda)(5\lambda)}=\frac{1}{5}$$

Similarly, $\cos B=\frac{19}{35}, \cos C=\frac{5}{7}$

$$\Rightarrow \quad \cos A{:}\cos B{:}\cos C=\frac{1}{5}{:}\frac{19}{35}{:}\frac{5}{7}$$

$\Rightarrow \quad \cos A : \cos B : \cos C = 7 : 19 : 25$

Thus, $\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$

11.

12. Given $\cot A, \cot B, \cot C \in$ AP

$$\Rightarrow \quad \frac{\cos A}{\sin A}, \frac{\cos B}{\sin B}, \frac{\cos C}{\sin C} \in AP$$

$$\Rightarrow \quad \frac{(b^2+c^2-a^2)}{2abc}, \frac{(c^2+a^2-b^2)}{2abc}, \frac{(a^2+b^2-c^2)}{2abc} \in \text{AP}$$

$$\Rightarrow \quad \frac{(b^2+c^2+a^2-2a^2)}{2abc}, \frac{(c^2+a^2+b^2-2b^2)}{2abc}, \frac{(a^2+b^2+c^2-2c^2)}{2abc} \in \text{AP}$$

$\Rightarrow \quad (b^2 + c^2 + a^2 - 2a^2), (c^2 + a^2 + b^2 - 2b^2), (a^2 + b^2 + c^2 - 2c^2) \in$ AP

$\Rightarrow \quad (-2a^2), (-2b^2), (-2c^2) \in$ AP

$\Rightarrow \quad a^2, b^2, c^2 \in$ AP

Hence, the result.

13. Let, $x = a^2 + 2a$, $y = 2a + 3$, $z = a^2 + 3a + 8$

Here, $x > 0$, $y > 0$ and $z > 0$

since $z = a^2 + 2a + 8 > 0$ for every $a$ in $R$

$$\Rightarrow \quad a > 0 \left(\because a < -2, a > 0 \text{ and } a > -\frac{3}{2}\right)$$

Also, $z - x = a + 8 > 0$, $z - y = a^2 + a + 5 > 0$

Thus, $x + y > z$

$\Rightarrow \quad a^2 + 3a + 8 < (a^2 + 2a) + (2a + 3)$

$\Rightarrow \quad a > 5$

$\Rightarrow \quad a \in (5, \infty)$

14. Let,

$$M=\frac{1}{3}\left(\cos\left(\alpha+\frac{\pi}{2}\right)+\cos\left(\beta+\frac{\pi}{2}\right)+\cos\left(\gamma+\frac{\pi}{2}\right)\right)$$

$$=-\frac{1}{3}(\sin\alpha+\sin\beta+\sin\gamma)$$

$M$ will be least, when $(\sin\alpha + \sin\beta + \sin\gamma)$ is providing us the greatest value.

Let $z = \sin\alpha + \sin\beta + \sin\gamma$

$= \sin\beta + \sin\alpha + \sin(2\pi - (\alpha + \sin\beta))$

$= \sin\alpha + \sin\beta - \sin(\alpha + \beta)$

$$=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$=2\sin\left(\frac{\alpha+\beta}{2}\right)\left(\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}\right)\right)$$

$$=2\sin\left(\frac{\alpha+\beta}{2}\right)\times 2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$$

$$=4\sin\left(\pi-\frac{\gamma}{2}\right)\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$$

$$=4\sin\left(\frac{\gamma}{2}\right)\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$$

$$=4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$$

It will be greatest, when,

$$\frac{\alpha}{2}=\frac{\beta}{2}=\frac{\gamma}{2}\Rightarrow \alpha=\beta=\gamma$$

Thus, $\alpha=\frac{2\pi}{3}=\beta=\gamma$

Therefore, the least value of $M$ is

$$=-\frac{1}{3}\left(3\sin\left(\frac{2\pi}{3}\right)\right)=-\frac{\sqrt{3}}{2}$$

15. We have $\cos A \cos B + \sin A \sin B \sin C = 1$

$$\Rightarrow \quad \frac{1-\cos A\cos B}{\sin A\sin B}=\sin C$$

$$\Rightarrow \quad \frac{1-\cos A\cos B}{\sin A\sin B}=\sin C\le 1$$

$\Rightarrow \quad 1 \cos A \cos B \le \sin A \sin B$

$\Rightarrow \quad 1 \le \sin A \sin B + \cos A \cos B$

$\Rightarrow \quad \cos(A - B) \ge 1$

$\Rightarrow \quad \cos(A - B) = 1$

$\Rightarrow \quad \cos(A - B) = \cos(0)$

$\Rightarrow \quad A - B = 0$

$\Rightarrow \quad A = B$

Therefore,

$$\sin C=\frac{1-\cos A\cos B}{\sin A\sin B}$$

$$=\frac{1-\cos A\cos A}{\sin A\sin A}=\frac{1-\cos^2 A}{\sin^2 A}=\frac{\sin^2 A}{\sin^2 A}=1$$

$\Rightarrow \quad C = 90°$

Hence, $A = 45° = B$, $C = 90°$

Now, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

$\Rightarrow \quad \dfrac{a}{\sin 45°} = \dfrac{b}{\sin 45°} = \dfrac{c}{\sin 90°}$

$\Rightarrow \quad \dfrac{a}{\frac{1}{\sqrt{2}}} = \dfrac{b}{\frac{1}{\sqrt{2}}} = \dfrac{c}{1}$

$\Rightarrow \quad \dfrac{a}{1} = \dfrac{b}{1} = \dfrac{c}{\sqrt{2}}$

$\Rightarrow \quad a:b:c = 1:1:\sqrt{2}$

Hence, the result.

16. Ans. (a, d)

From sine rule, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$b \sin A = a \sin B$

$b \sin A = a \sin B \le a$, since $\sin B \le 1$

In this case, $B = \dfrac{\pi}{2}, A < \dfrac{\pi}{2}$

Also $b \sin A < a$, if $0 < B < \pi, B \ne \dfrac{\pi}{2}$

If $A < \dfrac{\pi}{2}, B > A, B \ne \dfrac{\pi}{2}$,

then $b \sin A < a, A < \dfrac{\pi}{2}, b > a$

17. Ans. (a, d)

Let $\angle A = \alpha - \theta, \angle B = \theta, \angle C = \pi + \theta$

since $\angle A + \angle B + \angle C = \pi, \theta = 60°$

Thus, the largest angle of $\Delta ABC$ is $A$ and the smallest angle is $C$.

Let $x$ be the smallest side of the triangle.

Therefore, $\cos(60°) = \dfrac{x^2 + 10^2 - 9^2}{2 \cdot x \cdot 10}$

$\Rightarrow \quad x^2 + 19 = 10x$

$\Rightarrow \quad x^2 - 10x + 19 = 0$

$\Rightarrow \quad (x - 5)^2 = 6$

$\Rightarrow \quad (x - 5) = \pm\sqrt{6}$

$\Rightarrow \quad x = 5 \pm \sqrt{6}$

18. Let, $\angle B = 30°, \angle C = 45°$ and $a = (\sqrt{3} + 1)$

Then, $\angle A = (\pi - (\angle B + \angle C))$

$= (\pi - (30° + 45°)) = 105°$

From sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$\Rightarrow \quad \dfrac{a}{\sin(105°)} = \dfrac{b}{\sin(30°)} = \dfrac{c}{\sin(45°)}$

$\Rightarrow \quad \dfrac{(\sqrt{3}+1)}{\sin(105°)} = \dfrac{b}{\sin(30°)} = \dfrac{c}{\sin(45°)}$

$\Rightarrow \quad \dfrac{(\sqrt{3}+1)}{\left(\dfrac{\sqrt{3}+1}{2\sqrt{2}}\right)} = \dfrac{b}{\frac{1}{2}} = \dfrac{c}{\frac{1}{\sqrt{2}}}$

$\Rightarrow \quad 2\sqrt{2} = 2b = \sqrt{2}c$

$\Rightarrow \quad b = \sqrt{2}, c = 2$

Thus, the area of the given triangle

$= \dfrac{1}{2} \times bc \times \sin A$

$= \dfrac{1}{2} \times \sqrt{2} \times 2 \times \dfrac{(\sqrt{3}+1)}{2\sqrt{2}}$

$= \dfrac{(\sqrt{3}+1)}{2}$ sq cm

19. We have $BC = 2BD, AD = h, OD = h - r$, so that $BC = 2\sqrt{r^2 - (h-r)^2} = 2\sqrt{2rh - h^2}$

Thus, $P = 2AB + BC$

$\Rightarrow \quad 2AB = P - BC$

$\Rightarrow \quad 2AB = 2(\sqrt{2hr - h^2} + \sqrt{2hr}) - 2\sqrt{2hr - h^2}$

$\Rightarrow \quad AB = \sqrt{2hr}$

The area of the $\Delta ABC = A$

$= BD \times AD$

$= h\sqrt{2hr - h^2}$

Now, $\dfrac{A}{P^3} = \dfrac{h\sqrt{2hr - h^2}}{8(\sqrt{2hr - h^2} + \sqrt{2hr})^3}$

$= \dfrac{\sqrt{2r - h}}{8(\sqrt{2r - h} + \sqrt{2r})^3}$

Thus, $\lim\limits_{h \to 0}\left(\dfrac{A}{P^3}\right) = \dfrac{\sqrt{2r}}{8 \times (2\sqrt{2r})^3} = \dfrac{1}{128r}$

20. Ans. (c)

The given equation is

$k = 3 \sin x - 4 \sin^3 x = \sin 3x$

Thus, $k = \sin 3A, k = \sin 3B$

$\Rightarrow \quad \sin 3A = \sin 3B = \sin(\pi - 3B)$

$\Rightarrow \quad 3A = (\pi - 3B)$

$\Rightarrow \quad 3(A + B) = \pi$

$\Rightarrow \quad (A + B) = \dfrac{\pi}{3}$

Therefore, $\angle C = \pi - (A + B) = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

21. Given $A$, $B$ and $C$ are in AP

$\Rightarrow \quad 2B = A + C$

$\Rightarrow \quad 3B = A + B + C = \pi$

$\Rightarrow \quad B = \dfrac{\pi}{3}$

Now, $\sin(2A + B) = \dfrac{1}{2} = \sin\left(\dfrac{5\pi}{6}\right)$

$\Rightarrow \quad (2A + B) = \left(\dfrac{5\pi}{6}\right)$

$\Rightarrow \quad 2A = \left(\dfrac{5\pi}{6} - B\right) = \dfrac{5\pi}{6} - \dfrac{\pi}{3} = \dfrac{3\pi}{6} = \dfrac{\pi}{2}$

$\Rightarrow \quad A = \dfrac{\pi}{4}$

Also, $\sin(C - A) = \dfrac{1}{2} = \sin\left(\dfrac{\pi}{6}\right)$

$\Rightarrow \quad (C - A) = \left(\dfrac{\pi}{6}\right)$

$\Rightarrow \quad C = \left(\dfrac{\pi}{6} + A\right) = \left(\dfrac{\pi}{6} + \dfrac{\pi}{4}\right) = \dfrac{5\pi}{12} = 75°$

Thus, $A = 45°$, $B = 60°$, $C = 75°$

22. Let the sides of a triangle are $a - 1$, $a$, $a + 1$, where $a \in I^+ - \{1\}$
Let $\theta$ is the smallest angle and $2\theta$ is the greatest angle of the triangle.
By the sine rule,

$$\frac{\sin\theta}{a-1} = \frac{\sin(2\theta)}{a+1}$$

$\Rightarrow \quad \dfrac{a+1}{a-1} = \dfrac{\sin(2\theta)}{\sin\theta} = 2\cos\theta$

$\Rightarrow \quad \cos\theta = \dfrac{a+1}{2(a-1)}$

Again by the cosine rule,

$$\cos\theta = \frac{(a+1)^2 + a^2 - (a-1)^2}{2 \cdot a \cdot (a+1)}$$

$\Rightarrow \quad \cos\theta = \dfrac{a^2 + 4a}{2a(1+a)} = \dfrac{a+4}{2(a+1)}$

Therefore, $\dfrac{a+1}{2(a-1)} = \dfrac{a+4}{2(a+1)}$

$\Rightarrow \quad (a + 1)^2 = (a + 4)(a - 1)$

$\Rightarrow \quad a^2 + 2a + 1 = a^2 + 3a - 4$

$\Rightarrow \quad a = 5$

Hence, the sides of the triangle are 4, 5, 6.

23. Let $a = BC$, $b = CA$, $c = AB$ and $p = AD$.

$$\Delta ABC = \frac{1}{2}ap = \frac{1}{2}bc\sin A$$

$\Rightarrow \quad p = \dfrac{bc}{a}\sin A$

$\Rightarrow \quad p = \dfrac{abc}{a^2}\sin A$

$\Rightarrow \quad p = \dfrac{abc(\sin^2 B - \sin^2 C)}{a^2(\sin^2 B - \sin^2 C)}\sin A$

$\Rightarrow \quad p = \dfrac{abc\sin(B+C)\sin(B-C)}{(b^2 - c^2)\sin^2 A}\sin A$

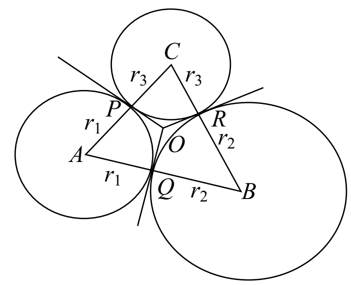
$\Rightarrow \quad p = \dfrac{abc\sin(B-C)}{(b^2 - c^2)}$

$\Rightarrow \quad p = \dfrac{ab^2 r\sin(B-C)}{(b^2 - b^2 r^2)}$, where $c = b$

$\Rightarrow \quad p = \left(\dfrac{ar}{1 - r^2}\right)\sin(B - C)$

$\Rightarrow \quad p \le \left(\dfrac{ar}{1 - r^2}\right)$, since $\sin(B - C) \le 1$

24.




Consider three circles with centres at $A$, $B$ and $C$ with radii $r_1$, $r_2$, $r_3$, respectively, which touch each other externally at $P$, $Q$, $R$.
Let the common tangents at $P$, $Q$, $R$ meet each other at $O$.
Then $OP = OQ = QR = 4$
Also, $OP \perp AB$, $OQ \perp AC$, $OR \perp BC$
Here, $O$ is the incentre of the triangle $ABC$
For $\Delta ABC$,

$$s = \frac{(r_1 + r_2) + (r_3 + r_2) + (r_1 + r_3)}{2} = r_1 + r_2 + r_3$$

and $\Delta = \sqrt{(r_1 + r_2 + r_3)r_1 r_2 r_3}$

Now, from the relation $r = \dfrac{\Delta}{s}$, we get,

$$\frac{\sqrt{(r_1 + r_2 + r_3)r_1 r_2 r_3}}{r_1 + r_2 + r_3} = 4$$

$\Rightarrow \quad \sqrt{\dfrac{r_1 r_2 r_3}{r_1 + r_2 + r_3}} = 4$

$\Rightarrow \quad \dfrac{r_1 r_2 r_3}{r_1 + r_2 + r_3} = 16 = \dfrac{16}{1}$

$\Rightarrow \quad (r_1 r_2 r_3) : (r_1 + r_2 + r_3) = 16 : 1$

25. We have

$\Rightarrow \quad \dfrac{2\cos A}{a} + \dfrac{2\cos B}{b} + \dfrac{2\cos C}{c} = \dfrac{1}{bc} + \dfrac{b}{ca}$

$$\frac{2bc\cos A}{abc} + \frac{ac\cos B}{abc} + \frac{2bc\cos C}{abc} = \frac{a^2}{abc} + \frac{b^2}{abc}$$

$\Rightarrow \quad 2bc\cos A + ac\cos B + 2bc\cos C = a^2 + b^2$

$\Rightarrow \quad (b^2 + c^2 - a^2) + \dfrac{1}{2}(a^2 + c^2 - b^2)$

$\quad + (a^2 + b^2 - c^2) = a^2 + b^2$

$\Rightarrow \quad 2b^2 - 2a^2 + c^2 + a^2 - b^2 = 0$

$\Rightarrow \quad a^2 = b^2 + c^2$

$\Delta ABC$ is a right angled triangle at $A$.
Thus, $\angle A = 90°$

26. Let $O$ be the centre $r$ be the radius of the circle passing through $A_i$, where $i = 1, 2, 3, \ldots, n$

Here, $\angle A_iOA_{i+1} = \dfrac{2\pi}{n}$, where $i = 1, 2, 3, \ldots, n$

Now, $\dfrac{1}{A_1A_2} + \dfrac{1}{A_1A_3} = \dfrac{1}{A_1A_2}$

$\Rightarrow \quad \dfrac{1}{2r\sin\left(\dfrac{\pi}{n}\right)} = \dfrac{1}{2r\sin\left(\dfrac{2\pi}{n}\right)} + \dfrac{1}{2r\sin\left(\dfrac{3\pi}{n}\right)}$

$\Rightarrow \quad \dfrac{1}{\sin\left(\dfrac{\pi}{n}\right)} = \dfrac{1}{\sin\left(\dfrac{2\pi}{n}\right)} + \dfrac{1}{\sin\left(\dfrac{3\pi}{n}\right)}$

$\Rightarrow \quad \dfrac{1}{\sin\theta} = \dfrac{1}{\sin 2\theta} + \dfrac{1}{\sin 3\theta}, \left(\dfrac{\pi}{n}\right) = \theta$

$\Rightarrow \quad \dfrac{1}{\sin\theta} - \dfrac{1}{\sin 3\theta} = \dfrac{1}{\sin 2\theta}$

$\Rightarrow \quad \dfrac{\sin 3\theta - \sin\theta}{\sin\theta \sin 3\theta} = \dfrac{1}{\sin 2\theta}$

$\Rightarrow \quad \dfrac{2\cos 2\theta \sin\theta}{\sin\theta \sin 3\theta} = \dfrac{1}{\sin 2\theta}$

$\Rightarrow \quad 2\sin 2\theta \cos 2\theta = \sin 3\theta$

$\Rightarrow \quad \sin(4\theta) = \sin(3\theta)$

$\Rightarrow \quad \sin(4\theta) = \sin(\pi - 3\theta)$

$\Rightarrow \quad (4\theta) = (\pi - 3\theta)$

$\Rightarrow \quad \theta = \dfrac{\pi}{7}$

$\Rightarrow \quad \dfrac{\pi}{n} = \dfrac{\pi}{7}$

$\Rightarrow \quad n = 7$

27. (i) $\Rightarrow$ (ii)

Suppose $a, b, c$ and area $\Delta$ are rational.

Thus, $s = \dfrac{a+b+c}{2}$ = rational

Since $\tan\left(\dfrac{B}{2}\right) = \dfrac{\Delta}{s(s-b)}$

and $\tan\left(\dfrac{C}{2}\right) = \dfrac{\Delta}{s(s-c)}$

and $\Delta, a, s, c-a, s-b$ all are rational.

Therefore $a, \tan\left(\dfrac{B}{2}\right), \tan\left(\dfrac{C}{2}\right)$ are rational.

(ii) $\Rightarrow$ (iii)

Consider $a, \tan\left(\dfrac{B}{2}\right), \tan\left(\dfrac{C}{2}\right)$ are rational.

and $\sin B = \dfrac{2\tan(B/2)}{1+\tan^2(B/2)}$ = rational

$\sin C = \dfrac{2\tan(C/2)}{1+\tan^2(C/2)}$ = rational

Also, $\tan\left(\dfrac{A}{2}\right) = \tan\left(\dfrac{\pi}{2} - \left(\dfrac{B+C}{2}\right)\right)$

$= \cot\left(\dfrac{B+C}{2}\right)$

$= \dfrac{1-\tan\left(\dfrac{B}{2}\right)\tan\left(\dfrac{C}{2}\right)}{\tan\left(\dfrac{B}{2}\right)+\tan\left(\dfrac{C}{2}\right)}$ = rational

Thus, $\sin A$ = rational

Hence, $a, \sin A, \sin B, \sin C$ are rational.

(iii) $\Rightarrow$ (i)

Suppose $a, \sin A, \sin B, \sin C$ are rational.

By the sine rules,

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$\Rightarrow \quad b = a \cdot \dfrac{\sin B}{\sin A}, c = a \cdot \dfrac{\sin C}{\sin A}$

$b, c$ = rational

Also, $\Delta = \dfrac{1}{2}bc\sin A$ = rational

This completes the proof.

28. Figure

From the figure, $AD = b\sin(23°)$

$\Rightarrow \quad \dfrac{abc}{b^2-c^2} = b\sin(23°)$

$\Rightarrow \quad \dfrac{a}{b^2-c^2} = \dfrac{\sin(23°)}{c}$

$\Rightarrow \quad \dfrac{a}{b^2-c^2} = \dfrac{\sin A}{a}$

$\Rightarrow \quad \sin A = \dfrac{a^2}{b^2-c^2}$

$\Rightarrow \quad \sin A = \dfrac{\sin^2 A}{\sin^2 B - \sin^2 C}$

$\Rightarrow \quad \sin A = \sin^2 B - \sin^2 C$

$\Rightarrow \quad \sin A = \sin(B+C)\sin(B-C)$

$\Rightarrow \quad \sin A = \sin(\pi - A)\sin(B-C)$

$\Rightarrow \quad \sin A = \sin(A)\sin(B-C)$

$\Rightarrow \quad \sin(B-C) = 1 = \sin(90°)$

$\Rightarrow \quad (B-C) = (90°)$

$\Rightarrow \quad (B-23°) = (90°)$

$\Rightarrow \quad B = 113°$

29. As we know that, the largest angle is the opposite to the largest side.

Let $a = BC = 3$, $b = CA = 5$, $c = AB = 7$

Thus, $\cos C = \dfrac{3^2+5^2-7^2}{2\cdot 3\cdot 5} = -\dfrac{15}{30} = -\dfrac{1}{2}$

$\Rightarrow \quad \cos C = -\dfrac{1}{2} = \cos\left(\dfrac{2\pi}{3}\right)$

$\Rightarrow \quad C = \dfrac{2\pi}{3}$

30. Let $\angle B = \dfrac{\pi}{3}, \angle C = \dfrac{\pi}{4}, \angle BAD = \theta, \angle CAD = \varphi$

By the sine rule,

$$\frac{\sin\theta}{BD} = \frac{\sin(\pi/3)}{AD}$$

$\Rightarrow \quad \sin\theta = \dfrac{\sqrt{3}}{2}\cdot\dfrac{BD}{AD}$

and $\dfrac{\sin\varphi}{DC} = \dfrac{\sin(\pi/4)}{AD}$

$\Rightarrow \quad \sin\varphi = \dfrac{1}{\sqrt{2}}\cdot\dfrac{DC}{AD}$

Now, $\dfrac{\sin\angle BAD}{\sin\angle CAD} = \dfrac{\sin\theta}{\sin\varphi}$

$$= \frac{\sqrt{3}}{2}\cdot\frac{BD}{AD}\cdot\sqrt{2}\cdot\frac{AD}{DC}$$

$$= \frac{BD}{DC}\cdot\sqrt{\frac{3}{2}}$$

$$= \frac{1}{3}\times\sqrt{\frac{3}{2}} = \frac{1}{\sqrt{6}}$$

31. Let $a = 4k$, $b = 5k$, $c = 6k$

$$s = \frac{1}{2}(a+b+c) = \frac{15}{2}k$$

$$s - a = \frac{15}{2}k - 4k = \frac{7}{2}k$$

$$s - b = \frac{15}{2}k - 5k = \frac{5}{2}k$$

$$s - c = \frac{15}{2}k - 6k = \frac{3}{2}k$$

Now, $\dfrac{R}{r} = \dfrac{abc}{4\Delta}\times\dfrac{s}{\Delta}$

$$= \frac{abcs}{4\Delta^2}$$

$$= \frac{abcs}{4s(s-a)(s-b)(s-c)}$$

$$= \frac{abc}{4(s-a)(s-b)(s-c)}$$

$$= \frac{(4k)(5k)(6k)}{4\left(\frac{7}{2}k\right)\left(\frac{5}{2}k\right)\left(\frac{3}{2}k\right)}$$

$$= \frac{16}{7}$$

32. Here, $B + C = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

$\Rightarrow \quad \tan(B+C) = \tan\left(\dfrac{3\pi}{4}\right) = -1$

$\Rightarrow \quad \dfrac{\tan B + \tan C}{1 - \tan B\tan C} = -1$

$\Rightarrow \quad \tan B + \tan C = -1 + \tan B\tan C = -1 + p$

Let $\tan B$ and $\tan C$ are the roots, then

$x^2 - (\tan B + \tan C)x - \tan B\cdot\tan C = 0$

$\Rightarrow \quad x^2 - (p-1)x + p = 0$

It has real roots.

So, $D \geq 0$

$\Rightarrow \quad (p-1)^2 - 4p \geq 0$

$\Rightarrow \quad p^2 - 2p + 1 - 4p \geq 0$

$\Rightarrow \quad p^2 - 6p + 1 \geq 0$

$\Rightarrow \quad (p-3)^2 - 8 \geq 0$

$\Rightarrow \quad (p-3)^2 - (2\sqrt{2})^2 \geq 0$

$\Rightarrow \quad (p - 3 + 2\sqrt{2})(p - 3 - 2\sqrt{2}) \geq 0$

$\Rightarrow \quad p \leq (3 - 2\sqrt{2}),\ p \geq (3 + 2\sqrt{2})$

$\Rightarrow \quad p \in (-\infty, (3 - 2\sqrt{2})) \cup ((3 + 2\sqrt{2}), \infty)$

33. Let $ABC$ be an equilateral triangle

Then $\angle A = \dfrac{\pi}{3} = \angle B = \angle C$

Therefore $\tan A + \tan B + \tan C$

$$= \tan\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)$$

$$= 3\tan\left(\frac{\pi}{3}\right)$$

$$= 3\sqrt{3}$$

Conversely, let $\tan A + \tan B + \tan C = 3\sqrt{3}$

Here, $A + B + C = \pi$

$\Rightarrow \quad A + B = \pi - C$

$\Rightarrow \quad \tan(A+B) = \tan(\pi - C) = -\tan C$

$\Rightarrow \quad \dfrac{\tan A + \tan B}{1 - \tan A\cdot\tan B} = -\tan C$

$\Rightarrow \quad \tan A + \tan B = \tan C + \tan A\cdot\tan B\cdot\tan C$

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A\cdot\tan B\cdot\tan C$

Thus,

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A\cdot\tan B\cdot\tan C = 3\sqrt{3}$

It is possible only when, $A = B = C = \dfrac{\pi}{3}$

Thus, the triangle is equilateral.

34. Ans. (b)

Here, $\Delta = \dfrac{1}{2}\times p\times p_1 = \dfrac{1}{2}\times q\times p_2 = \dfrac{1}{2}\times r\times p_3$

$\Rightarrow \quad p = \dfrac{2\Delta}{p_1}, q = \dfrac{2\Delta}{p_2}, r = \dfrac{2\Delta}{p_3}$

From sine rule of a triangle,

$$\frac{\sin P}{p} = \frac{\sin Q}{q} = \frac{\sin R}{r}$$

Given, $\sin P, \sin Q, \sin R$ are in AP

$\Rightarrow \quad p, q, r \in$ AP

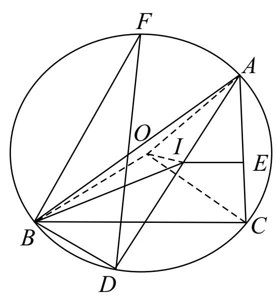
$$\Rightarrow \quad \frac{2\Delta}{p_1}, \frac{2\Delta}{p_2}, \frac{2\Delta}{p_3} \in \text{AP}$$

$$\Rightarrow \quad \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3} \in \text{AP}$$

$\Rightarrow \quad p_1, p_2, p_3 \in$ HP

Thus, the altitudes are in HP

35.




Let $O$ be the circumcentre and $OF$ be perpendicular to $AB$.

Let $I$ be the incentre and $IE$ perpendicular to $AC$.

Then $\angle OAF = 90° - C$

$$\angle OAI = \angle IAF - \angle OAF$$

$$= \frac{A}{2} - (90° - C)$$

$$= \frac{A}{2} + C - \frac{A+B+C}{2}$$

$$= \frac{C-B}{2}$$

Also, $AI = \dfrac{IE}{\sin\left(\frac{A}{2}\right)} = \dfrac{r}{\sin\left(\frac{A}{2}\right)}$

$$= 4R \sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$= 1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{B}{2}+\frac{C}{2}\right)\right)$$

$$= 1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad OI = R\sqrt{1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)}$$

Also, $OI^2$

$$= R^2 - 2R \times 4R \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$= R^2 - 2Rr$$

$$\Rightarrow \quad OI = \sqrt{R^2 - 2Rr}$$

Hence, the result.

36. We have $A + B + C = \pi$

$$\Rightarrow \quad \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \pi$$

$$\Rightarrow \quad \frac{A}{2} + \frac{B}{2} = \pi - \frac{C}{2}$$

$$\Rightarrow \quad \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\pi - \frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right)}{1 - \tan\left(\frac{A}{2}\right)\cdot\tan\left(\frac{B}{2}\right)} = \cot\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right)}{1 - \tan\left(\frac{A}{2}\right)\cdot\tan\left(\frac{B}{2}\right)} = \frac{1}{\tan\left(\frac{C}{2}\right)}$$

$$\Rightarrow \quad \tan\left(\frac{A}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)$$

$$= 1 - \tan\left(\frac{A}{2}\right)\cdot\tan\left(\frac{B}{2}\right)$$

$$\Rightarrow \quad \tan\left(\frac{A}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)$$

$$+ \tan\left(\frac{A}{2}\right)\cdot\tan\left(\frac{B}{2}\right) = 1$$

Dividing both sides by $\tan\left(\frac{A}{2}\right)\cdot\tan\left(\frac{B}{2}\right)\cdot\tan\left(\frac{C}{2}\right)$, we get,

$$\Rightarrow \quad \frac{1}{\tan\left(\frac{C}{2}\right)} + \frac{1}{\tan\left(\frac{C}{2}\right)} + \frac{1}{\tan\left(\frac{C}{2}\right)}$$

$$= \frac{1}{\tan\left(\frac{C}{2}\right)} \cdot \frac{1}{\tan\left(\frac{C}{2}\right)} \cdot \frac{1}{\tan\left(\frac{C}{2}\right)}$$

$$\Rightarrow \quad \cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)$$

$$= \cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$$

37. Figure

Here, $HE = JK = r_1$

But $IE = r$

So, $IH = r - r_1$

In a right $\Delta IHJ$, $\angle JIH = \left(\dfrac{\pi}{2} - \dfrac{A}{2}\right)$

$$\Rightarrow \quad \tan\left(\frac{\pi}{2} - \frac{A}{2}\right) = \frac{r_1}{r - r_1}$$

$$\Rightarrow \quad \cot\left(\frac{A}{2}\right) = \frac{r_1}{r - r_1}$$

Similarly, $\cot\left(\dfrac{B}{2}\right) = \dfrac{r_2}{r - r_2}, \cot\left(\dfrac{C}{2}\right) = \dfrac{r_3}{r - r_3}$.

In a $\Delta ABC$, we have

$$\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)$$
$$=\cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1}{r-r_1}\cdot\frac{r_2}{r-r_2}\cdot\frac{r_3}{r-r_3}$$

$$\Rightarrow \quad \frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}$$

**[IIT-JEE, 2000]**

38. We have $2ac\sin\left(\frac{A-B+C}{2}\right)$

$$=2ac\sin\left(\frac{A+C}{2}-\frac{B}{2}\right)$$
$$=2\,ac\sin\left(\frac{\pi}{2}-\frac{B}{2}-\frac{B}{2}\right)$$
$$=2ac\sin\left(\frac{\pi}{2}-B\right)$$
$$=2ac\cos B$$
$$=2ac\left(\frac{a^2+c^2-b^2}{2ac}\right)$$
$$=(a^2+c^2-b^2)$$

39. Ans. (a)

As $ABC$ be a right angled triangle, circum radius of this triangle is half of its hypotenuse.

Thus, $R=\frac{1}{2}\sqrt{a^2+b^2}$

Also, $r=(s-c)\tan\left(\frac{C}{2}\right)=(s-c)\tan\left(\frac{\pi}{4}\right)=(s-c)$

Now, $2(R+r)$

$$=\sqrt{a^2+b^2}+2s-2c$$
$$=c+2s-2c$$
$$=2s-c$$
$$=a+b+c-c$$
$$=a+b$$

40.

41. By the sine rule,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin(\pi-(A+B))}$$
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin(A+B)}$$

Here, we can easily find out $b$, $c$ and $C$, if we know $a$, $\sin A$, $\sin B$.

Also, we can find the values of $A$, $B$ and $C$ by using the half angle formulae, if we know the values of $a$, $b$ and $c$.

By using, $\frac{a}{\sin A}=\frac{\text{b}}{\sin B}=\frac{c}{\sin C}=2R$

We can find, $\angle B$, $\angle C$ and the sides $a$, $b$ and $c$, if we know $a$, $\sin B$ and $R$.

We cannot find $\angle B$, $\angle C$ and the sides $b$ and $c$, if we just know $a$, $\sin A$, $R$, since

$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$ gives

$\frac{b}{\sin B}$ and $\frac{c}{\sin C}$ from which we cannot obtain $b$, $c$, $\angle B$ and $\angle C$.

42. Let the angles of the $\Delta ABC$ are $4\theta$, $\theta$ and $\theta$

Also $4\theta+\theta+\theta=180°$

$\Rightarrow \quad 6\theta=180°$

$\Rightarrow \quad \theta=\frac{180°}{6}=30°$

By sine rules,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{a}{\sin(120°)}=\frac{b}{\sin(30°)}=\frac{c}{\sin(30°)}$$

$$\Rightarrow \quad \frac{a}{\sin(60°)}=\frac{b}{\sin(30°)}=\frac{c}{\sin(30°)}$$

$$\Rightarrow \quad \frac{a}{\frac{\sqrt{3}}{2}}=\frac{b}{\frac{1}{2}}=\frac{c}{\frac{1}{2}}$$

$$\Rightarrow \quad \frac{a}{\sqrt{3}}=\frac{b}{1}=\frac{c}{1}$$

Hence, the required ratio is

$$=\frac{a}{a+b+c}=\frac{\sqrt{3}}{2+\sqrt{3}}$$

43. From figure (i)

$$I_n=n\cdot\frac{1}{2}\cdot(OA_1)\cdot(OA_1)\sin\left(\frac{2\pi}{n}\right)$$
$$=\frac{\pi}{2}\sin\left(\frac{2\pi}{n}\right)$$

From figure (ii)

$$B_1B_2=2(B_1L)=2(OL)\tan\left(\frac{\pi}{n}\right)$$
$$=2\cdot1\cdot\tan\left(\frac{\pi}{n}\right)$$
$$=2\tan\left(\frac{\pi}{n}\right)$$

Thus, $O_n = n\left(\frac{1}{2}(B_1B_2)(OL)\right) = n \tan\left(\frac{\pi}{n}\right)$

Now, $\frac{I_n}{O_n}$

$$= \frac{(n/2)\sin(2\theta)}{n\tan\theta} \text{ where, } \theta = \frac{\pi}{n}$$

$$= \frac{2\tan\theta}{(1+\tan^2\theta)} \cdot \frac{1}{2\tan\theta}$$

$$= \cos^2\theta$$

$$= \frac{1}{2}(2\cos^2\theta)$$

$$= \frac{1}{2}(1+\cos 2\theta)$$

$$= \frac{1}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$$

$$I_n = \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$$

44. Ans. (d)

Let the sides of the triangle be $\lambda, \sqrt{3}\lambda, 2\lambda$

By the cosine rule,

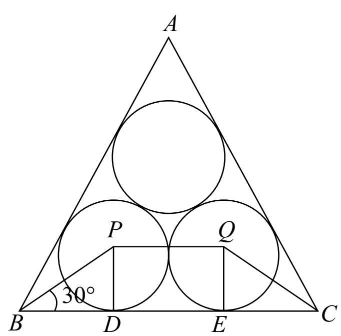
$$\cos A = \frac{(\sqrt{3}\lambda)^2 + (2\lambda)^2 - (\lambda)^2}{2\cdot(\sqrt{3}\lambda)\cdot(2\lambda)} = \frac{6\lambda^2}{4\sqrt{3}\lambda^2} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad A = \frac{\pi}{6}$$

Similarly, $B = \frac{\pi}{3}, C = \frac{\pi}{2}$

$A : B : C = 30° : 60° : 90° = 1 : 2 : 3$

45.




We have $\frac{BD}{PD} = \cot(30°) = \sqrt{3}$

$$BD = \sqrt{3}\,PD$$

$$DE = PQ = PR + RQ = 2$$

$$BC = BD + DE + EC = \sqrt{3} + 2 + \sqrt{3} = 2(\sqrt{3}+1)$$

Area of $\Delta ABC$

$$= \frac{\sqrt{3}}{4} \times (BC)^2$$

$$= \frac{\sqrt{3}}{4} \times 4(\sqrt{3}+1)^2$$

$$= \sqrt{3}(\sqrt{3}+1)^2$$

$$= \sqrt{3}(3+1+2\sqrt{3})$$

$$= \sqrt{3}(4+2\sqrt{3})$$

$$= (6+4\sqrt{3})$$

46. We have $\frac{b-c}{a}$

$$= \frac{\sin B - \sin C}{\sin A}$$

$$= \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

Thus, $\frac{b-c}{a} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$

$$(b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B-C}{2}\right)$$

47. We have from sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{a}{\sin(120°)} = \frac{b}{\sin(30°)} = \frac{c}{\sin(30°)}$$

$$\Rightarrow \quad \frac{a}{\frac{\sqrt{3}}{2}} = \frac{b}{\frac{1}{2}} = \frac{c}{\frac{1}{2}}$$

$$\Rightarrow \quad \frac{a}{\sqrt{3}} = \frac{b}{1} = \frac{c}{1} = \lambda \text{ (say)}$$

Now, $r = \frac{\Delta}{s} = \frac{bc\sin A}{2s} = \frac{bc\sin A}{(a+b+c)}$

$$\Rightarrow \quad \sqrt{3} = \frac{\left(\frac{\sqrt{3}}{2}\right)\lambda^2}{(\sqrt{3}+1+1)\lambda}$$

$$\Rightarrow \quad \sqrt{3}(2+\sqrt{3}) = \left(\frac{\sqrt{3}}{2}\right)\lambda$$

$$\Rightarrow \quad \lambda = 2(2+\sqrt{3})$$

Thus, the area of the $\Delta ABC$

$$= \frac{1}{2}\cdot bc \cdot \sin A$$

$$= \frac{1}{2}\cdot \lambda \cdot \lambda \cdot \sin(120°)$$

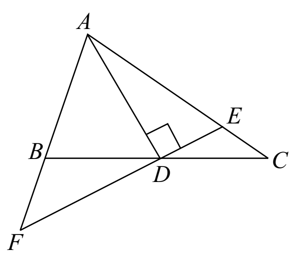
$$= \frac{\sqrt{3}}{4}\times \lambda^2$$

$$= \frac{\sqrt{3}}{4}\times 4(2+\sqrt{3})^2$$

$$= \sqrt{3}(7+4\sqrt{3})$$

$$= (12+7\sqrt{3})$$

48. Ans. (a, b, c)




Let $AD = p$

$ar(\Delta ABC) = ar(\Delta ABD) + (\Delta ADC)$

$$\frac{1}{2}bc\sin A = \frac{1}{2}bp\sin\left(\frac{A}{2}\right) + \frac{1}{2}cp\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad bc\sin A = p(b+c)\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad bc\,2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) = p(b+c)\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad p = \frac{2bc}{(b+c)}\cos\left(\frac{A}{2}\right)$$

$$AD = \frac{2bc}{(b+c)}\cos\left(\frac{A}{2}\right)$$

Also, $\dfrac{AD}{AE} = \cos\left(\dfrac{A}{2}\right)$

$$AE = AD\sec\left(\frac{A}{2}\right) = \frac{2bc}{b+c} = \frac{2}{\frac{1}{b}+\frac{1}{c}}$$

Thus, $AE$ is the HM of $b$ and $c$

Again, $\dfrac{DE}{AD} = \sin\left(\dfrac{A}{2}\right), \dfrac{FD}{AD} = \sin\left(\dfrac{A}{2}\right)$

$$EF = DE + FD = 2AD\sin\left(\frac{A}{2}\right)$$

$$= \frac{4bc}{b+c}\cos\left(\frac{A}{2}\right)\sin\left(\frac{A}{2}\right)$$

$$= \frac{2bc}{b+c}\times \sin A$$

49. Ans. (b)

Given, $AB \parallel CD$, $CD = 2AB$.

Let $AB = a$, $CD = 2a$ and the radius of the circle be $r$.

Let the circle touches at $P$, $BC$ at $Q$, $AD$ at $R$ and $CD$ at $S$.

Then $AR = AP = r$, $BP = BQ = a - r$,

$DR = DS = r$ and $CQ = CS = 2a - r$

In $\Delta BEC$, $BC^2 = BE^2 + EC^2$

$\Rightarrow \quad (a - r + 3a - r)^2 = (2r)^2 + a^2$

$\Rightarrow \quad (3a = 2r)^2 = (2r)^2 + a^2$

$\Rightarrow \quad 9a^2 + 4r^2 - 12ar = 4r^2 + a^2$

$$\Rightarrow \quad a = \frac{3}{2}r$$

Also, $ar(\text{Quad. } ABCD) = 18$

$\Rightarrow \quad ar(\text{Quad. } ABED) + ar(\Delta BCE) = 18$

$$\Rightarrow \quad a\cdot 2r + \frac{1}{2}\cdot a\cdot 2r = 18$$

$\Rightarrow \quad 3ar = 18$

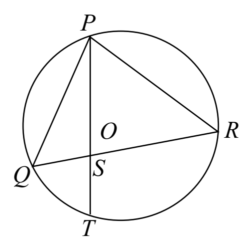
$$\Rightarrow \quad 3\times\frac{3}{2}\times r^2 = 18$$

$\Rightarrow \quad r^2 = 4$

$\Rightarrow \quad r = 2$

Thus, the radius is $r = 2$.

50. Ans. (b, d)




Here, $PS \times ST = QS \times SR$

Now, AM > GM

$$\Rightarrow \quad \frac{\frac{1}{PS}+\frac{1}{ST}}{2} > \sqrt{\frac{1}{PS}\cdot\frac{1}{ST}}$$

$$\Rightarrow \quad \frac{1}{PS}+\frac{1}{ST} > \frac{2}{\sqrt{QS\times SR}}$$

Also, $\dfrac{QS+SR}{2} > \sqrt{QS\times SR}$

$$\Rightarrow \quad \frac{1}{\sqrt{QS\times SR}} > \frac{2}{QR}$$

$$\Rightarrow \quad \frac{1}{PS}+\frac{1}{ST}>\frac{4}{QR}$$

51. We have c $\cos B+\cos C=4\sin^2\left(\frac{A}{2}\right)$

$$\Rightarrow \quad 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)=4\sin^2\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad \cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)=2\sin^2\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad \sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)=2\sin^2\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad \cos\left(\frac{B-C}{2}\right)=2\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad \cos\left(\frac{B-C}{2}\right)=2\sin\left(\frac{\pi}{2}-\frac{B+C}{2}\right)$$

$$\Rightarrow \quad \cos\left(\frac{B-C}{2}\right)=2\cos\left(\frac{B+C}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)}=\frac{2}{1}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{B+C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)-\cos\left(\frac{B+C}{2}\right)}=\frac{2+1}{2-1}$$

$$\Rightarrow \quad \frac{2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}{2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}=3$$

$$\Rightarrow \quad \cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)=3$$

$$\Rightarrow \quad \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\cdot\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=3$$

$$\Rightarrow \quad \frac{s}{s-a}=3$$

$\Rightarrow \quad 3s-3a=s$

$\Rightarrow \quad 2s=3a$

$\Rightarrow \quad a+b+c=3a$

$\Rightarrow \quad b+c=2a$

Hence, the result.

52. We have $2\cos\left(\frac{\pi}{2k}\right)+2\cos\left(\frac{\pi}{k}\right)=\sqrt{3}+1$

$$\Rightarrow \quad \cos\left(\frac{\pi}{2k}\right)+\cos\left(\frac{\pi}{k}\right)=\frac{\sqrt{3}+1}{2}$$

$$\Rightarrow \quad \cos\left(\frac{\theta}{2}\right)+\cos(\theta)=\frac{\sqrt{3}+1}{2},\text{ where }\frac{\pi}{k}=\theta$$

$$\Rightarrow \quad 2\cos^2\left(\frac{\theta}{2}\right)-1+\cos\left(\frac{\theta}{2}\right)=\frac{\sqrt{3}+1}{2}$$

$$\Rightarrow \quad 2\cos^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)=\frac{\sqrt{3}+1}{2}+1=\frac{\sqrt{3}+3}{2}$$

$$\Rightarrow \quad 2t^2+t-\frac{\sqrt{3}+3}{2}=0,\text{ where }t=\cos\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad t=\frac{-1\pm\sqrt{1+4(3+\sqrt{3})}}{4}=\frac{-1\pm(2\sqrt{3}+1)}{4}$$

$$\Rightarrow \quad t=\frac{-2-2\sqrt{3}}{4},\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad t=\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \cos\left(\frac{\theta}{2}\right)=\frac{\sqrt{3}}{2}=\cos\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad \left(\frac{\theta}{2}\right)=\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad \theta=\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad \frac{\pi}{k}=\left(\frac{\pi}{3}\right)\Rightarrow k=3$$

53. We have, $ar(\Delta ABC)=\frac{1}{2}ab\sin C$

$$\Rightarrow \quad 15\sqrt{3}=\frac{1}{2}\cdot 6\cdot 10\cdot\sin C$$

$$\Rightarrow \quad \sin C=\frac{\sqrt{3}}{2}=\sin\left(\frac{2\pi}{3}\right),$$

since $C$ is obtuse.

$$\Rightarrow \quad C=\left(\frac{2\pi}{3}\right)$$

Also, $c^2=a^2+b^2-2ab\cos C$

$=10^2+6^2-2\cdot 10\cdot 6\cdot\cos\left(\frac{2\pi}{3}\right)$

$=100+36+60$

$=196$

$\Rightarrow \quad c=14$

Now, $2s=a+b+c=6+10+14=30$

$\Rightarrow \quad s=15$

Therefore, $r=\frac{\Delta}{s}=\frac{15\sqrt{3}}{15}=\sqrt{3}$

$\Rightarrow \quad r^2=3$

54. Ans. (b)

We have $a^2+b^2-c^2$

$=(x^2+x+1)^2+(x^2-1)^2-(2x+1)^2$

$=x^4+x^2+1+2x^3+2x^2+2+x^4-2x^2+1-4x^2-4x-1$

$=2x^4+2x^3-2x^2-2x+1$

$=(x-1)(x+1)(2x^2+2x-1)$

Now, $\cos\left(\frac{\pi}{6}\right)=\frac{a^2+b^2-c^2}{2ab}$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{(x^2-1)(2x^2+2x-1)}{2(x^2+x+1)(x^2-1)}$$

$$\Rightarrow \quad \sqrt{3}(x^2+x+1) = 2(2x^2+2x-1)$$

$$\Rightarrow \quad (2-\sqrt{3})x^2 + (2-\sqrt{3})x - (1+\sqrt{3}) = 0$$

$$\Rightarrow \quad x = -(2+\sqrt{3}), (\sqrt{3}+1)$$

Thus, $x = (\sqrt{3}+1)$.

55. Ans. (c)

We have $\dfrac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$

$$= \frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P}$$

$$= \frac{1-\cos P}{1+\cos P}$$

$$= \frac{2\sin^2(P/2)}{2\cos^2(P/2)}$$

$$= \tan^2\left(\frac{P}{2}\right)$$

$$= \frac{(s-b)(s-c)}{s(s-a)}$$

$$= \frac{(s-b)^2(s-c)^2}{s(s-a)(s-b)(s-c)}$$

$$= \frac{((s-b)(s-c))^2}{\Delta^2}$$

$$= \frac{\left(\left(4-\frac{7}{2}\right)\left(4-\frac{5}{2}\right)\right)^2}{\Delta^2}, \text{ where } s = 4$$

$$= \frac{\left(\frac{1}{2}\left(\frac{3}{2}\right)\right)^2}{\Delta^2}$$

$$= \left(\frac{3}{4\Delta}\right)^2$$

57. Ans. (b)

Given $\quad a + b = a,\ ab = y$

Also, $x^2 - c^2 = y$

$$\Rightarrow \quad (a+b)^2 - c^2 = ab$$

$$\Rightarrow \quad a^2 + b^2 - c^2 = -ab$$

$$\Rightarrow \quad \frac{a^2+b^2-c^2}{2ab} = -\frac{ab}{2ab} = -\frac{1}{2}$$

$$\Rightarrow \quad \cos C = -\frac{1}{2}$$

$$\Rightarrow \quad C = 120°$$

Now, $R = \dfrac{abc}{4\Delta},\ r = \dfrac{\Delta}{s}$

Thus, $\dfrac{r}{R} = \dfrac{4\Delta^2}{s(abc)}$

$$= \frac{4\left(\frac{1}{2}ab\sin 120°\right)^2}{\frac{x+c}{2}\cdot y\cdot c}$$

$$= \frac{4\left(\frac{1}{2}\cdot y\cdot\frac{\sqrt{3}}{2}\right)^2}{\frac{x+c}{2}\cdot y\cdot c}$$

$$= \frac{3y}{2(x+c)c}$$